REINFORCED CONCRETE
A FUNDAMENTAL APPROACH
FIFTH EDITION
ACI 318-05 CODE EDITION

EDWARD G. NAWY
The table includes various symbols and equations, which are meant to depict different aspects of structural engineering, particularly related to concrete structures. Here is a breakdown of the symbols used:

- $a$: depth of equivalent rectangular stress block.
- $A_{eq}$: area enclosed by outside perimeter of concrete cross section.
- $A_T$: gross area of section, in.$^2$.
- $A_y$: area of shear reinforcement parallel to flexural tension reinforcement, in.$^2$.
- $A_e$: Effective cross-sectional area within a joint, in.$^2$. In a plane parallel to plane of reinforcement generating shear in the joint. The joint depth shall be the overall depth of the column. Where a beam frame into a support of larger width, the effective width of the joint shall not exceed the smaller of:
  (a) beam width plus the joint depth
  (b) twice the smaller perpendicular distance from the longitudinal axis of the beam to the column side.
- $A_r$: total area of longitudinal reinforcement to resist torsion, in.$^2$.
- $A_o$: area of reinforcement in bracket or corbel resisting tensile force $N_{br}$, in.$^2$.
- $A_s$: gross area enclosed by shear flow path, in.$^2$.
- $A_{cm}$: area enclosed by centerline of the outermost closed transverse torsional reinforcement, in.$^2$.
- $A_{pr}$: area of prestressed reinforcement in tension zone, in.$^2$.
- $A_t$: area of non prestressed tension reinforcement, in.$^2$.
- $A'_t$: area of compression reinforcement, in.$^2$.
- $A_{ts}$: total cross-sectional area of transverse reinforcement (including cross-rectangular) within spacing $s$ and perpendicular to dimension $l$. $A_l$: area of one leg of a closed stirrup resisting torsion within a distance $s$, in.$^2$.
- $A_{ts}^e$: total cross-sectional area of transverse reinforcement (stirrup or tie) within a spacing $s$ and perpendicular to plane of bars being spliced or developed, in.$^2$.
- $A_{sh}$: area of shear reinforcement within a distance $s$, or area of shear reinforcement perpendicular to flexural tension reinforcement within a distance $s$ for deep flexural members, in.$^2$.
- $A_{sh}$: area of shear friction reinforcement, in.$^2$.
- $A_{sh}$: area of shear reinforcement parallel to flexural tension reinforcement within a distance $s$, in.$^2$.
- $b$: width of compression face of member, in.
- $b_c$: perimeter of critical section for slabs and footings, in.
- $b_w$: width of that part of cross section containing the closed stirrups resisting torsion.
- $b_x$: width of cross section at contact surface being investigated for horizontal shear.
- $b_w$: web width, or diameter of circular section, in.
- $c$: distance from extreme compression fiber to neutral axis, in.
- $c_e$: size of rectangular or equivalent rectangular column, capital, or bracket measured transverse to the direction of the span for which moments are being determined, in.
- $c_e$: size of rectangular or equivalent rectangular column, capital, or bracket measured transverse to the direction of the span for which moments are being determined, in.
- $d$: distance from extreme compression fiber to centroid of tension reinforcement, in.
- $d'$: distance from extreme compression fiber to centroid of compression reinforcement, in.
- $d_{br}$: nominal diameter of bar, wire, or prestressing strand, in.
- $d_{cpu}$: thickness of concrete cover measured from extreme tension fiber to center of bar or wire located closest thereto, in.
- $d_{ct}$: distance from extreme compression fiber to centroid of prestressed reinforcement.
- $e$: eccentricity of load parallel to axis of member measured from centroid of cross section.
- $E_c$: modulus of elasticity of concrete, psi.
- $E_r$: modulus of elasticity of bar reinforcement, psi.
- $E_{pr}$: modulus of elasticity of prestressing reinforcement.
- $f'_c$: specified compressive strength of concrete, psi.
- $f_{ac}$: average strength to be used as basis for selecting concrete proportions, psi.
- $f_{rv}$: required average compressive strength of concrete used as the basis for selection of concrete proportions, psi.
- $f_{crv}$: square root of specified compressive strength of concrete, psi.
- $f_{ec}$: compressive strength of concrete at time of initial prestress, psi.
- $f_{crv}$: square root of compressive strength of concrete at time of initial prestress, psi.
- $f_{sp}$: average splitting tensile strength of lightweight aggregate concrete, psi.


\[ f_a = \text{stress due to unfactored dead load, at extreme fiber of section where tensile stress is caused by externally applied loads, psi.} \]

\[ f_c = \text{compressive stress in concrete due to effective prestress forces only (after allowance for all prestress losses) at extreme fiber of section where tensile stress is caused by externally applied loads, psi.} \]

\[ f_p = \text{stress in prestressed reinforcement at nominal strength, psi.} \]

\[ f_{sy} = \text{specified yield strength of prestressing tendons, psi.} \]

\[ f_{tp} = \text{specified yield strength of prestressed tendons, psi.} \]

\[ f_c = \text{modulus of rupture of concrete, psi.} \]

\[ f_t = \text{tensile strength of concrete, psi.} \]

\[ f_{ys} = \text{specified yield strength of non prestressed reinforcement, psi.} \]

\[ f_y = \text{specified yield strength of transverse reinforcement, psi.} \]

\[ h = \text{overall thickness of member, in.} \]

\[ I = \text{moment of inertia of section resisting externally applied factored loads, in.}^4 \]

\[ I_e = \text{moment of inertia about centroidal axis of gross section of beam, in.}^4 \]

\[ I = \text{moment of inertia of cracked section transformed to concrete, in.}^4 \]

\[ I_e = \text{moment of inertia about centroidal axis of gross section of beam, in.}^4 \]

\[ I = \text{moment of inertia about centroidal axis of gross section of beam, in.}^4 \]

\[ k = \text{effective length factor for compression members.} \]

\[ K_b = \text{flexural stiffness of beam: moment per unit rotation.} \]

\[ K_c = \text{flexural stiffness of column: moment per unit rotation.} \]

\[ K_{es} = \text{flexural stiffness of equivalent column: moment per unit rotation.} \]

\[ K_t = \text{flexural stiffness of slab: moment per unit rotation.} \]

\[ K_{tb} = \text{torsional stiffness of torsional member: moment per unit rotation.} \]

\[ L_b = \text{basic development length of standard hook in tension, measured from critical section to outside end of hook (straight embedment length between critical section and start of hook [point of tangency] plus radius of bend and one bar diameter), in.} \]

\[ L = f_a \times \text{applicable modification factors.} \]

\[ M = \text{maximum moment in member at stage deflection is computed.} \]

\[ M_u = \text{factored moment to be used for design of compression member.} \]

\[ M_d = \text{moment due to dead load.} \]

\[ M_c = \text{cracking moment.} \]

\[ M_n = \text{nominal moment strength.} \]

\[ M_{ux} = \text{maximum factored moment at section due to externally applied loads.} \]

\[ M_s = \text{factored moment at section.} \]

\[ a = \text{modulus of elasticity, } E. \]

\[ E_{cl} = \text{factor for axial load normal to cross section occurring simultaneously with } V_{ts}, \text{ to be taken as positive for compression, negative for tension, and to include effects of tension due to creep and shrinkage.} \]

\[ V_{ts} = \text{factor for tensile force applied at top of bracket or corbel acting simultaneously with } V_{ts}, \text{ to be taken as positive for tension.} \]

\[ P_a = \text{nominal axial load strength at balanced strain conditions.} \]

\[ P_c = \text{critical buckling load.} \]

\[ P_e = \text{nominal axial load strength at given eccentricity.} \]

\[ P_o = \text{outside perimeter of the concrete cross-section } A_{oc}, \text{ in.} \]

\[ P_e = \text{perimeter of centerline of outermost closed transverse transversal reinforcement, in.} \]

\[ r = \text{radius of gyration of cross section of a compression member.} \]

\[ s = \text{spacing of shear or torsion reinforcement in direct parallel to longitudinal reinforcement, in.} \]

\[ t = \text{thickness of a wall of a hollow section, in.} \]

\[ T_s = \text{factored torsional moment at section.} \]

\[ V_s = \text{nominal shear strength provided by concrete.} \]

\[ V_{ns} = \text{nominal shear strength provided by concrete when diagonal cracking results from combined shear and moment.} \]

\[ V_{ts} = \text{nominal shear strength provided by concrete when diagonal cracking results from excessive principal tensile stress in web.} \]

\[ V_f = \text{shear force at section due to unfactored dead load.} \]

\[ V_p = \text{vertical component of effective prestress force at section.} \]

\[ V_s = \text{nominal shear strength provided by shear reinforcement.} \]

\[ V_f = \text{factored shear force at section.} \]
"Reflections"—High-strength polymer concrete sculpture at Rutgers University. Work by R. H. Keck, the civil engineering class of 1962, and the author.
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To
Rachel E. Nawy

For her high-limit state of endurance over the years, which made the writing of this book in its several editions a reality.
Reinforced concrete is a widely used material for structural systems. Hence, graduates of every civil engineering program must have, as a minimum requirement, a basic understanding of the fundamentals of reinforced concrete. Additionally, design of the members of a typical structure is achieved only by trial and adjustment, assuming a section, and then analyzing it. In other words, every design is essentially an analysis. Consequently, design and analysis were combined rather than不同merging the two processes in separate chapters as many other books do. In such a manner, it becomes simpler for the student first introduced to the subject of reinforced concrete design to inter-relate the two aspects of the engineering thinking process.

This edition of the book reflects the new AASHTO 318-05 Code version, including extensive changes in notations and the inclusion of new examples and diagrams in the strut and tie method. The load factors and the strength reduction factors are consistent with the ASCE 7-05 Standard and the International Building Code on Seismic Design (IBC 2000-2003). The Limit Strain Approach, which is sometimes referred to as the "unified method" in the code, is the basis of the design process in this edition, with numerous analysis and design examples applying the Limit Strain hypothesis. As a consequence, all examples in the book use the new load factors and the new strength reduction factors for flexure, shear, torsion, and those to be used for columns and other compression members.

In addition to the prolific number of analysis and design examples in the book as well as the numerous flow charts, another significant feature of this book is the inclusion of examples in SI units in most of the chapters and a listing of the relevant equation in the SI format. This manner, the student as well as the practicing engineer can avail themselves of the tools for transition from the lb-ft. (SI) system to the System International (SI) where needed.

The text is an outgrowth of the author's lecture notes evolved in teaching the subject at Rutgers University over the past forty-five years and the experience accumulated over the years in teaching and research in the areas of reinforced and prestressed concrete inclusive of the Ph.D. level. The material is presented in such a manner that the student can be familiarized with the properties of plain concrete and its components, both for normal strength and high performance concrete, prior to embarking on the study of structural behavior. The book is uniquely different from other textbooks in that a good segment of its contents can be covered in one semester at the undergraduate level in spite of the in-depth discussions of some of its major topics. The book can effectively be suited to the senior year level, and continue to be used at more advanced and graduate levels.

The concise discussion presented in Chapters 1 through 4 on the historical development of concrete, the proportioning of the constituent materials, long-term behavior, and the development of safety factors, should give an adequate introduction to the subject of reinforced concrete. It should also aid in developing fundamental laboratory experiments and essential knowledge of mixture proportioning, strength and behavioral
requirements, and the concepts of reliability of performance of structures to which every engineering student and engineer should be exposed. The discussion of quality assurance should also give the reader a good introduction to a systematic approach needed for administering the development of concrete structural systems from conception to turnkey use.

Since concrete is a non-elastic material, with the non-linearity of its behavior starting at a very early stage of loading, only the ultimate strength approach, or what is sometimes termed the "limit state at failure approach," is given in this book. The working load approach was eliminated in the ACI 318-05 Code. Adequate coverage is given of the serviceability checks in terms of cracking and deflection behavior, as well as long-term effects. In this manner, the design should satisfy all the service-load-level requirements while ensuring that the theory used in the analysis (design) truly describes the actual behavior of the designed component.

Chapters 5, 6, 7, and 8 cover the flexural, diagonal tension, torsion, and serviceability behavior of one-dimensional members: beams and one-way slabs. Full emphasis has been placed on giving the student and the engineer a feeling for the internal strain distribution in structural reinforced concrete elements and a basic understanding of the reserve strength and the safety factors inherent in the design expressions. Chapter 9, on the analysis and design of columns and other compression members, treats the subject of strain compatibility and strain distribution in a manner similar to that in Chapter 5 on flexural analysis and design of beams. It includes a detailed discussion of how to construct interaction diagrams for columns as well as proportioning columns subjected to bi-axial bending and buckling as well as the P-delta effect.

It is important to mention that Chapter 10, on diagonal tension, also contains detailed coverage of the behavior of deep beams, corbels, and brackets, with sufficient design examples in supplement the theory. This topic has been included in view of the increased use of precast construction, the widespread understanding of the effects of reduced horizontal loads on floors, and the frequent need for including shear walls and deep beams in today's multilevel structures. A new section was added on the strut-and-tie modeling of structures, with particular emphasis on deep beams and corbels, which appears as an appendix in the ACI 318-05 Code. It includes an extensive deep beam design example, to aid the designer who elects this method for the design of deep beams, and an additional new example on the strut-and-tie design of corbels. Additionally, Chapter 7 treats the topic of torsion in some detail considering the space constraints of the book. The discussion ranges from the basic fundamentals of pure torsion in elastic and plastic materials to the design of reinforced concrete members subjected to combined torsion, shear, and bending. The material presented and the accompanying illustrative examples should give the reader a basic understanding of the complex behavior of reinforced concrete systems.

Chapter 11 presents an extensive coverage of the subject of analysis and design of two-way slab and plane floor systems. Following a discussion of fundamental behavior, it gives detailed design examples using both the ACI procedures and yield-line theory for the flexural design of reinforced concrete slabs. It also includes ultimate load solutions to most floor shapes and possible gravity loading patterns. Detailed discussion of the deflection behavior and evaluation of two-way panels, as well as the cracking mechanism of such panels, with appropriate analysis examples, makes this chapter another unique feature of this concise textbook.

Chapter 12 deals with continuous reinforced concrete structures. It presents a review of the various methods of analysis for continuity of multi-span beams and portals and gives relevant examples including those on the topics of limit theory and plastic hinges. Chapter 14 is an introduction to prestressed concrete consists with ACI and PCI standards. It should serve as a brief treatment of the subject in order to illustrate the
fundamental differences between reinforced and prestressed concrete. Chapter 13 on the LRFD design of bridge deck structures with extensive example gives a snapshot of the relevant AASHTO requirements for truckloads, expressions for flexural design and the modified compression field approach for shear and torsion as presented in AASHTO 2002-2003 provisions. Chapter 16, dealing with the seismic behavior of concrete structures, is one of the highlights of this book. It presents the subject in a concise manner as possible, yet is comprehensive enough to give several examples on the proportioning of elements of a frame and a shear wall with boundary elements, conforming to the latest ACI and IBC provisions.

The numerous flowcharts for every topic presented in the book should aid the user in developing the logic and step-by-step thinking in easily comprehending the analysis and design procedure for efficient reinforced concrete systems, supplemented with numerous charts and design tables in Appendix A.

Selected photographs of various areas of structural behavior of concrete elements at failure are included in all the chapters. They are taken from the extensive published research work by the author with many of his M.S. and Ph.D. students at Rutgers University over the past four decades. These photographs of tests and failure of various types of structural elements should aid the reader in visualizing the behavior of structural elements under load, particularly in departments where undergraduate testing of structural members is cost prohibitive. Additionally, photographs of landmark structures mainly in the United States are included throughout the book to illustrate the versatility of design in reinforced concrete.

The textbook fully conforms to the provisions of ACI 318-05 with an eye to stressing the basics rather than sifting every step to the code, which changes once every three years. Consequently, no attempt was made to tie any design or analysis step to the particular equation number in the code, but rather, the student is expected to gain the habit of getting familiar with the provisions and sections numbers of the ACI Code as a dynamic, ever-changing document. Conversions to SI units are included in the illustrative examples throughout the book, in addition to the separate solutions in SI Units, which have been added to most chapters in this edition.

The various topics have been presented in as concise a manner as possible but without sacrificing the need for the instructional details by students first exposed to reinforced concrete design. Hence, the topic of prestressed concrete has been only briefly covered in Chapter 14 and the reader is left to pursue more advanced works such as the author’s book Prestressed Concrete: A Fundamental Approach, Fourth Edition, 2005, also conforming to the ACI 318-05 Code.

Portions of this book are intended for a first course at the junior or senior level of the standard college or university curriculum in civil engineering, while the advanced topics can be adequately covered for use at the graduate level. The contents should also serve as a valuable guide to the practicing engineer who has to keep abreast of the state of the art in concrete, as well as the designer who is interested in a concise treatment of the fundamentals.

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1.1 HISTORICAL DEVELOPMENT OF STRUCTURAL CONCRETE

Concrete and its cementitious (volcanic) constituents, such as pozzolanic ash, have been used since the days of the Greeks, the Romans, and possibly earlier ancient civilizations. However, the early part of the nineteenth century marks the start of more intensive use of the material. In 1801, F. Coignet published his statement of principles of construction, recognizing the weakness of the material in tension. J. L. Lambot in 1850 constructed for the first time a small cement boat for exhibition in the 1855 World's Fair in Paris. J. Monier, a French gardener, patented in 1867 metal frames as reinforcement for concrete garden plant containers, and Koenen in 1886 published the first manuscript on the theory and design of concrete structures. In 1906, C. A. F. Turner developed the first flat slab without beams.

Thereafter, considerable progress occurred in this field such that by 1910 the German Committee for Reinforced Concrete, the Austrian Concrete Committee, the American Concrete Institute, and the British Concrete Institute were already established. Many buildings, bridges, and liquid containers of reinforced concrete were already constructed by 1920, and the era of linear and circular prestressing began.

Photo 1.1 Chaochow Bridge on Huaolo River, China (a.e. 605-617).
Chapter 1 Introduction

The rapid developments in the art and science of reinforced and prestressed concrete analysis, design, and construction have resulted in unique structural systems, such as the Krege Auditorium, Boston; the 1951 Festival of Britain Dome; Martin Tower and Lake Point Tower, Chicago; the Trump Tower, New York; Two Union Square Towers, Seattle; and many, many others.

Ultimate-strength theories were codified in 1938 in the USSR and in 1936 in England and the United States. Limit theories have also become a part of codes of several countries throughout the world. New constituent materials and composites of concrete have become prevalent, including the high-strength concretes of a strength in compression up to 20,000 psi (137.9 MPa) and 1800 psi (12.41 MPa) in tension. Steel reinforcing bars of strength in excess of 60,000 psi (413.7 MPa) and high-strength welded wire fabric in excess of 100,000 psi (699.5 MPa) ultimate strength are being used. Additionally, deformed bars of various forms have been produced. Such deformations help develop the maximum possible bond between the reinforcing bars and the surrounding concrete as a requisite for the viability of concrete as a structural medium. Prestressing steel of ultimate strength in excess of 300,000 psi (2068 MPa) is available.

All these developments and the massive experimental and theoretical research that has been conducted, particularly in the last two decades, have resulted in rigorous theories and codes of practice. Consequently, a simplified approach has become necessary to understand the fundamental structural behavior of reinforced concrete elements.

1.2 BASIC HYPOTHESIS OF REINFORCED CONCRETE

Plain concrete is formed from a hardened mixture of cement, water, fine aggregate, coarse aggregate (crushed stone or gravel), air, and often other admixtures. The plastic mix is placed and consolidated in the forms and, then cured to facilitate the acceleration of the chemical hydration reaction of the cement-water mix, resulting in hardened concrete. The finished product has high compressive strength and low resistance to tens-
sion, such that its tensile strength is approximately one-tenth of its compressive strength. Consequently, tensile and shear reinforcement in the tensile regions of sections has to be provided to compensate for the weak-tension regions in the reinforced concrete element.

It is this deviation in the composition of a reinforced concrete section from the homogeneity of standard wood or steel sections that requires a modified approach to the basic principles of structural design, as will be explained in subsequent chapters of this book. The two components of the heterogeneous reinforced concrete section are to be so arranged and proportioned that optimal use is made of the materials involved. This is possible because concrete can easily be given any desired shape by placing and compacting the wet mixture of the constituent ingredients into suitable forms in which the plastic mass hardens. If the various ingredients are properly proportioned, the finished product becomes strong, durable, and, in combination with the reinforcing bars, adaptable for use as main members of any structural system.

1.3 ANALYSIS VERSUS DESIGN OF SECTIONS

From the foregoing discussion, clearly a large number of parameters have to be dealt with in proportioning a reinforced concrete element, such as geometrical width, depth, area of reinforcement, steel strain, concrete strain, and steel stress. Consequently, trial and adjustment are necessary in the choice of concrete sections, with assumptions based on conditions at site, availability of the constituent materials, particular demands of the owners, architectural and headroom requirements, applicable codes, and environmental conditions. Such an array of parameters has to be considered because of the fact that reinforced concrete is often a site-constructed composite, in contrast to the standard mill-fabricated beam and column sections in steel structures.
Chapter 1  Introduction

Photo 1.4  Rockefeller Empire State Plaza, Albany, New York—Ammann & Whitney design. (Courtesy of New York Office of General Services.)

A trial section has to be chosen for each critical location in a structural system. The trial section has to be analyzed to determine if its nominal resisting strength is adequate to carry the applied factored load. Since more than one trial is often necessary to arrive at the required section, the first design input step generates a series of trial-and-adjustment analyses.

The trial-and-adjustment procedures for the choice of a concrete section lead to the convergence of analysis and design. Hence every design is an analysis once a trial section is chosen. The availability of handbooks, charts, desktop and handheld personal computers and programs supports this approach as a more efficient, compact, and speedy instructional method, compared with the traditional approach of treating the analysis of reinforced concrete separately from pure design.

Photo 1.5  Empire State Performing Arts Center, Albany, New York—Ammann & Whitney design. (Courtesy of New York Office of General Services.)
Photo 1.6  Toronto City Hall, Toronto, Canada. (Courtesy of Portland Cement Association.)

Photo 1.7  Two Union Square Towers, Seattle, Washington, 72 stories and 759 ft. high. Concrete strength is 30,000 psi. Design by the NBBI Group, Architects, Seattle, Washington. (Courtesy of Dr. Weston Huston and Turner Construction Company.)
Chapter 1  Introduction

Photo 1.8  The Trump Towers, Fifth Avenue, New York City: concrete strength in excess of 8000 psi. (Courtesy of Concrete Industry Board.)

2.1 INTRODUCTION

To understand and interpret the total behavior of a composite element requires a knowledge of the characteristics of its components. Concrete is produced by the collective mechanical and chemical interaction of a large number of constituent materials. Hence a discussion of the functions of each of these components is vital prior to studying concrete as a finished product. In this manner, the designer and the materials engineer can develop skills for the choice of the proper ingredients and so proportion them as to obtain an efficient and desirable concrete satisfying the designer's strength and serviceability requirements.

This chapter presents a brief account of the concrete-producing materials: cement, fine and coarse aggregate, water, air, and admixtures. The cement manufacturing process, the composition of cement, the type and gradation of fine and coarse aggregate, and the function and importance of the water, air, and admixtures are reviewed. The reader is referred to books on concrete, such as the selected references at the end of this chapter, for further information.

Photo 2.1 LaGuardia Airport parking garage ramps, New York.
2.2 PORTLAND CEMENT

2.2.1 Manufacture

Portland cement is made of finely powdered crystalline minerals composed primarily of calcium and aluminum silicate. The addition of water to these minerals produces a paste that, when hardened, becomes of stone-like strength. Its specific gravity ranges between 3.12 and 3.16 and it weighs 94 lb/ft³, which is the unit weight of a commercial sack or bag of cement.

The raw materials that make cement are:

1. Lime (CaO), from limestone
2. Silica (SiO₂), from clay
3. Alumina (Al₂O₃), from clay

(with very small percentages of magnesia; MgO and sometimes some alkalies). Iron oxide is occasionally added to the mixture to aid in controlling its composition.

The process of manufacture can be summarized as follows:

1. The raw mix of CaO, SiO₂, and Al₂O₃ is ground with other added minor ingredients either in dry or wet form. The wet form is called a slurry.
2. The mixture is fed into the upper end of a slightly inclined rotary kiln.
3. As the heated kiln operates, the material passes from its upper to its lower end at a predetermined, controlled rate.
4. The temperature of the mixture is raised to the point of incipient fusion, that is, the clinkering temperature. It is kept at that temperature until the ingredients combine
to form at 2700°F the portland cement pellet product. The pellets, which range in size from 4 to 2 in., are called clinkers.

5. The clinkers are cooled and ground to a powdery form.

6. A small percentage of gypsum is added during grinding to control or retard the setting time of the cement in the field.

7. Most of the final portland cement goes into silos for bulk shipment; some is packed in 94-lb bags for retail marketing.

Figure 2.1 illustrates schematically the manufacturing process of portland cement. The form and properties of the manufactured compound are described in the following sections.

2.2.2 Strength

The strength of cement is the result of a process of hydration. This chemical process results in recrystallization in the form of interlocking crystals producing the cement gel, which has high compressive strength when it hardens. Table 2.1 shows the relative contribution of each component of the cement toward the rate of gain in strength. The early strength of portland cement is higher with higher percentages of C₃S. If moist curing is continuous, later strength levels will be greater, with higher percentages of C₃S. C₃A contributing to the strength developed during the first day after placing the concrete because it is the earliest to hydrate.

**Table 2.1 Properties of Cements**

<table>
<thead>
<tr>
<th>Component</th>
<th>Rate of Reaction</th>
<th>Heat Liberated</th>
<th>Ultimate Cementing Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tricalcium silicate, C₃S</td>
<td>Medium</td>
<td>Medium</td>
<td>Good</td>
</tr>
<tr>
<td>Dicalcium silicate, C₂S</td>
<td>Slow</td>
<td>Small</td>
<td>Good</td>
</tr>
<tr>
<td>Tricalcium aluminate, C₃A</td>
<td>Fast</td>
<td>Large</td>
<td>Poor</td>
</tr>
<tr>
<td>Tetracalcium aluminoferrite, C₅AF</td>
<td>Slow</td>
<td>Small</td>
<td>Poor</td>
</tr>
</tbody>
</table>
Chapter 2  Concrete-Producing Materials

When portland cement combines with water during setting and hardening, lime is liberated from some of the compounds. The amount of lime liberated is approximately 20% by weight of the cement. Under unfavorable conditions, this might cause disintegration of a structure owing to leaching of the lime from the cement. Such a situation should be prevented by adding a siliceous mineral such as pozzolan to the cement. The added mineral reacts with the lime in the presence of moisture to produce strong calcium silicate.

2.2.3 Average Percentage Composition

Since there are different types of cement for various needs, it is necessary to study the percentage variation in the chemical composition of each type in order to interpret the reasons for variation in behavior. Table 2.2, studied in conjunction with Table 2.1, gives concise reasons for the difference in reaction of each type of cement when in contact with water.

2.2.4 Influence of Fineness of Cement on Strength Development

The size of the cement particles strongly influences the rate of reaction of cement with water. For a given weight of finely ground cement, the surface area of the particles is greater than that of the coarsely ground cement. This results in a greater rate of reaction with water and a more rapid hardening process for larger surface areas. This is one reason for the high early-strength type III cement giving in 3 days a strength that type I gives in 7 days and a strength in 7 days that type I gives in 28 days.

2.2.5 Influence of Cement on the Durability of Concrete

Disintegration of concrete due to cycles of wetting, freezing, thawing, and drying and the propagation of resulting cracks is a matter of great importance. The presence of minute air voids throughout the cement paste increases the resistance of concrete to disintegration. This can be achieved by the addition of air-entraining admixtures to the concrete while mixing.

Disintegration due to chemicals in contact with the structure, such as in the case of port structures and substructures, can also be slowed down or prevented. Since the concrete in such cases is exposed to chlorides and sometimes sulfates of magnesium and

<table>
<thead>
<tr>
<th>Type of Cement</th>
<th>Component (%)</th>
<th>General Characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C₃S</td>
<td>C₅S</td>
</tr>
<tr>
<td>Normal: I</td>
<td>49</td>
<td>25</td>
</tr>
<tr>
<td>Modified: II</td>
<td>45</td>
<td>29</td>
</tr>
<tr>
<td>High early strength: III</td>
<td>56</td>
<td>15</td>
</tr>
<tr>
<td>Low heat: IV</td>
<td>30</td>
<td>46</td>
</tr>
<tr>
<td>Sulfate resisting: V</td>
<td>43</td>
<td>36</td>
</tr>
</tbody>
</table>
sodium, it is sometimes necessary to specify sulfate-resisting cements. Usually, type II cement will be adequate for use in seawater structures.

2.2.6 Heat Generation during Initial Set
Since the different types of cement generate different degrees of heat at different rates, the type of structure governs the type of cement to be used. The bulkier and heavier in cross section the structure is, the less the generation of heat of hydration that is desired. In massive structures such as dams, piers, and caissons, type IV cement is more advantageous to use. From this discussion it is seen that the type of structure, the weather, and other conditions under which it is built and will exist are the governing factors in the choice of the type of cement that should be used.

2.3 WATER AND AIR

2.3.1 Water
Water is required in the production of concrete in order to precipitate chemical reaction with the cement, to wet the aggregate, and to lubricate the mixture for easy workability. Normally, drinking water can be used in mixing. Water having harmful ingredients, contamination, silt, oil, sugar, or chemicals is destructive to the strength and setting properties of cement. It can disrupt the affinity between the aggregate and the cement paste and can adversely affect the workability of a mixture.

Since the character of the colloidal gel or cement paste is the result only of the chemical reaction between cement and water, it is not the proportion of water relative to the whole of the mixture of dry materials that is of concern, only the proportion of water relative to the cement. Excessive water leaves an uneven honeycombed skeleton in the finished product after hydration has taken place, while too little water prevents complete chemical reaction with the cement. The product in both cases is a concrete that is weaker than and inferior to normal concrete.

2.3.2 Entrained Air
With the gradual evaporation of excess water from the mix, pores are produced in the hardened concrete. If evenly distributed, these could give improved characteristics to the product. Very uneven distribution of pores by artificial introduction of finely divided, uniformly distributed air bubbles throughout the product is possible by adding air-entraining agents such as viscosol resin. Air entrainment increases workability, decreases density, increases durability, reduces bleeding and segregation, and reduces the required sand content in the mix. For these reasons, the percentage of entrained air should be kept at the required optimum value for the desired quality of the concrete. The optimum air content is 9% of the mortar fraction of the concrete. Air entraining in excess of 5 to 6% of the total mix proportionally reduces the concrete strength.

2.3.3 Water/Cement Ratio
To summarize the preceding discussion, strict control has to be maintained on the water/cement ratio and the percentage of air in the mixture. As the water/cement ratio is the real measure of the strength of the concrete, it should be the principal criterion governing the design of most structural concretes. It is usually given as the ratio of weight of water to the weight of cement in the mixture.
2.3.4 Water/Cementitious Ratio

For high-strength high-performance concrete, mineral pozzolane or chemical admixtures are used replacing part of the cement in a particular mixture design. Hence, the water/cement ratio (w/c) would not be the governing criteria for strength requirement, but the water/cementitious ratio, \( W/(C + P) \).

2.4 AGGREGATES

Aggregates are those parts of the concrete that constitute the bulk of the finished product. They comprise 50 to 80% of the volume of the concrete and have to be so graded that the whole mass of concrete acts as a relatively solid, homogeneous, dense combination, with the smaller sizes acting as an inert filler of the voids that exist between the larger particles.

Aggregates are of two types:

1. Coarse aggregate: gravel, crushed stone, or blast-furnace slag
2. Fine aggregate: natural or manufactured sand

Since the aggregate constitutes the major part of the mixture, the more aggregate in the mix, the cheaper is the cost of the concrete, provided that the mixture is of reasonable workability for the specific job for which it is used.

2.4.1 Coarse Aggregate

Coarse aggregate is classified as such if the smallest size of the particles is greater than \( \frac{1}{2} \) in. (6 mm). Properties of the coarse aggregate affect the final strength of the hardened concrete and its resistance to disintegration, weathering, and other destructive effects. The mineral coarse aggregate must be clean of organic impurities and must bond well with the cement gel.

The common types of coarse aggregate are:

1. Natural crushed stone. This is produced by crushing natural stone or rock from quarries. The rock could be of igneous, sedimentary, or metamorphic type. Although crushed rock gives higher concrete strength, it is less workable in mixing and placing than are the other types.
2. Natural gravel. This is produced by the weathering action of running water on the beds and banks of streams. It gives less strength than crushed rock but is more workable.
3. Artificial coarse aggregates. These are mainly slag and expanded shale and are frequently used to produce lightweight concrete. They are by-products of other manufacturing processes, such as blast-furnace slag or expanded shale, or pumice for lightweight concrete.
4. Heavyweight and nuclear-shielding aggregates. With the specific demands of our atomic age and the hazards of nuclear radiation due to the increasing number of atomic reactors and nuclear power stations, specialized concretes have had to be produced to shield against x-rays, gamma rays, and neutrons. In such concretes, economic and workability considerations are not of prime importance. The main heavy, coarse aggregate types are steel punchings, ballast, magnetite, and limonite.
Whereas concrete with ordinary aggregate weighs about 144 lb/ft\(^3\), concrete made with these heavy aggregates weighs from 225 to 330 lb/ft\(^3\). The property of high density radiation-shielding concrete depends on the density of the compact product rather than primarily on the water-cement ratio criteria. In certain cases, high density is the only consideration, whereas in others both density and strength govern.

2.4.2 Fine Aggregate

Fine aggregate is a smaller filler made of sand. It ranges in size from No. 4 to No. 100 (4.75 mm to 150 \(\mu\)m) U.S. standard sieve sizes. A good fine aggregate should always be free of organic impurities, clay, or any deleterious material or excessive filler of size smaller than No. 100 sieve. It should preferably have a well-graded combination conforming to the American Society for Testing and Materials (ASTM) sieve analysis standards. For radiation-shielding concrete, fine steel shot and crushed iron ore are used as fine aggregate.

2.4.3 Grading for Normal-weight Concrete Mixtures

The recommended grading of coarse and fine aggregates for normal-weight concretes is presented in Table 2.3.

2.4.4 Grading for Lightweight Concrete Mixtures

The grading requirements for lightweight aggregate for structural concrete are given in Table 2.4.

2.4.5 Grading of heavyweight and Nuclear-shielding Aggregates

The grading requirements to ensure heavyweight concrete are given in Table 2.5.

<table>
<thead>
<tr>
<th>Table 2.3 Grading Requirements for Aggregates in Normal-Weight Concrete (ASTM C-33)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>percent passing</strong></td>
</tr>
<tr>
<td><strong>U.S. Standard Sieve Size, in (mm)</strong></td>
</tr>
<tr>
<td>2 in. (50.8)</td>
</tr>
<tr>
<td>1-1/2 in. (37.5)</td>
</tr>
<tr>
<td>1 in. (25.0)</td>
</tr>
<tr>
<td>1/2 in. (19.0)</td>
</tr>
<tr>
<td>1/4 in. (12.5)</td>
</tr>
<tr>
<td>1/8 in. (9.5)</td>
</tr>
<tr>
<td>No. 4 (4.75)</td>
</tr>
<tr>
<td>No. 8 (2.36)</td>
</tr>
<tr>
<td>No. 16 (1.18)</td>
</tr>
<tr>
<td>No. 30 (600 (\mu)m)</td>
</tr>
<tr>
<td>No. 50 (300 (\mu)m)</td>
</tr>
<tr>
<td>No. 100 (150 (\mu)m)</td>
</tr>
<tr>
<td>Size Designation</td>
</tr>
<tr>
<td>------------------</td>
</tr>
<tr>
<td>Fine aggregate</td>
</tr>
<tr>
<td>No. 4 to 0</td>
</tr>
<tr>
<td>Coarse aggregate</td>
</tr>
<tr>
<td>1 in. to No. 4</td>
</tr>
<tr>
<td>1/2 in. to No. 4</td>
</tr>
<tr>
<td>1/4 in. to No. 4</td>
</tr>
<tr>
<td>1/8 in. to No. 8</td>
</tr>
<tr>
<td>Combined fine</td>
</tr>
<tr>
<td>and coarse</td>
</tr>
<tr>
<td>aggregate</td>
</tr>
<tr>
<td>3/4 in. to 0</td>
</tr>
</tbody>
</table>
2.5 Admixtures

Table 2.5 Grading Requirements for Coarse Aggregate for Heavyweight Concrete
(ASTM C-637)

<table>
<thead>
<tr>
<th>Sieve Size</th>
<th>Grading 1: for 1 in. (37.5 mm)</th>
<th>Grading 2: for 1 in. (19.0 mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coarse Aggregate</td>
<td>Maximum-size Aggregate</td>
</tr>
<tr>
<td>2 in. (50 mm)</td>
<td>100</td>
<td>—</td>
</tr>
<tr>
<td>1 1/2 in. (37.5 mm)</td>
<td>95-100</td>
<td>100</td>
</tr>
<tr>
<td>1 in. (25.0 mm)</td>
<td>40-80</td>
<td>55-100</td>
</tr>
<tr>
<td>3/4 in. (19.0 mm)</td>
<td>20-45</td>
<td>40-80</td>
</tr>
<tr>
<td>3/8 in. (12.5 mm)</td>
<td>0-10</td>
<td>0-15</td>
</tr>
<tr>
<td>3/16 in. (9.5 mm)</td>
<td>0-2</td>
<td>0-2</td>
</tr>
<tr>
<td>Fine Aggregate</td>
<td>No. 5 (2.36 mm)</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>No. 6 (1.18 mm)</td>
<td>95-100</td>
</tr>
<tr>
<td></td>
<td>No. 10 (0.45 mm)</td>
<td>55-95</td>
</tr>
<tr>
<td></td>
<td>No. 20 (0.045 mm)</td>
<td>30-55</td>
</tr>
<tr>
<td></td>
<td>No. 100 (0.005 mm)</td>
<td>10-30</td>
</tr>
<tr>
<td></td>
<td>No. 200 (0.002 mm)</td>
<td>0-10</td>
</tr>
<tr>
<td>Fineness modulus</td>
<td>1.30-2.10</td>
<td>1.00-1.60</td>
</tr>
</tbody>
</table>

Data in Table 2.5 to 2.7 reprinted with permission from the American Society for Testing and Materials, Philadelphia, Pa.

2.4.6 Unit Weights of Aggregates

The unit weight of the concrete depends on the unit weight of the aggregate, which in turn depends on the type of aggregate: whether it is normal, lightweight, or heavyweight (for radiation shielding). Table 2.6 gives the unit weights of the various aggregates and the corresponding unit weight of the concrete.

2.5 ADMIXTURES

Admixtures are materials other than water, aggregate, or hydraulic cement that are used as ingredients of concrete and that are added to the batch immediately before or during

Table 2.6 Unit Weight of Aggregates

<table>
<thead>
<tr>
<th>Type</th>
<th>Unit Weight of Dry-rodded Aggregate (lb/ft³)*</th>
<th>Unit Weight of Concrete (lb/ft³)*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insulating concrete</td>
<td>15-50</td>
<td>20-90</td>
</tr>
<tr>
<td>(perlite, vermiculite, etc.)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Structural lightweight</td>
<td>40-70</td>
<td>90-110</td>
</tr>
<tr>
<td>Normal weight</td>
<td>70-110</td>
<td>130-160</td>
</tr>
<tr>
<td>Heavyweight</td>
<td>&gt;125</td>
<td>180-380</td>
</tr>
</tbody>
</table>

*1 lb/ft³ = 0.0160 kglm⁻³.
Chapter 2  Concrete-Producing Materials

the mixing. Their function is to modify the properties of the concrete so as "to make it more suitable for the work at hand, or for economy, or for other purposes such as saving energy" (Ref. 2.6). The major types of admixtures can be summarized as follows:

1. Accelerating admixtures
2. Air-entraining admixtures
3. Water-reducing admixtures and set controlling admixtures
4. Finely divided mineral admixtures
5. Admixtures for no-slump concretes
6. Polymers
7. High-range water-reducing admixtures (HRWRA)

2.5.1 Accelerating Admixtures

These admixtures are added to the concrete mix to reduce the time of setting and accelerate early strength development. The best known are calcium chlorides. Other accelerating chemicals include a wide range of soluble salts, such as chlorides, bromides, carbonates, and silicates, and some other organic compounds, such as triethanolamine.

It must be stressed that calcium chlorides should not be used where progressive corrosion of steel reinforcement can occur. The maximum dosage is 1% by weight of the portland cement, and preferably ½%.

2.5.2 Air-entraining Admixtures

These admixtures form minute bubbles 1 mm in diameter or smaller in the concrete or mortar during mixing and are used to increase the workability of the mixture during placing and the frost resistance of the finished product. Most air-entraining admixtures are in

Photo 2.3 Scanning electron microscope photograph of polymer-cement mortar fracture surface under tension. (Tests by Nasmyth, Sun, and Sauer.)
2.5 Admixtures

Photo 2.4 Scanning electron microscope photograph of concrete fracture surface.
(Tests by Newby, Sun, and Sauer.)

liquid form, although a few are powders, flakes, or semisolids. The amount of the admixture required to obtain a given air content depends on the shape and the grading of the aggregate used. The finer the size of the aggregate, the larger is the percentage of admixture needed. It is also governed by several other factors, such as type and condition of the mixer, use of fly ash or other pozzolans, and the degree of agitation of the mixture. It can be expected that air entrainment reduces the strength of the concrete. Maintaining cement content and workability, however, offsets the partial reduction of strength because of the resulting reduction in the water/cement ratio.

2.5.3 Water-reducing and Set-controlling Admixtures

These admixtures increase the strength of the concrete. They also enable reducing the cement content in proportion to the reduction in the water content.

Most admixtures of the water-reducing type are water soluble. The water they contain becomes part of the mixing water in the concrete and is added to the total weight of water in the design of the mix. It has to be emphasized that the proportion of the mortar to the coarse aggregate should always remain the same. Changes in the water content, air content, or cement content are compensated for by corresponding changes in the fine aggregate content so that the volume of the mortar remains the same.

2.5.4 Finely Divided Admixtures

These are mineral admixtures used to rectify deficiencies in the concrete mixture by providing missing fines from the fine aggregate; improving one or more qualities of the concrete, such as reducing permeability or expansion; and reducing the cost of concretemaking materials. Such admixtures include hydraulic lime, slag cement, fly ash, and raw or calcined natural pozzolan.
2.5.5 Admixtures for No-slump Concrete

No-slump concrete is defined in Ref. 2.6 as a concrete with a slump of 1 in. (25 mm) or less immediately after mixing. The choice of the admixture depends on the desired properties of the finished product, such as its effect on the plasticity, setting time and strength development, freeze-thaw effects, and strength and cost.

2.5.6 Polymers

There are new types of admixtures that enable producing concretes of very high strength up to a compressive strength of 15,000 psi or higher and a tensile splitting strength of 1500 psi or higher. Such concretes are generally produced using a polymerizing material through (1) modifying the concrete property through water reduction in the field or (2) impregnating and irradiating under elevated temperature in laboratory environment.

Polymers-modified concrete (PMC) is concrete made through the addition of resin and hardener as an “admixture.” The principle is to replace part of the mixing water by the polymer so as to attain the high compressive strength and other qualities reported in detail in Ref. 2.7. The optimum polymer/cement ratio by weight seems to lie within the range of 0.3 to 0.45 to achieve such high compressive strengths.

2.5.7 Superplasticizers

These are admixtures which can be termed “high-range, water-reducing chemical admixtures.” There are four types of plasticizers:

1. Sulfonated melamine formaldehyde condensates, with a chloride content of 0.005% (MSF)
2. Sulfonated naphthalene formaldehyde condensates, with negligible chloride content (NSF)
3. Modified lignosulfonates, which contain no chlorides

These admixtures are made from organic sulfonates and are termed superplasticizers in view of their considerable ability to facilitate reducing the water content in a concrete mixture which simultaneously increasing the slump up to 8 in. (203 mm) or more. A dosage of 1% to 2% by weight of cement is advisable. Higher dosages can result in a reduction in compressive strength.

4. Other superplasticizers, such as sulfonic acid esters or other carbohydrate esters

A dosage of 1% to 2% by weight of cement is advisable. Higher dosages can result in a reduction in compressive strength unless the cement content is increased to balance this reduction effect. It should be noted that the superplasticizers exert their action by decreasing the surface tension of water and by equidirectional charging of the cement particles. These properties, coupled with the addition of silica fume, help the concrete achieve high strength and water reduction without loss of workability.

2.5.8 Silica-fume Admixture Use in High-strength Concrete

Silica fume is generally accepted as an efficient admixture for high-strength concrete mixtures. It is a pozzolanic material that has received considerable attention in both research and application. Silica fume is a by-product resulting from the use of high-purity quartz with coal in the electric arc furnace in the production of silicon and ferrosilicon alloys. Its main constituent, fine spherical particles of silicon dioxide, makes it an ideal cement replacement, simultaneously raising the concrete strength. Being a waste product...
with relative ease of collection as compared to fly ash or slag, silica fume has gained rapid popularity. Norway first experimented with this product, followed by other Scandinavian countries in the 1970s. Canada and the United States have embarked on extensive use of this product since the early 1980s.

Proportions of silica fume in concrete mixtures vary from 3% to 30% by weight of the cement depending on strength and workability requirements. However, water demand is greatly increased with increasing proportion of silica fume, and high-range water reducers are essential to keep the water/cement ratio low in order to produce higher-strength, yet workable, concrete. Silica fume seems to attain a high early strength in about 3 to 7 days with relatively less increase in strength at 28 days. The strength-development pattern of flexural and tensile splitting strengths is similar to that of compressive strength gain for silica-fume-added concrete. The addition of silica fume to the mixture can produce significant increase in strength, increased modulus of elasticity, and increased flexural strength.

2.5.9 Corrosion Inhibitors

Corrosion inhibitors are usually organic compounds that can migrate through the concrete, forming a protective film around the reinforcing bars thereby inhibiting corrosion. Several types are available such as DCI, Rheocrete, and Cortec MCI. The MCI 2000 series is a water soluble concentrate that forms both anodic and cathodic protection, is environmentally safe, and seems to have an effective corrosion inhibition mechanism for long-term durability of the reinforcement.

SELECTED REFERENCES

2.8. American Concrete Institute, Super-plasticizers in Concrete, Special Publication SP-62, ACI, Detroit, 1979, 427 pp.
3.1 INTRODUCTION

The general knowledge gained from Chapter 2 can now be utilized to design and obtain a concrete of characteristics and functions to suit a definite purpose. As should be realized by now, the proportioning and types of ingredients establish in part the quality of the concrete and hence the quality of the total structural system. Not only must good materials be chosen, but uniformity must be maintained in the whole product. The general characteristics of good concrete are summarized in the following sections.

3.1.1 Compactness

The space occupied by the concrete should, as much as possible, be filled with solid aggregate and cement gel free of honeycombing. Compactness may be the primary criterion for those types of concrete that intercept nuclear radiation.

3.1.2 Strength

Concrete should always have sufficient strength and internal resistance to the various types of failure.

Photo 3.1 Lake Point Tower, Chicago. (Courtesy of Portland Cement Association.)
3.1.3 Water/Cement Ratio and Water/Cementitious Ratio

The water/cement ratio should be suitably controlled to give the required design strength.

3.1.4 Texture

Exposed concrete surfaces should have a dense and hard texture that can withstand adverse weather conditions.

3.1.5 Parameters Affecting Concrete Quality

To achieve the aforementioned properties, good quality control has to be exercised on the factors shown in Fig. 3.1. The following are the most important parameters:

1. Quality of cement
2. Proportion of cement in relation to water in the mixture
3. Strength and cleanliness of aggregate
4. Interaction or adhesion between cement paste and aggregate
5. Adequate mixing of the ingredients
6. Proper placing, finishing, and compaction of the fresh concrete
7. Curing at a temperature not below 50°F while the placed concrete gains strength
8. Chloride content not to exceed 0.15% in reinforced concrete exposed to chlorides in service and 1-1% for dry protected concrete

A study of these requirements shows that most of the control actions have to be taken prior to placing the fresh concrete. Since such control is governed by the proportions and the mechanical ease or difficulty in handling and placing, the development of criteria based on the theory of proportioning for each mixture should be studied. Most mixture design methods have become essentially only of historical and academic value.

The two universally accepted methods for mixture proportioning for normal-weight and lightweight concrete are the American Concrete Institute’s methods of proportioning. However, described in the recommended practice for selecting proportions for normal-weight,
3.2 PROPORTIONING THEORY—NORMAL STRENGTH CONCRETE

Water/cement ratio (w/c ratio) theory states that for a given combination of materials and as long as workable consistency is obtained, the strength of concrete at a given age depends on the ratio of the weight of mixing water to the weight of cement. In other words, if the ratio of water to cement is fixed, the strength of concrete at a certain age is also essentially fixed, as long as the mixture is plastic and workable and the aggregate sound, durable, and free of deleterious materials. Whereas strength depends on the w/c ratio, economy depends on the percentage of aggregate present that would still give a workable mixture. The aim of the designer should always be to get concrete mixtures of optimum strength at minimum cement content and acceptable workability. The lower the w/c ratio is, the higher the concrete strength.
3.2 Proportioning Theory—Normal Strength Concrete

![Flowchart for normal-strength concrete mixture design](image)

Figure 3.2 Flowchart for normal-strength concrete mixture design.

Once the w/c ratio is established and the workability or consistency needed for the specific design is chosen, the rest should be simple manipulation with diagrams and tables based on large numbers of trial mixes. Such diagrams and tables allow an estimate of the required mix proportions for various conditions and permit predetermination on small unrepresentative batches.

### 3.2.1 ACI Method of Mixture Design for Normal Strength Concrete

The flowchart in Fig. 3.2 and the following design example best illustrate the mixture design process using the ACI mixture design method. One aim of the design is to produce workable concrete that is easy to place in the forms. A measure of the degree of consistency and extent of workability is the slump. In the slump test, the plastic concrete specimen is mixed into a conical metal mold as described in ASTM Standard C-143. The mold is lifted, leaving the concrete to "slump." That is, to spread or drop in height. This drop in height is the slump measure of the degree of workability of the mix. Figure 3.3 gives a flowchart for high-strength concrete mixture proportioning.

### 3.2.2 Example 3.1: Mixture Design of Normal-weight Concrete

Design a concrete mixture using the following details:
Figure 3.3 Flowchart for mixture proportioning of high-strength high-performance concrete.

Required strength: 4000 psi (27.6 MPa)
Type of structure: beam
Maximum size of aggregate = 1 in. (19 mm)
Fineness modulus of sand = 2.6
Dry-cast weight of coarse aggregate = 100 lb/ft^3
Moisture absorption 3% for coarse aggregate and 2% for fine aggregate

Solution: Required slump for beams (Table 3.1) = 3 in.
maximum aggregate size (given) = \( \frac{3}{4} \) in.

For a slump between 3 and 4 in. and a maximum aggregate size of 1 in.,
weight of water required per cubic yard of concrete (Table 3.2) = 340 lb/yd^3
For the specified compression strength $f' = 4000$ psi,

\[
\text{w/c ratio (Table 3.3)} = 0.57
\]

Table 3.4 is also needed if volumes instead of weights are used in the mixture design calculations. Therefore,

amount of cement required per cubic yard of concrete = \(\frac{340}{0.57} = 596.5 \text{ lb/yd}^3\)

Using a sand fineness value of 2.6 and Table 3.4,

volume of coarse aggregate = 0.64 yd\(^3\)

Using the dry-rodded weight of 100 lb/ft\(^3\) for coarse aggregate,

weight of coarse aggregate = \((0.64) \times (27 \text{ lb/ft}^3) \times 100\)

= 1728 lb/yd\(^3\)

Estimated weight of fresh concrete for aggregate of 1-in. maximum size (Table 3.5) = 3960 lb/yd\(^3\)

weight of sand = [weight of fresh concrete - weights of (water + cement + coarse aggregate)]

= 3960 - 340 - 596.5 - 1728 = 1299.5 lb

net weight of sand to be taken = 1.02 \times 1299.5 = 1321.4 lb

(moisture absorption 2%) = 1321.4 lb

<table>
<thead>
<tr>
<th>Table 3.1 Recommended Slumps for Various Types of Construction</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Slump (in.)</strong>*</td>
</tr>
<tr>
<td><strong>Types of Construction</strong></td>
</tr>
<tr>
<td>Maximum(^b)</td>
</tr>
<tr>
<td>Reinforced foundation walls and footings</td>
</tr>
<tr>
<td>Plain footings, caissons, and substructure walls</td>
</tr>
<tr>
<td>Beams and reinforced walls</td>
</tr>
<tr>
<td>Building columns</td>
</tr>
<tr>
<td>Pavements and slabs</td>
</tr>
<tr>
<td>Mass concrete</td>
</tr>
</tbody>
</table>

\(^a\) 1 in. = 25.4 mm.

\(^b\) May be increased 1 in. for methods of consolidation other than vibration.
### Table 3.2 Approximate Mixing Water and Air Content Requirements for Different Stumpas and Nominal Maximum Sizes of Aggregates

<table>
<thead>
<tr>
<th>Slump (in.)</th>
<th>1 in.</th>
<th>1 1/2 in.</th>
<th>2 in.</th>
<th>3 in.</th>
<th>4 in.</th>
<th>6 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 2</td>
<td>350</td>
<td>320</td>
<td>290</td>
<td>260</td>
<td>220</td>
<td>190</td>
</tr>
<tr>
<td>3 to 4</td>
<td>385</td>
<td>365</td>
<td>340</td>
<td>320</td>
<td>300</td>
<td>280</td>
</tr>
<tr>
<td>6 to 7</td>
<td>410</td>
<td>385</td>
<td>360</td>
<td>340</td>
<td>320</td>
<td>300</td>
</tr>
</tbody>
</table>

**Approximate amount of entrapped air in non-air-entrained concrete (%):**

<table>
<thead>
<tr>
<th>Slump (in.)</th>
<th>1 in.</th>
<th>1 1/2 in.</th>
<th>2 in.</th>
<th>3 in.</th>
<th>4 in.</th>
<th>6 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 2</td>
<td>3.5</td>
<td>2.5</td>
<td>1.5</td>
<td>1</td>
<td>0.5</td>
<td>0.3</td>
</tr>
<tr>
<td>3 to 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6 to 7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Air-entrained Concrete:**

<table>
<thead>
<tr>
<th>Slump (in.)</th>
<th>1 in.</th>
<th>1 1/2 in.</th>
<th>2 in.</th>
<th>3 in.</th>
<th>4 in.</th>
<th>6 in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 2</td>
<td>365</td>
<td>350</td>
<td>320</td>
<td>290</td>
<td>260</td>
<td>240</td>
</tr>
<tr>
<td>3 to 4</td>
<td>340</td>
<td>320</td>
<td>300</td>
<td>280</td>
<td>260</td>
<td>240</td>
</tr>
<tr>
<td>6 to 7</td>
<td>365</td>
<td>345</td>
<td>320</td>
<td>300</td>
<td>280</td>
<td>260</td>
</tr>
</tbody>
</table>

**Recommended average total air content (% of air level of exposure):**

<table>
<thead>
<tr>
<th>Exposure</th>
<th>Mild</th>
<th>Moderate</th>
<th>Extreme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.5</td>
<td>6.0</td>
<td>7.5</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>5.5</td>
<td>7.0</td>
</tr>
<tr>
<td></td>
<td>3.5</td>
<td>4.5</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>3.0</td>
<td>4.0</td>
<td>5.5</td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>2.0</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>1.5</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>1.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

*These quantities of mixing water are for use in computing cement factors for trial batches. They are maximum for reasonably well shaped angular coarse aggregates. Curved shapes and pitteness are objectionable. The water content for well graded well shaped aggregate is used. They are average for reasonably well shaped coarse aggregate, well graded from coarse to fine.

**Additional recommendations for air content and necessary tolerances and air content for control in the field are given in a number of ACI documents, including ACI 210.1-55, 314.3, 316, and 502.**

For concrete containing large aggregates that will be wet-sieved over the 1 1/2-in. sieve prior to testing for air content, the percentage of air expected in the 1 1/2-in. material should be tabulated as in the 1 in. column. However, initial proportioning calculations should include the air content as a percent of the total batch.

*When using large aggregates in low-cement factor concrete, air entrainment need not be detrimental to strength. In most cases the mixing water requirement is increased sufficiently to improve the water/cement ratio that would compensate for the strength-softening effect of entrained air. Generally, therefore, for these large maximum sizes of aggregate, air contents recommended for extreme exposure should be considered even though there may be little or no exposure to moisture and freezing.

*These values are based on the criteria that 9% nominally of the mortar phase of the concrete. If the mortar volume is substantially different from that determined at the recommended practice, it may be desirable to calculate the needed air content by taking 9% of the actual mortar volume.
### Table 3.3 Relationship Between Water/Cement Ratio and Compressive Strength of Concrete

<table>
<thead>
<tr>
<th>Compressive Strength at 28 days (ksi)</th>
<th>Water/Cement Ratio, by Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Non-air-entrained Concrete</td>
</tr>
<tr>
<td>6000</td>
<td>0.41</td>
</tr>
<tr>
<td>5000</td>
<td>0.48</td>
</tr>
<tr>
<td>4000</td>
<td>0.57</td>
</tr>
<tr>
<td>3000</td>
<td>0.68</td>
</tr>
<tr>
<td>2000</td>
<td>0.82</td>
</tr>
</tbody>
</table>

*Values are estimated average strengths for concrete containing not more than the percentage of air shown in Table 3.2. For a constant water/cement ratio, the strength of concrete is reduced as the air content is increased.

Strength is based on a 6 in. x 12 in. cylinders moist-cured 28 days at 73.4 ± 3°F (23 ± 1°C) in accordance with Section 9(b) of ASTM C-31, "Making and Curing Concrete Compression and Flexure Test Specimens in the Field."

Relationship assumes minimum size of aggregate about 3/16 in.; for given sources, strength produced for a given water/cement ratio will increase as maximum size of aggregate decreases.

*1000 psi = 6.9 MPa.

### Table 3.4 Volume of Coarse Aggregate per Unit of Volume of Concrete

<table>
<thead>
<tr>
<th>Maximum Size of Aggregate (in.)*</th>
<th>Volume of Dry-rolled Coarse Aggregate* Per Unit Volume of Concrete for Different Faenness Moduli of Sand</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2.40</td>
</tr>
<tr>
<td>1/2</td>
<td>0.50</td>
</tr>
<tr>
<td>1</td>
<td>0.59</td>
</tr>
<tr>
<td>11/2</td>
<td>0.66</td>
</tr>
<tr>
<td>1</td>
<td>0.71</td>
</tr>
<tr>
<td>11/2</td>
<td>0.73</td>
</tr>
<tr>
<td>2</td>
<td>0.78</td>
</tr>
<tr>
<td>3</td>
<td>0.82</td>
</tr>
<tr>
<td>6</td>
<td>0.87</td>
</tr>
</tbody>
</table>

*1 in. = 25.4 mm.

*Volumes are based on aggregates in dry-rolled condition as described in ASTM C-96, "Unit Weight of Aggregate." These volumes are selected from empirical relationships to produce concrete with a degree of workability suitable for usual reinforced construction. For less workable concrete, the coarse aggregate content may be decreased up to 10%, provided that the slump and water/cement ratio requirements are satisfied.
Table 3.5 First Estimate of Weight of Fresh Concrete

<table>
<thead>
<tr>
<th>Maximum size of aggregate (in.)</th>
<th>Non-air-entrained concrete</th>
<th>Air-entrained concrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>2840</td>
<td>3660</td>
</tr>
<tr>
<td>1/2</td>
<td>3800</td>
<td>3760</td>
</tr>
<tr>
<td>1</td>
<td>4010</td>
<td>3900</td>
</tr>
<tr>
<td>1 1/2</td>
<td>4070</td>
<td>3980</td>
</tr>
<tr>
<td>2</td>
<td>4120</td>
<td>4000</td>
</tr>
<tr>
<td>3</td>
<td>4160</td>
<td>4040</td>
</tr>
<tr>
<td>6</td>
<td>4230</td>
<td>4120</td>
</tr>
</tbody>
</table>

1 in. = 25.4 mm.

Values calculated and presented below are for concrete of medium richness (550 lb of cement per cubic yard) and medium slump with aggregate specific gravity of 2.7. Water requirements are based on values for 3 to 4-in. slump in Table 5.3.2 of ASTM C-143. If desired, the estimated weight may be refined as follows if necessary information is available: for each 10-lb difference in mixing water from Table 5.3.2, values for 3 to 4-in. slump, correct the weight per cubic yard 15 lb in the opposite direction; for each 100-lb difference in cement content from 550 lb, correct the weight per cubic yard 15 lb in the same direction; for each 0.1 by which aggregate specific gravity deviates from 2.7, correct the concrete weight 100 lb in the same direction.

Weight of fresh concrete per cubic yard, lb

\[ = 16,565 (100 - A) + C \left( \frac{G_s}{G_c} \right) - W(G_c - 1) \]

where:
- \( G_s \) = weighted average specific gravity of combined fine and coarse aggregate, bulk saturated surface dry density
- \( G_c \) = specific gravity of cement (generally 3.15)
- \( A \) = air content, %
- \( W \) = mixing water requirement, lb/yd³
- \( C \) = cement requirement, lb/yd³

Net weight of gravel = 103 \times 1728

(moisture absorption 3%) = 1779.84 lb

Net weight of water = 340 \times 0.02 \times 1295.5 = 0.03 \times 1728

= 262.25 lb

For 1 yd³ of concrete:

- cement = 596.5 lb = 600 lb (273 kg)
- sand = 1321.41 lb = 1320 lb (600 kg)
- gravel = 1779.84 lb = 1760 lb (810 kg)
- water = 262.25 lb = 290 lb (120 kg)
3.3 High-Strength High-Performance Concrete Mixtures Design

3.3.3 Mixture Design for Structural Lightweight Concrete

Structural lightweight concrete can be defined as concrete having a 28-day compressive strength in excess of 2000 psi and an air-dry unit weight less than 115 lb/ft³. The coarse aggregate used is primarily expanded shale, slate, slags, and so on, and the same principles and procedures used in normal-weight concrete are applicable to this type of concrete. Air entrainment is very desirable, if not mandatory. A recommended percentage of air-entraining agents of at least 6% is necessary to give the product acceptable weathering qualities.

3.3 HIGH-STRENGTH HIGH-PERFORMANCE CONCRETE MIXTURES DESIGN

3.3.1 Introduction

High-strength concrete by present ACI definitions covers concretes whose cylinder compressive strength exceeds 6000 psi (41.4 MPa). Proportioning concrete mixtures is more critical for high-strength concrete than for normal-strength concrete. The procedure is similar to the proportioning process for normal-strength concrete except that adjustments have to be made for the admixtures that replace part of the cement content in the mixture and the need to use often smaller size aggregates in very-high-strength concretes (Refs. 3.3, 3.5).

Several types of strength-modifying admixtures can be used: high-range water reducers (superplasticizers), polymers and pozzolanic mineral admixtures such as fly ash, blast-furnace slag, and silica fume. However, in mixture proportioning for very-high-strength concrete, isolating the water/cementitious materials ratio w/cm from the paste/aggregate ratio due to the very low water content can be more effective in arriving at the optimum mixture with a lesser number of trial mixtures and field trial batches.

A few other methods are available today. The very low w/cm ratio required for strengths in the range of 20,000 psi (138 MPa) or higher requires major modifications to the present ACI standards approach used in mixture proportioning that seems to work well for strengths up to 12,000 psi (83 MPa); see Refs. 3.3, 3.5. The optimum mixture that can be chosen with minimum trials has to produce a satisfactory concrete product both in its plastic and its hardened state. An approach presented in Ref. 3.7 is based on mortar volume/stone volume ratio, proportioning the solids in the mortar on the basis of the ratio:

\[
\frac{\text{solid sand volume} + \text{cementitious solid volume}}{\text{mortar volume}}
\]

The ACI standard is well established for fly ash concretes (FAC) as in Ref. 3.3 to 3.6. Ample mixture proportioning results are available for polymers. The same is true for silica fume concretes (SFC) and slag concrete (SC or GGBFSC). They are, however, not established in the form of a standard. The computational example using fly ash as a mineral admixture for concrete mixture design (FAC) for strengths up to 12,000 psi (83 MPa) should serve as a systematic step-by-step guide for proportioning mixtures using polymers, silica fume, and granulated blast-furnace slag within the strength range possible in the use of other admixtures such as silica fume.

The age at test is a governing criteria for selecting mixture proportions. The standard 28 days strength for normal-strength concrete penalizes high-strength concrete since the latter continues gaining strength after that age. One has also to consider that a structure is subjected to service load at 60 to 90 days age at the earliest. Consequently, mixture proportioning has to be based in this case on these latter age levels and also on either field experience or laboratory batch trials. The average compressive field strength
results should exceed the specified design compressive strength by a sufficiently high margin so as to reduce the probability of lower test results.

3.3.2 Selecting Mixture Proportions on the Basis of Laboratory Trial Batches

In this case, the laboratory trial batches should give

\[
f'_c = \frac{(f'_{c} + 1400)}{0.90} \text{ksi} \tag{3.1a}
\]

In SI units,

\[
f'_c = \frac{(f'_{c} + 9.7)}{0.90} \text{MPa} \tag{3.1b}
\]

It is important to note that high-strength high-performance concrete requires special attention to the selection and control of the ingredients in the mixture in order to obtain optimum proportioning and maximum strength. To achieve this aim, care in the choice of the particular cement, admixture brand, dosage rate, mixing procedure, and quality and size of aggregate becomes paramount. Since all the cement does not hydrate, it is advisable that the cement content be kept minimum for optimum mixture proportioning.

3.3.2.1 Cement and Other Cementitious Ingredients. A proper selection of types and source of cement is extremely important. ASTM cement requirements are only minimum requirements and certain brands are better than others due to the variations in the physical and chemical properties of the various cements. High-strength concrete requires high cementitious materials content, namely a low water/cementitious materials ratio (w/cm), and the fineness of the cementitious materials has a major effect on the workability of the fresh mix and the strength of the hardened concrete. They contribute to the reduction in water demand and lower the temperature of hydration. Hence, a determination has to be made whether to choose fly ash class F or G, silica fume, or granulated slag.

3.3.2.2 Coarse Aggregate. Aggregates greatly influence the strength of the hardened concrete as they comprise the largest segment of all the constituents. Consequently, only hard aggregate should be used for normal-weight high-strength concrete so that the aggregate would or at least have the strength of the cement gel. As higher strength is sought, the aggregate size should be decreased. It is advisable to limit aggregate size to 1 in. (25 mm) maximum size for strengths up to 900 psi (62 MPa). For higher strengths, a 3/8 in. or preferably 1/2 in. size aggregate should be used (12.7–19.0 mm). For strengths in the range of 15,000 to 20,000 psi (103–138 MPa), higher strength trap rock from selected quarries should be used in order to achieve such very high strengths. Beyond 20,000 to 30,000 psi strength, the aggregate size should not exceed 1 in. in structural components.

3.3.2.3 Fine Aggregate. A fineness modulus (FM) in the range of 2.5 to 3.2 is recommended for high-strength concrete to facilitate workability. Lower values result in decreased workability and a higher water demand. The mixing water demand depends on the void ratio in the sand. The basic void ratio is 0.35 and should be adjusted for other void ratios such that the void content \( V \) in percent can be evaluated from

\[
V = \left(1 - \frac{\text{Oven-dry rotted unit weight (lb/ft}^3\right)}{\text{Bulk dry specific gravity}} \times 62.4 \right) \times 100 \tag{3.2a}
\]

in SI units

\[
V = \left(1 - \frac{\text{Oven-dry rotted unit weight (kg/m}^3\right)}{\text{Bulk dry specific gravity}} \times 10^3 \right) \times 100 \tag{3.2b}
\]
3.3 High-Strength High-Performance Concrete Mixtures Design

Table 3.6 Required Average Compressive Strength When Data Are Not Available to Establish a Standard Deviation

<table>
<thead>
<tr>
<th>Specified Strength $f'_c$ psi (MPa)</th>
<th>Required Strength $f'_c$ psi (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&gt; 5000 (34.5)</td>
<td>$f'_c = 1400$ (97)</td>
</tr>
</tbody>
</table>

The mixing water has to be accordingly adjusted to account for the change in the basic void ratio such that the mixing water adjustment would be as follows:

Mixing Water Adjustment (lb/yd$^3$) $A = 8(V - 35)$ (3.3a)

in SI units,

Mixing Water Adjustment (kg/m$^3$) $A = 4.7(V - 35)$ (3.3b)

3.3.2.4 Workability-Enhancing Chemical Admixtures. High-strength mixtures have a rich cementitious content that require a high water content, with the knowledge that excessive water reduces the compressive strength of the concrete and affects its long-term performance. Thus, water-reducing admixtures become mandatory. High-range water-reducing admixtures (HRWR) are used. These are sometimes called superplasticizers. The dosage rate is usually based on fluid oz. per 100 lb (45 kg) of total cementitious materials if they are in liquid form. If the water-reducing agent is in powdered form, the dosage rate would be on weight ratio basis.

The optimum admixture percentage should be determined on trial and adjustment basis as they can reduce the water demand by almost 30 to 35% with a corresponding increase in compressive strength. A slump of 1 to 2 in. (25 to 35 mm) is considered adequate. If, however, no HRWR admixtures are used, the slump should be increased to 2 to 4 in. (50 to 100 mm). In addition, air-entraining admixtures are used if the concrete is exposed to freezing and thawing cycles in severe environmental conditions. For structural components in building systems, air entraining is unnecessary as these are usually not subjected to the type of frost action that exposed bridge decks or sea oil platforms endure.

3.3.3 Recommended Proportions

Tables 3.6 to 3.14 adapted from Ref. 3.3 recommend the necessary ingredient contents for proportioning mixtures for high-strength concrete. A flowchart giving the mixture proportioning sequence for high-strength concrete is shown in Figure 3.3.

3.3.4 Example 3.2: Triax Mixture Design

Design a high-strength concrete mixture for the columns in a multi-story structure for a specified 28 days compressive strength of 10,000 psi (69 MPa). A slump of 9 in. (229 mm) is required for workability needed in congested reinforcement in the columns. Do not use an

Table 3.7 Maximum Size Coarse Aggregate

<table>
<thead>
<tr>
<th>Required Concrete Strength $f'_c$ psi (MPa)</th>
<th>Maximum Aggregate Size in. (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt; 9000 (62)</td>
<td>1-1 (19-25)</td>
</tr>
<tr>
<td>≥ 9000 (62)</td>
<td>1-1 (9.5-12.7)</td>
</tr>
</tbody>
</table>
### Table 3.8 Coarse Aggregate to Concrete Fractional Volume Ratio (Sand Fineness Modulus 2.5–3.2)

<table>
<thead>
<tr>
<th>Nominal max. size in. (mm)</th>
<th>l/1 (8.5)</th>
<th>l/2 (12.7)</th>
<th>l/3 (19)</th>
<th>l/4 (25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractional volume or oven-dry soaked coarse aggregate</td>
<td>0.65</td>
<td>0.68</td>
<td>0.72</td>
<td>3.75</td>
</tr>
</tbody>
</table>

### Table 3.9 Recommended Slump

<table>
<thead>
<tr>
<th>With HRWR* in. (mm)</th>
<th>No HRWR in. (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2 (25-50)</td>
<td>2-4 (50-100)</td>
</tr>
</tbody>
</table>

*Adjust slump to that desired in the field by adding HRWR.

**HRWR = High Range Water Reducer**

### Table 3.10 Mixing Water Requirement and Air Content of Fresh Concrete Using Sand with 35% Void Ratio—First Trial Water Content

<table>
<thead>
<tr>
<th>Slump in. (mm)</th>
<th>l/1 (8.5)</th>
<th>l/2 (12.5)</th>
<th>l/3 (19)</th>
<th>l/4 (25)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2 (25-50)</td>
<td>310 (183)</td>
<td>295 (174)</td>
<td>285 (169)</td>
<td>280 (165)</td>
</tr>
<tr>
<td>2-3 (35-75)</td>
<td>320 (190)</td>
<td>310 (183)</td>
<td>295 (174)</td>
<td>290 (171)</td>
</tr>
<tr>
<td>3-4 (75-100)</td>
<td>330 (195)</td>
<td>320 (189)</td>
<td>305 (180)</td>
<td>300 (177)</td>
</tr>
</tbody>
</table>

\[ \Delta \text{Strapped Air %} = 3 \times (1.5) \times (2) \]

\[ l/100 = 0.05 \text{g/cm}^3 \]

\[ l/100 = \left( \frac{V}{1000} - V \right) \times \text{Oven-dry soaked mass} \times \text{Bulk Specific gravity (dry)} \times 62.4 \]

**Notes:**

- Water added to HRWR
- Adjust mixing water value for sand void ratio value over 35%.

For void content V, % = \( \left( 1 - \frac{\text{Oven-dry soaked mass}}{\text{Bulk Specific gravity (dry)} \times 62.4} \right) \times 100 \)

For mixing water adjustment, \( l/100 = \left( V - 35 \right) \times 0.05 \)

For density adjustment, \( l/100 = \left( V - 35 \right) \times 0.07 \)
### Table 3.11 w/cm Ratio for Concrete without High Range Water Reducer (without HRWR)

<table>
<thead>
<tr>
<th>Field Strength f'_c, sy</th>
<th>Maximum Size Coarse Aggregate, In. (mm)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 (9.5)</td>
<td>2 (12.7)</td>
<td>3 (19)</td>
<td>4 (25)</td>
</tr>
<tr>
<td>7000 (48)</td>
<td>0.42</td>
<td>0.41</td>
<td>0.40</td>
<td>0.39</td>
</tr>
<tr>
<td>56 day</td>
<td>0.46</td>
<td>0.45</td>
<td>0.44</td>
<td>0.43</td>
</tr>
<tr>
<td>8000 (55)</td>
<td>0.35</td>
<td>0.34</td>
<td>0.33</td>
<td>0.33</td>
</tr>
<tr>
<td>56 day</td>
<td>0.38</td>
<td>0.37</td>
<td>0.36</td>
<td>0.35</td>
</tr>
<tr>
<td>9000 (62)</td>
<td>0.30</td>
<td>0.29</td>
<td>0.29</td>
<td>0.28</td>
</tr>
<tr>
<td>56 day</td>
<td>0.33</td>
<td>0.31</td>
<td>0.31</td>
<td>0.30</td>
</tr>
<tr>
<td>10,000 (69)</td>
<td>0.26</td>
<td>0.26</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>56 day</td>
<td>0.29</td>
<td>0.28</td>
<td>0.27</td>
<td>0.26</td>
</tr>
</tbody>
</table>

* f'_c = f'_c + 1.400 (f'_c - f'_c, s)  
* These are average field values; enter into the table 0.9 (required f'_c, s)  

### Table 3.12 w/cm Ratio for Concrete with High-Range Water Reducer (with HRWR)

<table>
<thead>
<tr>
<th>Field Strength f'_c, sy</th>
<th>Maximum Size Coarse Aggregate, In. (mm)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 (9.5)</td>
<td>2 (12.7)</td>
<td>3 (19)</td>
<td>4 (25)</td>
</tr>
<tr>
<td>1000 (48)</td>
<td>0.50</td>
<td>0.48</td>
<td>0.45</td>
<td>0.43</td>
</tr>
<tr>
<td>56 day</td>
<td>0.51</td>
<td>0.47</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td>8000 (55)</td>
<td>0.44</td>
<td>0.42</td>
<td>0.46</td>
<td>0.38</td>
</tr>
<tr>
<td>56 day</td>
<td>0.48</td>
<td>0.45</td>
<td>0.42</td>
<td>0.40</td>
</tr>
<tr>
<td>9000 (62)</td>
<td>0.38</td>
<td>0.36</td>
<td>0.35</td>
<td>0.34</td>
</tr>
<tr>
<td>56 day</td>
<td>0.42</td>
<td>0.39</td>
<td>0.37</td>
<td>0.36</td>
</tr>
<tr>
<td>10,000 (69)</td>
<td>0.33</td>
<td>0.32</td>
<td>0.31</td>
<td>0.30</td>
</tr>
<tr>
<td>56 day</td>
<td>0.37</td>
<td>0.35</td>
<td>0.33</td>
<td>0.22</td>
</tr>
<tr>
<td>11,000 (76)</td>
<td>0.30</td>
<td>0.29</td>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
<td>56 day</td>
<td>0.37</td>
<td>0.31</td>
<td>0.29</td>
<td>0.29</td>
</tr>
<tr>
<td>12,000 (83)</td>
<td>0.27</td>
<td>0.26</td>
<td>0.25</td>
<td>0.26</td>
</tr>
<tr>
<td>56 day</td>
<td>0.30</td>
<td>0.28</td>
<td>0.27</td>
<td>0.26</td>
</tr>
</tbody>
</table>

* f'_c = f'_c + 1.400 (f'_c - f'_c, s)  
* These are average field values; enter into the table 0.9 (required f'_c, s)  

Note: A comparison of the values contained in Tables 3.6 and 3.7 permits, in particular, the following conclusions:
1. For a given water/cementious material ratio, the field strength of concrete is greater with the use of HRWR than without it, and this greater strength is reached within a shorter period of time.
2. With the use of HRWR, a given concrete field strength can be achieved in a given period of time using less cementitious material than would be required when not using HRWR.
aggregate size exceeding 1/4 in. (12.7 mm). Use a high-range water reducer (HRWR) to obtain the 9-in. slump and a set-retarding admixture. Assume that the ready-mix producer has no prior history with high-strength concrete.

Given the following sand properties:

- Fineness modulus FM
- Bulk specific gravity (dry), BS$_{dm}$ = 2.59
- Absorption based on dry weight, Abs = 1.1%
- Dry rodded unit weight, DRUW = 100 lb/ft$^3$ (1620 kg/m$^3$)
- Moisture content in sand = 6.4%

Solution: From the author's solution in Ref. 3.5.

1. Select Slump and Required Concrete Strength: Because a HRWR agent is used, choose strength on the basis of 1- to 2-in. slump prior to the addition of HRWR. Also, since the ready-mix producer has no prior history with high-strength concrete, laboratory trial mixes have to be designed for the selection of the optimum proportion. From Eq. 3.1(a),

   \[ f_c = 0.85 (1400 / 0.90) \]

   \[ = (10,000 + 1,400) / 0.90 = 12,670 \text{ psi (87 MPa)} \]

2. Select Maximum Aggregate Size: A crushed limestone graded 1/4 in. (12.7 mm) maximum size is selected with BS$_{dm}$ = 2.7%. Ads = 0.70 and DRUW = 101 lb/ft$^3$, stone moisture content = 0.5%.

Table 3.14 Modification Factor for Standard Deviation when Fewer than Thirty Tests are Available

<table>
<thead>
<tr>
<th>Number of Tests$^a$</th>
<th>Modification Factor for Standard Deviation$^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 15</td>
<td>Use Table 3.6</td>
</tr>
<tr>
<td>15</td>
<td>1.16</td>
</tr>
<tr>
<td>20</td>
<td>1.08</td>
</tr>
<tr>
<td>25</td>
<td>1.03</td>
</tr>
<tr>
<td>30 or more</td>
<td>1.00</td>
</tr>
</tbody>
</table>

$^a$Interpolate for intermediate number of tests.
$^b$Modified standard deviation to be used to determine required strength $f_c$ in Eq. 3.1(a).
3. Select Optimum Course Aggregate Content: From Table 3.8, fractional ratio = 0.68. Dry weight of coarse aggregate/yd³ of concrete is

\[ W_{na} = (\% \text{ DRUW}) \times (\text{DRUW} \times 27) \]

\[ = 0.68 \times 181 \times 27 = 1854 \text{ lb (841 kg)} \]

4. Estimate Mixing Water and Air Content: From Table 2.10, the first estimate of the required mixing water is 295 lb/yd³ (174 kg/m³) of concrete and the entrapped air content when HRWR is used = 2.5%.

From Eq. 3.2(a), the void content of the sand to be used is

\[ V = \left[ 1 - \frac{100}{2.59 \times 0.4} \right] \times 100 = 36\% \]

From Eq. 3.3(a), the mixing water adjustment

\[ A = 8(V - 35) = 8(36 - 35) = 8 \text{ lb/yd}³ \text{ (4.7 kg/m³) of concrete} \]

Hence, total mixing water \( W = 295 + 8 = 303 \text{ lb (138 kg)} \)

5. Select Water/Cementitious Materials Ratio where: The values in Table 3.11 and 3.12 are average field strengths values. Hence, strength \( f'c \), for which the w/cm ratio is to be found is:

\[ f'c = (3,900 \times 12,670 = 11,400 \text{ psi (77 MPa)} \]

From Table 3.12 for 1-in. size aggregate, the desirable

\[ W/(C + P) \text{ ratio = 0.272} \text{ by interpolation} \]

6. Compute Content of Cementsitious Materials: From before, mixing water \( W = 303 \text{ lb, hence, C + P = 303/0.272 = 1114 lb (505 kg)} \)

7. Proportion the Basic Mixture With Cement Only: Volumes of all materials per yd³ except sand are as follows:

- Cement = 1114 \times 3.15 = 5.47 ft³
- Stone = 1854 \times 2.76 = 10.77 ft³
- Water = 303 = 4.86
- Air = 0.02 \times 27 = 0.54

Total = \( \frac{21,770 \text{ ft}³}{(1 \text{ cu. ft} = 35.31 \text{ cu. ft})} \)

Hence the required volume of sand per yd³ of concrete = 27 – 21.77 = 5.23 ft³

Converting the sand volume to weight,

sand = 5.23 \times 62.4 \times 2.59 = 845 \text{ lb (384 kg)}

The mix proportions by weight for the no fly ash concrete would be:

- lb/yd³ (kg/m³)
  - Cement = 1114 (661)
  - Sand, dry = 845 (391)
  - Stone, dry = 1854 (1100)
  - Water, incl. 3 oz/cwt
  - Retarding admixture (cwt - hundred weight of cement) = 0 (100)

Total = 4116 lb/yd³ (2442 kg/yd³)
5. Proportion Companion Mixtures Using Cement and Fly Ash:

Use in this case ASTM Class C fly ash (FA) which has bulk specific gravity $s_g = 2.64$

From Table 3.13, the FA replacement = 20–35%.

Use four trial mixtures: 20, 25, 30, 35% levels

For trial mixture No. 1, FA = 0.20(1144) = 223 lb

hence, cement = 1144 – 223 = 921 lb.

In a similar manner, the weights of the remaining materials would be

<table>
<thead>
<tr>
<th>Mixture No.</th>
<th>Cement (lb)</th>
<th>Fly Ash (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>891 (404)</td>
<td>223 (101)</td>
</tr>
<tr>
<td>2</td>
<td>835 (379)</td>
<td>270 (126)</td>
</tr>
<tr>
<td>3</td>
<td>700 (354)</td>
<td>354 (151)</td>
</tr>
<tr>
<td>4</td>
<td>724 (328)</td>
<td>390 (177)</td>
</tr>
</tbody>
</table>

Taking mixture No. 1, the volumes of components except sand per yd$^3$ of concrete are:

\[
\text{Cement} = \frac{891}{(3.15 \times 0.424)} = 4.83 \text{ ft}^3
\]

\[
\text{FA} = \frac{223}{(2.64 \times 0.424)} = 1.35 \text{ ft}^3
\]

From before,

\[
\text{Stone} = 10.77
\]

\[
\text{Water (incl. 2.5 oz/cwt retarder)} = 4.86
\]

\[
\text{Air} = 0.54
\]

\[
\text{Total} = 22.05 \text{ ft}^3
\]

\[
\text{Sand Volume} = 27 - 22.05 = 4.95 \text{ ft}^3 = 4.95 \times 0.424 \times 2.59 = 800 \text{ lb}
\]

The mix proportions by weight for the fly ash concrete would be:

<table>
<thead>
<tr>
<th>lb/yd$^3$ (Kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cement = 891 (356)</td>
</tr>
<tr>
<td>Fly Ash = 223 (132)</td>
</tr>
<tr>
<td>Sand, dry = 800 (472)</td>
</tr>
<tr>
<td>Stone, dry = 1854 (1084)</td>
</tr>
<tr>
<td>Water, incl. retarder = 303 (170)</td>
</tr>
<tr>
<td>Total = 4071 (1620)</td>
</tr>
</tbody>
</table>

\[
1 \text{ lb/yd}^3 = (1.59 \text{ kg/m}^3)
\]

In a similar manner, the mix proportions for 25, 30, and 35% of fly ash content are computed to give the following companion mixtures (Table 3.15).

9. Trial Mixtures Adjustment for Absorbed Water Content in Aggregate: From before,

\[
\text{moisture content in sand} = 6.4\% \\
\text{moisture content in stone} = 0.3\%
\]
### Table 3.15 Mixture Proportions in Ex. 3.2 without Moisture Trial Batch Adjustment

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Basic Mix: C only</th>
<th>C + FA Mixes, lb</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Cement</td>
<td>1114</td>
<td>891</td>
</tr>
<tr>
<td>Fly Ash</td>
<td>0</td>
<td>223</td>
</tr>
<tr>
<td>Sand (dry)</td>
<td>845</td>
<td>800</td>
</tr>
<tr>
<td>Stone (dry)</td>
<td>1854</td>
<td>1854</td>
</tr>
<tr>
<td>Water (+ Ret.)</td>
<td>303</td>
<td>303</td>
</tr>
<tr>
<td>Total</td>
<td>4116</td>
<td>4051</td>
</tr>
<tr>
<td>kg/m³ concrete</td>
<td>2428</td>
<td>2402</td>
</tr>
</tbody>
</table>

1 kg/m³ = 1 lb/ft³

From Table 3.15, corrections in the basic mixture for the wetness of the aggregates.

wet sand = 845 (1 + 0.004) = 899 lb
wet stone = 1854 (1 + 0.005) = 1863 lb

From input data, sand absorption based on dry weight = 1.1% and stone absorption = 0.7% hence, the water correction is

= 303 - 845 (0.004 - 0.011) - 1854 (0.005 - 0.007)
= 303 - 45 + 4 = 262 lb (119 kg)

Accordingly, the batch weight of water has to be corrected to account for the excess moisture contributed by the aggregates = total moisture - aggregate absorbed moisture. Hence, Table 3.15 is modified to Table 3.16.

10. Size of Laboratory Trial Mixtures: The usual size of the trial mixture is 3.0 ft³ (0.085 m³). The reduced batch weights to yield 3.0 ft³ of concrete would be if the values tabulated in Table 3.16 to give (Table 3.17).

11. Adjustment of Trial Mixture Due to Slump Observation:
(a) Basic Mix
Assume that the water calculated to produce the 1- to 2-in. slump, namely, 29.11 lb from Table 3.17 was found not to be adequate and has to be increased to 30 lb/3 ft³ including the 2.5 oz corresponding admixture.

### Table 3.16 Moisture Adjusted Mixture Proportions in Ex. 3.2

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Basic Mix: C only</th>
<th>C + FA Mixes, lb</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Cement</td>
<td>1114</td>
<td>891</td>
</tr>
<tr>
<td>Fly Ash</td>
<td>0</td>
<td>223</td>
</tr>
<tr>
<td>Sand (dry)</td>
<td>845</td>
<td>800</td>
</tr>
<tr>
<td>Stone (dry)</td>
<td>1854</td>
<td>1854</td>
</tr>
<tr>
<td>Water (+ Ret.)</td>
<td>303</td>
<td>303</td>
</tr>
<tr>
<td>Total</td>
<td>4136</td>
<td>4090</td>
</tr>
<tr>
<td>kg/m³ concrete</td>
<td>2441</td>
<td>2413</td>
</tr>
</tbody>
</table>

1 kg/m³ = 0.039 ft³
Table 3.17 Reduced Batch Weight Yielding 3 P2 of Concrete

<table>
<thead>
<tr>
<th>Ingredient</th>
<th>Basic Mix: C only, lb</th>
<th>C + FA Mixes, lb</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
</tr>
<tr>
<td>Cement</td>
<td>123.78</td>
<td>99.00</td>
<td>92.78</td>
<td>96.67</td>
<td>89.44</td>
<td>80.44</td>
</tr>
<tr>
<td>Fly Ash</td>
<td>0</td>
<td>24.78</td>
<td>31.00</td>
<td>37.11</td>
<td>43.33</td>
<td></td>
</tr>
<tr>
<td>Sand (wet)</td>
<td>99.89</td>
<td>94.56</td>
<td>93.44</td>
<td>92.33</td>
<td>91.44</td>
<td></td>
</tr>
<tr>
<td>Stone (wet)</td>
<td>207.00</td>
<td>207.00</td>
<td>207.00</td>
<td>207.00</td>
<td>207.00</td>
<td></td>
</tr>
<tr>
<td>Water (+ Ret.)</td>
<td>28.11</td>
<td>29.11</td>
<td>29.11</td>
<td>29.11</td>
<td>29.11</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>460.00</td>
<td>455.00</td>
<td>453.00</td>
<td>453.00</td>
<td>451.00</td>
<td></td>
</tr>
<tr>
<td>lb/ft³ concrete</td>
<td>245.3</td>
<td>242.7</td>
<td>241.6</td>
<td>241.1</td>
<td>240.5</td>
<td></td>
</tr>
<tr>
<td>kg/(1/10) m³ concrete</td>
<td>123.78</td>
<td>99.00</td>
<td>92.78</td>
<td>96.67</td>
<td>89.44</td>
<td>80.44</td>
</tr>
</tbody>
</table>

The actual batch weights have therefore to be adjusted so that the actual batch weight for the basic mix (no fly ash) becomes:

- Cement = 123.78 lb
- Sand = 99.89 lb
- Stone = 207.00 lb
- Water = 30.00 lb

These values have to be adjusted for moisture correction to dry weight. The basic total added water = 30 x 9 = 270 lb. From before, the absorbed water in the aggregates = 45 - 4 = 41 lb. Actual total water content = 270 + 41 = 311 lb = 34.56 lb per 3 ft³.

Yield of Trial Batch:

Consequently the actual yield of the trial mixture becomes:

- Cement = 123.78 lb - (3.15 x 62.4) = 0.63
- Sand = 99.89 + (2.59 x 62.4) = 5.8
- Stone = 207.00 + (2.76 x 62.4) = 1.20
- Water = 34.56 x 62.4 = 0.55
- Air = 0.02 x 3 ft³ = 0.06

Total yield volume of trial batch = 3.02 ft³.

The yield in lb/ft³ of concrete is obtained by multiplying all the previous values by 9 and converting the volumes to weights giving:

- Cement = 1114 lb
- Sand, dry = 845 lb
- Stone = 1854 lb
- Water (incl. retarder) = 309 lb
The new mixture proportions result in a water-cementitious materials ratio w/cm = 309/1114 = 0.28

versus the desirable ratio of 0.272 previously obtained from Table 3.11.

In order to maintain the 0.272 ratio, the weight of cement should be increased to 390/0.272 = 1,136 lb/yd³ of concrete.

The increase in volume due to the adjustment of the weight of cement = (1,136 - 1,114) = (3.15 x 62.4) x 0.11 ft³.

This increase in volume should be adjusted for by the removal of an equal volume of sand. Hence, weight of sand to be removed = 0.11 x 2.59 x 62.4 = 17.79 lb/yd³, or 18 lb/yd³. The resulting adjusted mixture proportions become:

- Cement = 1,136 lb
- Sand, dry = 845 - 18 = 827 lb
- Stone, dry = 1,854 lb
- Water + 2.5 oz/cwt retarder = 311 lb

(b) Increasing Slump to 9 in. (229 mm): The required slump in this example is 9 in. (229 mm). To achieve this value without the addition of water, which will reduce the strength, a high-range water reducer, namely, a plasticizer is used.

The dosage recommended by the manufacturer of the HRWR ranged between 8 and 16 oz/100 lb of cementitious material. Laboratory tests in a laboratory with ambient temperature of 44°F, indicated the following:

- 8-oz dosage produced 5-in. slump
- 11-oz dosage produced 10-in. slump
- 16-oz dosage produced segregation of the fresh concrete.

In all these cases, a consistent dosage rate of retarding admixture of 2.5 oz/cwt was also added to the mixture with the mixing water.

The HRWR was added to the mixture about 13 min after initial mixing. It was determined that:

1. The mixture with 10 in. (255 mm) slump had adequate workability, hence no correction needed to the coarse aggregate content.
2. Air content of the HRWR concrete mixture was found to be 1.9%; hence, no correction needed.
3. The 28-day compressive strength of the basic mixture was found to be 12,700 psi, satisfying the required f'c = 12,670 psi.

Note: It is important to recognize if additional water at this stage was needed to produce the required slump and workability, then an additional cycle of corrections to actual batches of aggregate have to be executed in the same manner as in the previous steps.

12. Summary of Trial Mixtures Laboratory Performance: In addition, field trials must verify the chosen laboratory trial mixture. In this case, mixture No. 3 from Table 3.18 giving the highest 28 days compressive strength of 12,750 psi (88 kPa) is the closest to the required f'c = 12,670 psi that can give an average compressive strength f'c = 10,000 required in this example.

Table 3.18 summarizes the performance of the five mixtures, namely the basic no-FA concrete and the four concretes with FA at 20, 25, 30, and 35% content of the total cementitious material. Slump values for no-HRWR mixtures and those with HRWR were measured in the laboratory slump tests.

This section on high-strength high-performance concrete mixture design is a condensation of the author's section on this topic in Ref. 3.5.
Table 3.18 Laboratory Final Trial Mixtures

<table>
<thead>
<tr>
<th>Ingredient ID (1)</th>
<th>Basic Mix: C only lb (2)</th>
<th>#1 20% CF lb (3)</th>
<th>#2 25% CF lb (4)</th>
<th>#3 30% CF lb (5)</th>
<th>#4 35% CF lb (6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cement</td>
<td>1136</td>
<td>891</td>
<td>835</td>
<td>780</td>
<td>724</td>
</tr>
<tr>
<td>Fly Ash</td>
<td>233</td>
<td>279</td>
<td>334</td>
<td>390</td>
<td></td>
</tr>
<tr>
<td>Sand (dry)</td>
<td>827</td>
<td>78</td>
<td>772</td>
<td>763</td>
<td>755</td>
</tr>
<tr>
<td>Stone (dry)</td>
<td>1854</td>
<td>163</td>
<td>163</td>
<td>163</td>
<td>163</td>
</tr>
<tr>
<td>Water (m Ret.)</td>
<td>311</td>
<td>334</td>
<td>300</td>
<td>298</td>
<td>297</td>
</tr>
<tr>
<td>Slump, in. (mm)</td>
<td>1.10</td>
<td>.70</td>
<td>1.15</td>
<td>1.50</td>
<td>1.90</td>
</tr>
<tr>
<td>(273)</td>
<td>(151)</td>
<td>(294)</td>
<td>(381)</td>
<td>(481)</td>
<td></td>
</tr>
<tr>
<td>Retarder, oz/cwt</td>
<td>3.5</td>
<td>2.5</td>
<td>2.1</td>
<td>2.5</td>
<td>2.0</td>
</tr>
<tr>
<td>HRWR, oz/cwt</td>
<td>10.00</td>
<td>10.50</td>
<td>11.00</td>
<td>10.25</td>
<td>9.00</td>
</tr>
<tr>
<td>Slump, in. (mm)</td>
<td>10.00</td>
<td>10.75</td>
<td>8.75</td>
<td>10.50</td>
<td>8.25</td>
</tr>
<tr>
<td>(250)</td>
<td>(229)</td>
<td>(270)</td>
<td>(235)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28 day strength, psi (MPa)</td>
<td>12,600</td>
<td>12,400</td>
<td>12,550</td>
<td>12,750</td>
<td>12,250</td>
</tr>
<tr>
<td>(87)</td>
<td>(85)</td>
<td>(87)</td>
<td>(88)</td>
<td>(86)</td>
<td></td>
</tr>
</tbody>
</table>

3.4 PCA METHOD OF MIXTURE DESIGN

The mixture design method proposed by the Portland Cement Association (PCA) is essentially similar to the ACI method. Generally, results would be very close once trial batches are prepared in the laboratory. The PCA publication listed in the references gives the details of the method as well as other information on properties of the ingredients.

3.5 ESTIMATING COMpressive STRENGTH OF A TRIAL MIXTURE USING THE SPECIFIED COMpressive STRENGTH

The compressive strength for which the trial mixture is designed is not the strength specified by the designer. The mixture should be overdesigned to assure that the actual structure has concrete with specified minimum compressive strength. The extent of mixture overdesign depends on the degree of quality control available in the mixing plant.

ACI Committee 318 specifies a systematic way of determining the compressive strength for mixture designs using the specified compressive strength $f'_c$. The procedure is presented in a self-explanatory flowchart form in Fig. 3.4. The cylinder compressive strength $f'_c$ (see Section 3.7) is the test result at 28 days after casting normal-weight concrete. Mixture design has to be based on an adjusted higher value $f'_{cu}$. This adjusted cylinder compressive strength $f'_{cu}$ for which a trial mixture design is calculated depends on the extent of field data available.

1. No cylinder test results available. If field-strength test records for the specified class (or within 1000 psi of the specified class) of concrete are not available, the trial mixture strength $f'_{cu}$ can be calculated by increasing the cylinder compressive strength $f'_c$ by a reasonable value depending on the extent of spread in values expected in the supplied concrete. Such a spread can be quantified by the standard deviation values represented by the values in excess on $f'_c$ in Table 3.19. Table 3.20 can then be used to obtain the water/cement ratio needed for the required cylinder strength value $f'_{cu}$. 


3.5 Estimating Compressive Strength of a Trial Mix Using the Specified Compressive Strength

![Flowchart diagram]

Figure 3.4 Flowchart for selection and documentation of concrete proportions.

Table 3.15 Required Average Compressive Strength when Data Are Not Available to Establish a Standard Deviation

<table>
<thead>
<tr>
<th>Specified Compressive Strength, $f'_c$ (psi)</th>
<th>Required Average Compressive Strength, $f'_u$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 3000</td>
<td>$f'_u = 1000$</td>
</tr>
<tr>
<td>1000–5000</td>
<td>$f'_u = 1200$</td>
</tr>
<tr>
<td>More than 5000</td>
<td>$f'_u = 1400$</td>
</tr>
</tbody>
</table>

*1000 psi = 6.9 MPa*
Table 3.20  Maximum Permissible Water/Cement Ratios for Concrete when Strength Data from Field Experience or Trial Mixtures are Not Available

<table>
<thead>
<tr>
<th>Specified Compressive Strength, $f'_c$ (ksi)</th>
<th>Non-air-entrained Concrete</th>
<th>Air-entrained Concrete</th>
</tr>
</thead>
<tbody>
<tr>
<td>2500</td>
<td>0.67</td>
<td>0.54</td>
</tr>
<tr>
<td>3000</td>
<td>0.58</td>
<td>0.46</td>
</tr>
<tr>
<td>3500</td>
<td>0.51</td>
<td>0.40</td>
</tr>
<tr>
<td>4000</td>
<td>0.44</td>
<td>0.35</td>
</tr>
<tr>
<td>4500</td>
<td>0.38</td>
<td></td>
</tr>
<tr>
<td>5000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*28-day strength. With most materials, the water/cement ratios shown will provide average strengths greater than those calculated using Eqs. 3.1 and 3.2.

*100 psi = 0.69 MPa.

*For strengths above 4000 psi for non-air-entrained concrete and 4000 psi for air-entrained concrete, mixture proportions should be established using trial mixtures.

2. Data available on more than thirty consecutive cylinder tests. If more than thirty consecutive test results are available, Eqs. 3.4 – 3.5 in section 3.5.2 can be used to establish the required mixture strength, $f'_c$, from $f'_c$. If two groups of consecutive test results with a total of more than 30 are available, $f'_c$ can be obtained using Eqs. 3.4a, b, and c.

**Photo 3-4** Cylinder compression test.
3. Data available on fewer than thirty consecutive cylinder tests. If the number of consecutive test results available is fewer than thirty and more than 15, Eq. 3.4a should be used in conjunction with Table 3.21. Essentially, the designer should calculate the standard deviation \( s \) using Eq. 3.5a, multiply the \( s \) value by a modification factor provided in Table 3.21, and use the modified \( s \) in Eqs. 3.4(a), 3.4(b). In this manner, the expected degree of spread of cylinder test values as measured by the standard deviation \( s \) is well accounted for.

3.5.1 Recommended Proportions for Concrete Strength \( f_c' \)

Once the required average strength \( f_c' \) for mixture design is determined, the actual mixture can be established to obtain this strength using either existing field data or a basic trial mixture design.

1. Use of field data. Field records of existing \( f_c' \) values can be used if at least 10 consecutive test results are available. The test records should cover a period of time of at least 45 days. The materials and conditions of the existing field mixture data should be the same as the ones to be used in the proposed work.

2. Trial mixture design. If the field data are not available, trial mixtures should be used to establish the maximum water/cement ratio or minimum cement content for designing a mixture that produces a 28-day \( f_c' \) value. In this procedure, the following requirements have to be met:
   (a) Materials used and age of testing should be the same for the trial mixture and the concrete used in the structure.
   (b) At least three water/cement ratios or three cement contents should be tried in the mixture design. The trial mixtures should result in the required \( f_c' \). Three cylinders should be tested for each w/c ratio and each cement content tried.
   (c) The slump and air content should be within ±0.75 in. and 0.5% of the permissible limits.
   (d) A plot is constructed of the compressive strength at the designated age versus the cement content or water/cement ratio, from which one can then choose the w/c ratio or the cement content that can give the average \( f_c' \) value required.

<table>
<thead>
<tr>
<th>Number of tests*</th>
<th>Modification factor for standard deviation*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Less than 15</td>
<td>Use Table 3.19</td>
</tr>
<tr>
<td>15</td>
<td>1.16</td>
</tr>
<tr>
<td>20</td>
<td>1.08</td>
</tr>
<tr>
<td>25</td>
<td>1.03</td>
</tr>
<tr>
<td>30 or more</td>
<td>1.00</td>
</tr>
</tbody>
</table>

*Specify for intermediate number of tests.
*Modified standard deviation to be used in determining required average strength \( f_c' \), per Eqs. 3.1a and b or Eqs. 3.4a and b whichever apply.
3.5.2 Trial Mixture Design for Average Strength When Prior Field-strength Data Are Available

If field test data are available for more than thirty consecutive tests, the trial mix should be designed for compressive strength $f'_{c}$, calculated from

\[ f'_{c} = \begin{cases} 
5000 \text{psi} \\
2.33x - 500 \\
0.9f'_{c} + 2.33x 
\end{cases} \] (3.4a)

or

\[ f'_{c} = 1.34t_{c} \] (3.4a)

(b) $f'_{c} > 5000$ psi

The larger value of $f'_{c}$ from Eqs. 3.4a and 3.4b should be used in designing the mixture for $f'_{c} \leq 5000$ psi, with the expectation of attaining the minimum $f'_{c}$ specified design compressive strength and Equations 3.4a and 3.4c for $f'_{c} > 5000$ psi. When average compressive strength data is not available to establish a standard deviation, the following is used for the required average strength, $f_{c}$:

For $f_{c} < 3000$,

$ f_{c} = f'_{c} + 1000$,

For $3000 \leq f_{c} \leq 5000$,

$ f_{c} = f'_{c} + 1200$,

For $f_{c} > 5000$,

$ f_{c} = 1.10f'_{c} + 700$

The standard deviation $s$ is defined by the expression

$ s = \left( \frac{2(F_{c} - f')}{n - 1} \right)^{1/2} $ (3.5a)

where $F_{c}$ = individual strength

$ F_{c}$ = average of the $n$ specimens

If two test records are used to determine the average strength, the standard deviation becomes

$ s = \left( \frac{(n_{1} - 1)s_{1}^{2} + (n_{2} - 1)s_{2}^{2}}{n_{1} + n_{2} - 2} \right)^{1/2} $ (3.5b)

where $s_{1}, s_{2}$ = standard deviations calculated from two test records, 1 and 2, respectively

$n_{1}, n_{2}$ = number of tests in each test record, respectively

If the number of test results available is fewer than 30 and more than 15, the value of $s$ used in Eqs. 3.4a and 3.4b should be multiplied by the appropriate modification factor value given in Table 3.2a.

3.5.3 Example 3.3: Calculation of Design Strength for Trial Mixture

Calculate the average compressive strengths $f'_{c}$ for the design of a concrete mixture if the specified compressive strength $f'_{c}$ is 5000 psi (34.5 MPa) such that (a) the standard deviation obtained using more than 30 consecutive tests is 500 psi (3.45 MPa); (b) the standard deviation obtained using 15 consecutive tests is 450 psi (3.11 MPa); (c) records of prior cylinder test results are not available.

Solution: (a) Using Eq. 3.4a,

$ f'_{c} = 5000 + 1.34 \times 500 $

$ = 5670 $ psi
3.7 Quality Tests on Concrete

Using Eq. 3.4b,

\[ f'_{cu} = 5000 + 2.33 \times 500 - 510 = 5665 \text{ psi} \]

Hence the required trial mix strength \( f'_{cu} = 5670 \text{ psi} \) (39.2 MPa).

(b) \( f'_{t} = 450 \text{ psi in 15 tests. From Table 3.19, the modification factor for } s = 1.16. Hence the value of standard deviation to be used in Eqs. 3.4(a) and 3.4(b) is } 1.16 \times 450 = 522 \text{ psi (3.6 MPa). Using Eq. 3.4(a),} \]

\[ f'_{cu} = 5000 + 1.34 \times 522 = 5700 \text{ psi} \]

Using Eq. 3.4(b)

\[ f'_{t} = 5000 + 2.33 \times 522 - 500 = 5716 \text{ psi} \]

Hence the required trial mix strength \( f'_{t} = 5716 \text{ psi} \) (39.4 MPa).

(c) Records of prior test results are not available. Using Table 3.19,

\[ f'_{cu} = f'_{t} + 1200 \text{ for 5000 psi concrete} \]

Hence the trial mixture strength \( = 5000 + 1200 = 6200 \text{ psi (42.8 MPa).} \)

If the mixing plant keeps good records of its cylinder test results over a long period, the required trial mixture strength \( f'_{cu} \) can be reduced as a result of such quality control, hence reducing costs for the owner.

3.6 Mixture Designs for Nuclear-Shielding Concrete

Whereas from the foregoing discussion it is seen that the design criterion was the w/c ratio, in concrete used for shielding against x-rays, gamma rays, and neutrons, the criterion is compactness or density of the mixture, regardless of workability. To achieve maximum density, tests have been conducted on various mixtures using crushed magnetite ore or fine steel shot instead of sand and steel punchings, magnetites, barites, or limonites instead of stone, as discussed previously. Results of these tests for both compactness and strength have shown that the w/c ratio has to be limited to 3.5 to 4.0 gil of water per bag of cement.

3.7 Quality Tests on Concrete

3.7.1 Workability or Consistency

Possible tests for workability or consistency include:

2. Remolding tests using Power's flow table.

The first method is the accepted ASTM standard.

3.7.2 Air Content

Measurement of the air content in fresh concrete is always necessary, especially when air-entraining agents are used.
3.7.3 Compressive Strength of Hardened Concrete

This is done by loading cylinders 6 in. in diameter and 12 in. high in compression perpendicular to the axis of the cylinder. For high-strength concrete, cylinders 4 in. dia. × 8 in.-height can be used applying proper dimensional correction.

3.7.4 Flexural Strength of Plain Concrete Beams

This test is performed by three-point loading of plain concrete beams of size 6 in. × 6 in. × 18 in. that have spans three times their depth.

3.7.5 Tensile Splitting Tests

These tests are performed by loading the standard 6 in. × 12 in. cylinder by a line load perpendicular to its longitudinal axis, with the cylinder placed horizontally on the testing machine platen. The tensile splitting strength can be defined as

\[ f_t = \frac{2P}{\pi DL} \]  

(3.6)

where 

\[ P = \text{total value of the line load registered by the testing machine} \]

\[ D = \text{diameter of the concrete cylinder} \]

\[ L = \text{cylinder height} \]

The results of all these tests give the designer a measure of the expected strength of the designed concrete in the built structure.

2.8 PLACING AND CURING OF CONCRETE

3.8.1 Placing

The techniques necessary for placing concrete depend on the type of member to be cast; that is, whether it is a column, a beam, a wall, a slab, a foundation, a mass concrete dam, or an extension of previously placed and hardened concrete. For beams, columns, and walls, the forms should be well oiled after cleaning them, and the reinforcement should be cleared of rust and other harmful materials. In foundations, the earth should be compacted and thoroughly moistened to about 6 in. in depth to avoid absorption of the moisture present in the wet concrete. Concrete should always be placed in horizontal layers.
that are compacted by means of high-frequency, power-driven vibrators of either the immersion or external type, as the case requires, unless it is placed by pumping. Keep in mind, however, that overvibration can be harmful since it could cause segregation of the aggregate and bleeding of the concrete.

3.8.2 Curing

As seen in Chapter 2, hydration of the cement takes place in the presence of moisture at temperatures above 50°F. It is necessary to maintain such a condition in order that the chemical hydration reaction can take place. If drying is too rapid, surface cracking takes place. This would result in reduction of concrete strength due to cracking, as well as failure to attain full chemical hydration.

To facilitate good curing conditions, any of the following methods can be used:

1. Continuously sprinkling with water.
2. Ponding with water.
3. Covering the concrete with wet burlap, plastic film, or waterproof curing paper.
4. Using liquid membrane-forming curing compounds to retain the original moisture in the wet concrete.
5. Steam curing in cases where the concrete member is manufactured under factory conditions, such as in cases of precast beams and pipes and prestressed girders and poles. Steam-curing temperatures are about 150°F. Curing time is usually 1 day, compared to the 5 to 7 days necessary when using the other methods.

3.8.3 Internal Curing Through Water Entrainment

Proper curing is a significant factor in high performance of concrete and the control of early-age cracking. This is particularly important when high-strength high performance concrete is desired, both in reinforced and prestressed concrete structures. In addition to
the methods briefly enumerated in Sec. 3.8.2, water entrainment, enhanced hydration and reduction of autogenous shrinkage can be achieved through *internal curing*, using structural lightweight aggregate (LWA) as a supplement to the normal aggregate in the proportioned mixture.

As discussed by T. A. Holm (Ref. 3.25), structural lightweight aggregate containing high internal moisture content may partially substitute ordinary aggregate to provide "internal curing" in concrete containing high volume of cementitious materials. High cementitious concretes are vulnerable to self-desiccation. They can benefit significantly from the moisture available within the structural lightweight aggregate, especially at early age during the critical first seven days, when a large proportion of autogenous shrinkage takes place. This property is particularly helpful when curing vertical members and concretes containing high volumes of pozzolans that are sensitive to early drying.

This "internal curing" process is made possible when the moisture content of structural lightweight aggregate has a high degree of saturation during mixing (percent of internal pore volume occupied by water)—a fact known for many years that the absorbed moisture in structural lightweight aggregate acts not as part of the w/c ratio, but available for internal curing. It is important to stress that the benefits of internal curing by virtue of water entrainment go far beyond the improvements in long-term strength gain. A significant reduction in permeability is achieved by the major increase in the length of curing time available, hence a resulting high performance of the finished product.

At this time, it is reasonable to assume that the amount of absorbed water, entrained in the pores of structural lightweight aggregate substituted for an equal volume of normal density aggregate) that is necessary to fully cure low w/cm concrete mixtures may be estimated by:

\[
\frac{w}{cm} \text{ (high performance concrete)} = \left( \frac{\text{Entrained water}}{\text{Cementitious materials}} \right) \leq 0.45,
\]

where the entrained water = (Mass of LWA) x (Moisture content of LWA)

### 3.9 Properties of Hardened Concrete

The mechanical properties of hardened concrete can be classified as (1) short-term or instantaneous properties and (2) long-term properties. The short-term properties can be enumerated as (1) strength in compression, tension, and shear and (2) stiffness measured by modules of elasticity. The long-term properties can be classified in terms of creep and shrinkage. The following sections present some details of the aforementioned properties.

#### 3.9.1 Compressive Strength

Depending on the type of mixture, the properties of aggregate, and the time and quality of curing, compressive strengths of concrete can be obtained up to 20,000 psi or more. Commercial production of concrete with ordinary aggregate is usually in the range from 3000 to 10,000 psi, with the most common concrete strengths in the range from 3000 to 9000 psi.

The compressive strength, \( f'_c \), is based on standard 6 in. x 12 in. cylinders cured under standard laboratory conditions and tested at a specified rate of loading at 28 days of age. The standard specifications used in the United States are usually taken from ASTM C-59. It should be mentioned that the strength of concrete in the actual structure may not be the same as that of the cylinder because of the difference in compaction and curing conditions.

The ACI Code specifies for a strength test the average of two cylinders from the same sample tested at the same age, which is usually 28 days. As for the frequency of testing, the Code specifies that the strength level of an individual class of concrete can be
considered as satisfactory if (1) the average of all sets of three consecutive strength tests equals or exceeds the required $f'_c$ and (2) no individual strength test (average of two cylinders) falls below the required $f'_c$ by more than 500 psi. The average concrete strength for which a concrete mixture must be designed should exceed $f'_c$ by an amount that depends on the uniformity of plant production, as explained in Section 3.5.

It must be emphasized that the design $f'_c$ should not be the average cylinder strength. The design value should be chosen as the conceivable minimum cylinder strength.

### 3.9.2 Tensile Strength

The tensile strength of concrete is relatively low. A good approximation for the tensile strength $f_t$ is $0.10f'_c < f_t < 0.20f'_c$. It is more difficult to measure tensile strength than compressive strength because of the gripping problems with testing machines. A number of methods are available for tension testing, the most commonly used method being the cylinder splitting test or sometimes referred to as the Brazilian test.

For members subjected to bending, the value of the modulus of rupture $f'_r$, rather than tensile splitting strength $f'_t$ is used in design. The modulus of rupture is measured by testing to failure plain concrete beams 6 in. square in cross section, having a span of 18 in. and loaded at the third points (ASTM C-78). The modulus of rupture has a higher value than the tensile splitting strength. The ACI specifies a value of $7.5 \sqrt{f'_c}$. For the modulus of rupture of normal-weight normal-strength concrete,

In most cases, lightweight concrete has a lower tensile strength than does normal-weight concrete. Following are the ACI Code stipulations for lightweight concrete.

1. If the splitting tensile strength $f'_t$ is specified,
   $$ f'_t = 1.09f'_c = 7.5 \sqrt{f'_c} $$
2. If $f'_t$ is not specified, use a factor of 0.75 for all lightweight concrete and 0.85 for sand-lightweight concrete. Linear interpolation may be used for mixtures of natural sand and lightweight fine aggregate.

### 3.9.3 Shear Strength

Shear strength is more difficult to determine experimentally than the tests discussed previously because of the difficulty in isolating shear from other stresses. This is one reason for the large variation in shear-strength values reported in the literature, varying from 20% of the compressive strength in normal loading to a considerably higher percentage of up to 85% of the compressive strength in cases where direct shear exists in combination with compression. Control of a structural design by shear strength is significant only in rare cases, since shear stresses must ordinarily be limited to continually lower values in order to protect the concrete from failure in diagonal tension.

### 3.9.4 Stress–Strain Curve

Knowledge of the stress–strain relationship of concrete is essential for developing all the analysis and design terms and procedures in concrete structures. Figure 3.5a shows a typical stress–strain curve obtained from tests using cylindrical concrete specimens loaded in uniaxial compression over several minutes. The first portion of the curve, to about 40% of the ultimate strength $f'_c$, can be considered essentially linear for all practical purposes. After approximately 90% of the failure stress, the material loses a large portion of its
stiffness, thereby increasing the curvilinearity of the diagram. At ultimate load, cracks parallel to the direction of loading become distinctly visible, and most concrete cylinders (except those with very low strengths) fail suddenly shortly thereafter. Figure 3.5b shows the stress-strain curves of concrete of various strengths reported by the Portland Cement Association. It can be observed that (1) the lower the strength of concrete, the higher the failure strain; (2) the length of the initial relatively linear portion increases with the in-
Figure 3.5 (a) Typical stress-strain curve of concrete; (b) stress-strain curves for various concrete strengths.
crease in the compressive strength of concrete; and (3) there is an apparent reduction in ductility with increased strength.

### 3.9.5 Modulus of Elasticity

Since the stress–strain curve shown in Fig. 3.6 is curvilinear at a very early stage of its loading history, Young's modulus of elasticity can be applied only to the tangent of the curve at the origin. The initial slope of the tangent to the curve is defined as the initial tangent modulus, and it is also possible to construct a tangent modulus at any point of the curve. The slope of the straight line that connects the origin to a given stress (about 0.4\(f_c\)) determines the secant modulus of elasticity of concrete. This value, termed in design calculation the modulus of elasticity, satisfies the practical assumption that strains occurring during loading can be considered basically elastic (recoverable on unloading) and that any subsequent strain due to the load is regarded as creep.

The ACI Code gives the following expressions for calculating the secant modulus of elasticity of concrete \(E_s\) for \(f_c\) up to 6000 psi

\[
E_s (\text{psi}) = 33w^{1/2} \sqrt{f_c} \quad \text{for} \ 90 < w < 1551 \text{lb/ft}^2;
\]

\[
E_s (\text{MPa}) = 0.0143w^{1/2} \sqrt{f_c};
\]
where \( \rho_c \) is the density of concrete in pounds per cubic foot (1 \( \text{lb/ft}^3 = 16.02 \text{ kg/m}^3 \)) and \( f_c' \) is the compressive cylinder strength in psi. For normal-weight concrete,

\[
E_c = 57,000 \sqrt{f_c'} \text{ psi or } E_c = 4239 \sqrt{f_c'} \text{ N/mm}^2
\]

For concrete compressive strength \( f_c' = 6000 \to 12,000 \text{ psi}, \)

\[
E_c, (\text{psi}) = (40,000 \sqrt{f_c'} + 1.0 \times 10^6 \left( \frac{W_c}{145} \right)^{1.9}; E_c, (\text{MPa}) = (3.32 \sqrt{f_c'} + 6895 \left( \frac{W_c}{3220} \right))^{1.9}
\]

For \( f_c' > 12,000 \text{ psi}, \) reference has to be made to the research literature, or conduct control tests in large projects to establish a realistic \( E_c. \)

These expressions are valid only in general terms, since the value of the modulus of elasticity is also affected by factors other than loads, such as moisture in the concrete specimen, the water/cement ratio, age of the concrete, and temperature. Therefore, for special structures such as arches, tunnels, and tanks, and high-strength concretes, the modulus of elasticity needs to be determined from test results.

Limited work exists on the determination of the modulus of elasticity in tension because the low tensile strength of concrete is normally disregarded in calculations. It is, however, valid to assume within those limitations that the value of the modulus in tension is equal to that in compression.

### 3.9.6 Shrinkage

Basically, there are two types of shrinkage: plastic shrinkage and drying shrinkage. **Plastic shrinkage** occurs during the first few hours after placing fresh concrete in the forms. Exposed surfaces such as floor slabs are more easily affected by exposure to dry air because of their large contact surface. In such cases, moisture evaporates faster from the concrete surface than it is replaced by the bleed water from the lower layers of the concrete elements.

**Drying shrinkage**, on the other hand, occurs after the concrete has already attained its final set and a good portion of the chemical hydration process in the cement gel has been accomplished. Drying shrinkage is the decrease in the volume of a concrete element when it loses moisture by evaporation. The opposite phenomenon, that is, volume increase through water absorption, is termed swelling. In other words, shrinkage and swelling represent water movement out of or into the gel structure of a concrete specimen due to the difference in humidity or saturation levels between the specimen and the surroundings irrespective of the external load.
Chapter 3  Concrete

Figure 3.7 Shrinkage-time curve.

Shrinkage is not a completely reversible process. If a concrete unit is saturated with water after having fully shrunk, it will not expand to its original volume. Figure 3.7 relates the increase in shrinkage strain $\varepsilon_a$ with time. The rate decreases with time since older concretes are more resistant to stress and consequently undergo less shrinkage, such that the shrinkage strain becomes almost asymptotic with time.

Several factors affect the magnitude of drying shrinkage:

1. **Aggregate.** The aggregate acts to restrain the shrinkage of the cement paste; hence concretes with high aggregate content are less vulnerable to shrinkage. In addition, the degree of restraint of a given concrete is determined by the properties of aggregates; those with high modulus of elasticity or with rough surfaces are more resistant to the shrinkage process.

2. **Water/cement ratio.** The higher the water/cement or water/semenitious ratio, the higher the shrinkage effects. Figure 3.8 is a typical plot relating aggregate content to water/cement ratio.

3. **Size of the concrete element.** Both the rate and total magnitude of shrinkage decrease with an increase in the volume of the concrete element. However, the duration of shrinkage is longer for larger members since more time is needed for drying to reach the internal regions. It is possible that 1 year may be needed for the drying process to begin at a depth of 10 in. from the exposed surface and 10 years to begin at 24 in. below the external surface.

4. **Medium ambient conditions.** The relative humidity of the medium affects greatly the magnitude of shrinkage; the rate of shrinkage is lower at high states of relative humidity. The environment temperature is another factor, in that shrinkage becomes stabilized at low temperatures.

Figure 3.8 Water ratio and aggregate content effect on shrinkage.
5. **Amount of reinforcement.** Reinforced concrete shrinks less than plain concrete; the relative difference is a function of the reinforcement percentage.

6. **Admixtures.** This effect varies depending on the type of admixture. An acceleratesor such as calcium chloride, used to accelerate the hardening and setting of the concrete, increases the shrinkage. Pozzolans can also increase the drying shrinkage, whereas air-entraining agents have little effect.

7. **Type of cement.** Rapid-hardening cement shrinks somewhat more than other types, while shrinkage-compensating cements minimize or eliminate shrinkage cracking if used with restraining reinforcement.

8. **Carbonation.** Carbonation shrinkage is caused by the reaction between carbon dioxide (CO₂) present in the atmosphere and that present in the cement paste. The amount of the combined shrinkage varies according to the sequence of occurrence of carbonation and drying processes. If both phenomena take place simultaneously, less shrinkage develops. The process of carbonation, however, is dramatically reduced at relative humidities below 50%.

### 3.9.6.1 Shrinkage Prediction for Standard Conditions

The value of the ultimate shrinkage strain at standard conditions has the following range.

\[
(\varepsilon_{sh})_u = 415 \times 10^{-6} \text{ to } 1070 \times 10^{-6} \text{ in./in. (mm/mm)}
\]

An average value of \((\varepsilon_{sh})_u\), as recommended by ACI Committee 209 as follows:

- **moist-cured for seven days:** \((\varepsilon_{sh})_u = 800 \times 10^{-6} \text{ in./in. (mm/mm)}\)
- **steam-cured for 1-3 days:** \((\varepsilon_{sh})_u = 730 \times 10^{-6} \text{ in./in. (mm/mm)}\)

A common average shrinkage strain in standard conditions for both moist-cured and steam-cured concretes can be used with sufficient accuracy having a value

\[(\varepsilon_{sh})_u = 780 \times 10^{-6} \text{ in./in. (mm/mm)}\]

The shrinkage strain prediction expression for standard conditions become:

**after 7 days of moist curing:**

\[
(\varepsilon_{sh}) = \frac{t}{35} (\varepsilon_{sh})_u
\]

where \(t\) is the age of concrete in days after curing.

**after 1-3 days of steam curing:**

\[
(\varepsilon_{sh}) = \frac{t}{25} (\varepsilon_{sh})_u
\]

### 3.9.6.2 Shrinkage and Temperature Reinforcement Requirements for Building Structures

Reinforcement for shrinkage and temperature stresses normal to flexural reinforcement has to be provided in structural slabs, cast foundations and walls in accordance with ACI 318 Code for Building Structures, when the flexural reinforcement extends in one direction only. They serve to control cracking due to restraint of the structural element by adjacent members. The percentage of reinforcement required is:

(a) 0.20 percent for grades 40 and 50 ksi. Steel reinforcement,
(b) 0.18 percent for grade 60 ksi steel reinforcement,
(c) For stresses exceeding 60 ksi, the percentage is proportioned to give

\[
\frac{0.0048 \times 60,000}{f_t}
\]
The spacing of the reinforcement should not exceed 5 times the slab thickness, nor further apart than 18 in. Also, additional consideration needs to be given to effects of forces due to prestressing, vibration, impact, creep, and excessive restraint.

3.9.6.3 Shrinkage and Temperature Reinforcement Requirements for Liquid Retaining Structures, Sanitary Containment Structures. These are special structures where water tightness is essential to prevent leakage and loss of potable water or hazardous sewage liquids. The requirements for reinforcement percentage are set in the ACI 318 Code for Environmental Structures for structural slabs, walls and mat foundations, where the flexural reinforcement extends in one direction only. Where shrinkage and temperature movements are significantly restrained, additional consideration needs to be given to effects of forces due to prestressing, vibration, impact, creep.

Concrete sections that are 24 in. thick or greater may have the minimum shrinkage and temperature reinforcement based on 12 in. concrete layer at each face. The reinforcement in the bottom of the base slabs supported on soil may be reduced by 50 percent of that required in this table.

Table 3.21 Minimum Shrinkage and Temperature Reinforcement for Environmental Structures

<table>
<thead>
<tr>
<th>Length between Movement Joint(s), ft</th>
<th>Minimum shrinkage and temperature Reinforcement ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Grade 40</td>
</tr>
<tr>
<td>Less than 20 ft</td>
<td>0.003</td>
</tr>
<tr>
<td>20 to less than 30 to less than 40</td>
<td>0.004</td>
</tr>
<tr>
<td>40 and greater</td>
<td>0.005</td>
</tr>
</tbody>
</table>

*Maximum shrinkage and temperature reinforcement where movement joints are not provided.

Note: When using this table, the actual joint spacing should be multiplied by 1.5 if no more than 50 percent of the reinforcement passes through the joint.

3.9.7 Creep

Creep, or lateral material flow, is the increase in strain with time due to a sustained load. Initial deformation due to load is the elastic strain, while the additional strain due to the same sustained load is the creep strain. This practical assumption is acceptable since the initial recorded deformation includes few time-dependent effects.

Figure 3.9 illustrates the increase in creep strain with time, and as in the case of shrinkage, it can be seen that creep decreases with time. Creep cannot be observed directly and can be determined only by deducting elastic strain and shrinkage strain from the total deformation. Although shrinkage and creep are not independent phenomena, it can be assumed that superposition of strains is valid; hence

\[
\text{total strain} \ (e_t) = \text{elastic strain} \ (e_s) + \text{creep strain} \ (e_c) + \text{shrinkage strain} \ (e_a)
\]

An example of the relative numerical values of strain due to the foregoing three factors is presented for a normal concrete specimen subjected to 900 psi in compression:

- Immediate elastic strain, \( e_s = 250 \times 10^{-6} \text{ in./in.} \)
- Shrinkage strain after 1 year, \( e_a = 500 \times 10^{-6} \text{ in./in.} \)
- Creep strain after 1 year, \( e_c = 750 \times 10^{-6} \text{ in./in.} \)
- \( e_t = \frac{1500 \times 10^{-6} \text{ in./in.}}{e_t} \)
These relative values illustrate that stress-strain relationships for short-term loading lose their significance and long-term loadings become dominant in their effect on the behavior of a structure.

Figure 3.10 qualitatively shows in a three-dimensional model the three types of strain discussed resulting from sustained compressive stress and shrinkage. Since creep is time dependent, this model has to be such that its orthogonal axes are deformation, stress, and time.

Numerous tests have indicated that creep deformation is proportional to the applied stress, but the proportionality is valid only for low stress levels. The upper limit of the relationship cannot be determined accurately, but can vary between 0.2 and 0.5 of the ultimate strength $f'$. This range in the limit of the proportionality is expected due to the large extent of microcracks at about 40% of the ultimate load.

Figure 3.11a shows a section of the three-dimensional model in Fig. 3.10 parallel to the plane containing the stress and deformation axes at time $t_1$. It indicates that both elastic and creep strains are linearly proportional to the applied stress. In a similar manner, Fig. 3.11b illustrates a section parallel to the plane containing the time and strain axes at a stress $f'_{c}$; hence it shows the familiar creep-time and shrinkage-time relationships.

![Figure 3.9 Strain-time curve.](image)

![Figure 3.10 Three-dimensional model of time-dependent structural behavior.](image)
As in the case of shrinkage, creep is not completely reversible. If a specimen is unloaded after a period under a sustained load, an immediate elastic recovery is obtained that is less than the strain precipitated on loading. The instantaneous recovery is followed by a gradual decrease in strain, called creep recovery. The extent of the recovery depends on the age of the concrete when loaded with older concretes presenting higher creep recoveries, while residual strains or deformations become frozen in the structural element (see Fig. 3.12).

Creep is closely related to shrinkage and, as a general rule, a concrete that is resistant to shrinkage also presents a low creep tendency, as both phenomena are related to the hydrated cement paste. Hence creep is influenced by the composition of the concrete, the environmental conditions, and the size of the specimen, but principally creep depends on loading as a function of time.

The composition of a concrete specimen can be essentially defined by the water/cement ratio, aggregate and cement types, and aggregate and cement contents. Therefore, like shrinkage, an increase in the water/cement ratio and in the cement content increases creep. Also, as in shrinkage, the aggregate induces a restraining effect such that an increase in aggregate content reduces creep.
3.9.8 Creep Effects

As in shrinkage, creep increases the deflection of beams and slabs and causes loss of pre-stress. In addition, the initial eccentricity of a reinforced concrete column increases with time due to creep, resulting in the transfer of the compressive load from the concrete to the steel in the section.

Once the steel yields, additional load has to be carried by the concrete. Consequently, the resisting capacity of the column is reduced and the curvature of the column increases further, resulting in overstress in the concrete and leading to failure.

3.9.9 Rheological Models

Rheological models are mechanical devices that portray the general deformation behavior and flow of materials under stress. A model is basically composed of elastic springs and ideal dashpots denoting stress, elastic strain, delayed elastic strain, irrecoverable strain, and time. The springs represent the proportionality between stress and strain, and the dashpots represent the proportionality of stress to the rate of strain. A spring and a dashpot in parallel form a Kelvin unit, and in series they form a Maxwell unit.

Two rheological models will be discussed: the Burgers model and the Ross model. The Burgers model in Fig. 3.13 is shown since it can approximately simulate the stress-strain-time behavior of concrete at the limit of proportionality with some limitations. This model simulates the instantaneous recoverable strain (a); the delayed recoverable elastic strain in the spring (b); and the irreversible time-dependent strain in dashpots (c and d). The weakness in this model is that it continues to deform at a uniform rate as long as the load is sustained by the Maxwell dashpot, a behavior not similar to concrete, where creep reaches a limiting value with time, as shown in Fig. 3.9.

A modification in the form of the Ross rheological model in Fig. 3.14 can eliminate this deficiency. A in this model represents the Hookian direct proportionality of stress-to-strain element, D represents the Newtonian element, and B and C are the elastic springs that can transmit the applied load P(t) to the enclosing cylinder walls by direct friction. Since each coil has a defined frictional resistance, only those coils whose resistances equal the applied load P(t) are displaced; the others remain unstressed, symbolizing the irrecoverable deformation in concrete. As the load continues to increase, it overcomes the spring resistance of unit B, pulling out the spring from the dashpot and signifying failure in a concrete element. More rigorous models have been used, such as Roll's model to assist in predicting the creep strains. Mathematical expressions for such predictions can be very rigorous. One convenient expression due to Ross defines creep C under load after a time interval t as follows:

\[
C = \frac{t}{a + bt} \quad (3.7)
\]

where a and b are constants determinable from tests.

![Figure 3.13 Burgers model.](image-url)
Work by Branson (Refs. 3.15 and 3.16) has simplified creep evaluation. The additional strain $\varepsilon_c$ due to creep can be defined as

$$\varepsilon_c = \rho_c f_c$$  \hspace{1cm} (3.8a)

where $\rho_c$ = unit creep coefficient, generally called specific creep
d
$f_c$ = stress intensity in the structural member corresponding to strain $\varepsilon_c$.

If $C_u$ is the ultimate creep coefficient,

$$C_u = \rho_u E_u$$  \hspace{1cm} (3.8b)

An average value of $C_u = 2.35$.

Branson's model, verified by extensive tests, relates the creep coefficient $C_i$ at any time to the ultimate creep coefficient as follows:

$$C_i = \frac{\rho_i}{10 + \rho_i} C_u$$  \hspace{1cm} (3.9)

or, alternatively,

$$\rho_i = \frac{\rho_i}{10 + \rho_i}$$  \hspace{1cm} (3.10)

where $i$ is the time in days.

The selected references at the end of the chapter give detailed information on the creep coefficients and constants to be used to evaluate creep effect. The brief discussion in this section is intended to provide exposure to the procedures considered in any fundamental study of creep and shrinkage behavior.

3.10 HIGH-STRENGTH CONCRETE

3.10.1 General Principles

Concretes with compressive strength $f'_c$ of at least 6000 psi (44.4 MPa) can be classified as high-strength concrete, with the possibility today of achieving 20,000-psi (137.9-MPa) concrete under field conditions. To produce such concrete, chemical and mineral admixtures as well as air-entraining agents have to be used. Chemical retarders are used to retard the setting time for the cement-rich, high-strength concrete. Mineral admixtures such as fly ash, slag cement, and silica fume are also frequently used.

It is found that silica-fume admixtures in the range from 5 to 20% by weight of cement are an ideal additive for drastic increase in the compressive strength and considerable reduction in permeability. The increase in concrete density and strength is due to the dispersion of ultrafine particles of silica fume between the cement grains. This in turn results in a reduction of workability, which is enhanced further by the reduced water/cement ratios of the high-strength concrete mixture. Consequently, high-range, water-reducing admixtures, called plasticizers, would have to be added in the required proportions in order to increase workability appreciably while maintaining a low water/cement ratio.
As with other ingredients of high-strength concrete, the fine and coarse aggregates should be of good quality. For low w/c ratios, smaller-size coarse aggregates give better results. The grading of the aggregates is relatively unimportant in high-strength concrete compared to conventional concrete due to the high content of fine cementitious materials. However, it is sometimes helpful to increase the fineness modulus to make the concrete consistency less viscous. Gap grading provides better results than continuous grading. For compressive strength above 8000 psi, it is advisable to use a maximum size of aggregate less than 1 to 2 in. Cleanliness of both the fine and coarse aggregates deserves particular attention. In general, three characteristics of the coarse aggregate—compressive strength, bonding potential with cement pastes, and low water absorption capacity—are important in the production of high-strength concrete.

In addition to stringent quality control of materials, high-strength concrete requires proper proportioning to attain the desired mixture along with careful mixing, handling, placing, and curing. Available mixture proportions data could be used as guidelines for trial mixture designs. However, to attain the desired strength and characteristics, extensive trial mixture designs are required. In addition, the importance of curing increases due to the use of low w/c ratios, as one must not only avoid moisture escape but also provide extra water for hydration. Similarly, proper mixing, handling, and placing are important to prevent moisture loss and produce workable concrete.

As to water content, it is important to consider the total water content, including that from the coarse and fine aggregate and all admixtures. Whereas in conventional practice a range of 0.40 to 0.45 w/c is used, the following are the recommended values for higher-strength concretes:

<table>
<thead>
<tr>
<th>f_c (psi)</th>
<th>w/c</th>
</tr>
</thead>
<tbody>
<tr>
<td>6,000-9,000</td>
<td>0.35-0.40</td>
</tr>
<tr>
<td>10,000-12,000</td>
<td>0.30-0.35</td>
</tr>
<tr>
<td>12,000-20,000</td>
<td>0.30-0.22</td>
</tr>
</tbody>
</table>

The very low w/cm ratio ranges are achieved by utilizing large amounts of superplasticizers and high cement content.

In summary, four basic principles have to be considered in the production of high-strength concrete: (1) improved aggregate-matrix bond, (2) reduced porosity, (3) improved compaction, and (4) application of internal agents such as silica fume and plasticizers and external agents such as lateral confinement through internal steel hoops, heat or steam curing, proper handling, and strict quality control.

### 3.10.2 Design Criteria

Available expressions defining concrete properties are based primarily on experimental data of concrete with compressive strength below 6000 psi. Such expressions do not necessarily define the relevant parameters when high-strength concrete is being used. When the concrete compressive strength exceeds 6000 psi for high-strength concrete, particularly in the range from 8000 to 12,000 psi, engineering properties of the concrete, such as elasticity, flexural strength, tensile resistance, and bond strength, may be affected.

The principal mechanical properties of concrete are compressive and tensile strength, creep and shrinkage, and modulus of elasticity. The actual values of tensile strength, modulus of elasticity, creep, and shrinkage are a function of compressive strength for most low- and moderate-strength concretes. But such correlation is not always the case for high compressive strengths.
3.10.2.1 Modulus of Elasticity \( E_c \). The modulus of elasticity is strongly influenced by the concrete materials and proportions used. An increase in the modulus \( E_c \) is expected with the increase in compressive strength since the slope of the ascending branch of the stress-strain diagram becomes steeper for higher-strength concretes, but at a lower rate than the compressive strength. The value of the secant modulus \( E_c \) for normal-strength concrete at 28 days is usually approximately \( 4 \times 10^6 \) psi, whereas for higher-strength concretes values in the range from \( 7 \) to \( 8 \times 10^6 \) psi have been recorded. These higher values can be used to reduce short- and long-term deflection of flexural members and eccentricity of columns and other heavily loaded members.

For concretes in the strength range up to 6000 psi, the ACI Code empirical equation for the secant modulus of concrete \( E_c \), given in Section 3.9.5 is reasonably applicable. However, as the strength of concrete increases, in the range from 12,000 to 20,000 psi, the value of \( E_c \) increases at a faster rate than that generated by the ACI expression \( (E_c = 33 \times 10^6 \sqrt{f_c}) \), thereby underestimating the true \( E_c \) value.

Available expressions for \( E_c \) applicable to concrete strength up to 12,000 psi are inconclusive. The expression by Carrasquillo et al. (Ref. 3.19) for normal-weight concrete of strengths up to 12,000 psi and lightweight concrete up to 9000 psi is

\[
E_c(40,000 \sqrt{f_c} + 1 \times 10^6) \left( \frac{w_r}{145} \right)^{1.5} \tag{3.11}
\]

where \( w_r \) is the unit weight of the hardened concrete in pcf. Other investigations report that as \( f_c \) approaches 12,000 psi for normal-weight concrete and less for lightweight concrete, Eq. 3.19 can underestimate the true value of \( E_c \). At the present state of the art, it is advisable in cases of very high strength use in major structures where \( f_c \) is in the range of 20,000 psi or higher that adequate stress-strain cylinder compression tests be performed with stress-strain readings. In this manner, the deduced secant modulus value of \( E_c \) at an \( f_c = 6.45f \), intercept could predict more accurately the true value of the \( E_c \), for the particular mixture and aggregate size and properties until an acceptable expression is available to the designer. The long-term stiffness and deflection computations would thereby be more representative.

Work at Rutgers (Ref. 3.22) on high-strength composite construction has resulted in considerable enhancement of the ductility of high-strength reinforced concrete beams. Prestressed concrete prisms of high-strength concrete were used in place of the normal mild steel bar reinforcement. The mixture proportions in lb/yd\(^3\) were as shown in Table 3.23. The mixture was designed for 7-day compressive strength of 12,000 psi (84 MPa). The ratio of the cementations/fine/coarse aggregate was 1:1.22:2.06, and the slump varied between 4 and 6 in. (100 and 150 mm). The prestressing strands were stress relieved 270-kips (1900-MPa), 7-wire, 1-in.-dia (9.5-mm)-diameter strands.

Figure 3.15 shows the cross section of the composite beams, and Fig. 3.16 gives a typical stress-strain relationship of the concrete, which achieved in some of the mixes a

<table>
<thead>
<tr>
<th>Table 3.23 Mix Proportions (lb/yd(^3)) for Composite Beams</th>
<th>( f_c &gt; 12,000 ) psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coarse Aggregate,</td>
<td>Fine Aggregate (Natural Sand)</td>
</tr>
<tr>
<td>Size</td>
<td>(in.)</td>
</tr>
<tr>
<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>1851</td>
<td>1100</td>
</tr>
</tbody>
</table>

1 lb/yd\(^3\) = 0.99 Kg/m\(^3\).
Figure 3.15 Cross sections of high-strength concrete composite beams reinforced with high-strength prestressed concrete prisms.
7-day strength of 13,250 psi (91.4 MPa). The tested specimens were instrumented with a fiber-optic sensor system developed by the author using Bragg grating sensors both internally and externally.

3.10.2.2 Modulus of Rupture $f'$. An evaluation of the tensile behavior of concrete can be made by the modulus of rupture or bending test. A range of $7.5\sqrt{f'}$ to $12\sqrt{f'}$ has been reported for high-strength concrete modulus of rupture values with a reasonable expression in terms of compressive strength as follows for plain normal-weight concrete (Refs. 3.13 and 3.16):

$$f'_{(psi)} = 11.7 \sqrt{f}$$  \hspace{1cm} (3.12)

3.10.2.3 Tensile Splitting Strength $f'_s$. Tensile strength of concrete is an important parameter in determining when the first flexural crack may develop. A general expression for high-strength concrete for $f'_s$ range of 3000 to 12,000 psi is given as follows:

$$f'_s = 7.4 \sqrt{f}$$  \hspace{1cm} (3.13)

However, with deliberate selection of materials and proportions, including the use of silica fume and smaller coarse aggregates, the tensile strength may be increased to almost twice that predicted by this expression.

3.10.2.4 Creep and Shrinkage. With high-strength concrete, greater stress may be applied with little or no increase in long-term deformation above the level expected in moderate-strength concrete. Since high-strength concrete has low water/cement ratios that could be as low as 0.22 for $f'_c = 20,000$ psi, shrinkage can be very limited. With a range of shrinkage strain of $250 \times 10^{-6}$ to $500 \times 10^{-6}$ in./in.

3.10.3 Confining Effect on High-Strength Concrete

Use of high-strength concrete in compression members, such as in tall structures, leads to considerable reduction in the size of the concrete sections. Widely accepted properties of such concretes are their higher modulus $E$ values, less ductile mode of failure, and larger strain at maximum stress. The use of confining circular or rectangular spiral reinforcement leads to increased strength and ductility of the confined concrete. Published experi-
mental results on the effects of rectilinear confinement in very high strength concrete (in excess of 12,000 psi) are scarce. Results of tests in Ref. 3.20 for concretes of up to 25,600-psi compressive strength indicate general improvement of the behavior of the concrete when confined. Instead of collapsing in a very brittle fashion, the concrete failed in a more ductile and gradual manner.

The peak stress $f_c$ in Figure 3.27b, the strain $\varepsilon_c$, and especially the ductility increased with the increase in the volumetric ratio, but not proportionately. If the peak stress $f_c = K f_p$, where $K$ is the effective confinement, then $K$ can be expressed as

$$K = 1 + 0.0091 \left(1 - 0.245 \left(\frac{d}{h'}\right) \left(\frac{d}{8n p'}\right) \sqrt{\frac{f_p}{f_c'}}\right)$$

(3.14)

where $s$ = center-to-center spacing of the lateral ties, in.

$h'$ = length of one side of the rectangular ties, in.

$n$ = number of longitudinal steel bars

$d'$ = nominal diameter of lateral ties, in.

$p'$ = volumetric ratio of lateral reinforcement

$p = \frac{p'}{f_p}$ = volumetric ratio of longitudinal reinforcement

$f_p'$ = yielding stress of the lateral steel, psi

The peak strain $\varepsilon_c$ can be predicted by the following expression (Ref. 3.20):

$$\varepsilon_c = 0.00265 + \frac{0.0033 \left(1 - 0.734 \left(\frac{d}{h'}\right) \left(\frac{d}{8n p}\right)\right)}{\sqrt{f_c'}}$$

(3.15)

### 3.10.4 Mixture Proportions for 20,000-psi Concrete

For very high compressive strength in excess of 12,000 psi (82.74 MPa), it is essential to make a large number of trial mixtures (five or more) and take extra care in the selection of aggregate size and source. Steel cylinder molds are preferred for uniformity of test results, using 4 in. x 8 in. molds, and applying the appropriate dimensional correction. It is also necessary to grind the cylinder ends and then either cap them with high-strength casting compound prior to loading or apply the load directly to the ground ends through a removable steel cap with a hard neoprene pad bearing directly on the ground specimen ends. Preparation of the cylinders should resemble as closely as possible the field conditions of concrete placement. Mock-up placement of the high-strength concrete is advisable in order to evaluate the construction procedures and performance of the concrete in field conditions and to identify potential problems with batching, placement, and testing of the concrete at early ages. With corrective measures taken immediately.

A slump of 8 in. with $w/c = 0.22$ resulted from the mixture proportions indicated. A typical compressive age plot for the indicated mixture based on 4 in. x 8 in. cylinder tests is shown in Fig. 3.18.
Figure 3.17 (a) Normalized complete stress-strain curve by various authors; (b) stress and stain parameters. (From Ref. 3.15.)
Table 3.24 Mix Proportions for $f_c > 19,000$ psi

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<thead>
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<tr>
<td>1872</td>
<td>1165</td>
<td>957</td>
<td>747</td>
<td>13</td>
<td>2.1</td>
<td>9.8</td>
</tr>
<tr>
<td>1894</td>
<td>1165</td>
<td>956</td>
<td>317</td>
<td>13</td>
<td>2.1</td>
<td>16.4</td>
</tr>
<tr>
<td>1905</td>
<td>1165</td>
<td>950</td>
<td>(0.6 + 0.23)</td>
<td>(70 B)</td>
<td>(6.6)</td>
<td>(up to 24)</td>
</tr>
</tbody>
</table>

*Weight of solid silica fume only. Water contained as part of the emulsion must be subtracted from the total water allowed.

Figure 3.18 Compressive strength versus age of high-strength concrete.

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3.3. ACI Committee 211. Guide for Selecting Proportions for High Strength Concrete with Portland Cement and Fly Ash, ACI 211-88R-95, American Concrete Institute, Farmington Hills, MI.

3.4. ACI Manual of Concrete Practice 2005, Vols. 1, 2, 3, 4, 5, 6, American Concrete Institute, Farmington Hills, MI.


3.9. ACI Committee 318, "Building Code Requirements for Structural Concrete (ACI 318-05) and the Commentary (ACI 318R-05)," American Concrete Institute, Farmington Hills, MI, 2005, 442 pp.


3.27. ACI Committees 318, "Code Requirements for Environmental Engineering Concrete Structures (ACI 318-05) and Commentary (ACI 318R-05)," American Concrete Institute, Farmington Hills, MI, 2001, 347 pp.
PROBLEMS FOR SOLUTION

3.1. Design a concrete mix using the following data:

- Required strength: $f' = 5000$ psi (34.5 MPa)
- Type of structure: beam
- Maximum size of aggregate = 1 in. (18 mm)
- Fineness modulus of sand = 2.6
- Dry-rotted weight of coarse aggregate = 100 lb/ft$^3$
- Moisture absorption: 2% for coarse aggregate and 2% for fine aggregate

3.2. Using the data of Ex. 3.1, design a 6% air-entrained mixture.

3.3. Repeat Ex. 3.1 for a mixture design strength $f' = 3000$ psi (20.7 MPa).

3.4. Estimate the strength of the trial mixture, $f_{tu}$, for the following cases:

- (a) $f' = 3500$ psi (24.15 MPa); $s$ (using 40 consecutive tests) = 300 psi (2.07 MPa).
- (b) $f' = 3000$ psi (20.7 MPa); $s$ (using 20 consecutive tests) = 250 psi (1.73 MPa).
- (c) $f' = 3000$ psi (20.7 MPa); test results are not available.
- (d) $f' = 4000$ psi (27.6 MPa); $s$ (using 15 tests) = 375 psi (2.59 MPa).

3.5. Using the data in Ex. 3.2, design a high-strength high-performance concrete for a mixture design strength $f' = 8000$ psi (55 MPa) with a slump of 6 in. (152 mm) using fly ash as mineral admixture in addition to a reasonable content of high-range water-reducing agent (superplasticizer).

3.6. Using the data in Ex. 3.2, design a high-strength high-performance concrete for a mixture design strength $f' = 15,000$ psi (104 MPa) with a slump of 6 in. (152 mm) using silica fume as mineral admixture in addition to a reasonable content of high-range water-reducing agent.

3.7. Estimate the strength of a trial mixture, $f_{tu} = 7500$ psi (52 MPa); $s$ (using 20 consecutive tests) = 550 psi (4.3 MPa).

3.8. Solve problem 3.7 for a condition where no prior test results are available.
4.1 INTRODUCTION

Concrete is strong in compression but weak in tension. Therefore, reinforcement is needed to resist the tensile stresses resulting from the induced loads. Additional reinforcement is occasionally used to reinforce the compression zone of concrete beam sections. Such steel is necessary for heavy loads in order to reduce long-term deflections.

Whereas Chapters 2 and 3 dealt with plain concrete and its constituent materials, this chapter discusses composite reinforced concrete, which can withstand high tensile as well as compressive forces. A discussion of the types of reinforcing material, the variety of structural systems, and their components is presented.

Additionally, concrete structures have to perform adequately under service-load conditions in addition to having the necessary reserve strength to resist ultimate load. The subjects of reliability, safety, and load factors are also presented.

Photo 4.1 University of Illinois Assembly Hall at Urbana. (Courtesy of Azomm and Whitney.)
Steel reinforcement for concrete consists of bars, wires, and welded wire fabric, all of which are manufactured in accordance with ASTM standards. The most important properties of reinforcing steel are:

1. Young's modulus, $E$,
2. Yield strength, $f_y$,
3. Ultimate strength, $f_u$,
4. Steel grade designation
5. Size or diameter of the bar or wire

To increase the bond between concrete and steel, projections called deformations are rolled on the bar surface as shown in Fig. 4.1, in accordance with ASTM specifications. These deformations must satisfy ASTM Specification A615-10 to be accepted as deformed bars. The deformed wire has indentations pressed into the wire or bar to serve as deformations. Except for wire used in spiral reinforcement in columns, only deformed bars, deformed wires, or wire fabric made from smooth or deformed wire may be used in reinforced concrete under approved practice.

Figure 4.2 shows typical stress-strain curves for grade 40, 60, and 80 steel. They have corresponding yield strengths of 40,000, 60,000, and 80,000 psi (276, 414, and 552 N/mm², respectively) and generally have well-defined yield points. For steels that lack a well-defined yield point, the yield-strength value is taken as the strength corresponding to a unit strain of 0.005 for grades 40 and 60 steels and 0.0035 for grade 80 steel. The ultimate tensile strengths corresponding to the 40, 60, and 80 grade steels are 70,000, 90,000, and 100,000 psi (483, 621, and 689 N/mm²), and some steel types are given in Table 4.1.

The percent elongation at fracture, which varies with the grade, bar diameter, and manufacturing source, ranges from 4.5 to 12% over a 5-in. (203.2-mm) gage length.

For most steels, the behavior is assumed to be elastoplastic, and Young's modulus is taken as 29 x 10⁶ psi (200 x 10⁶ MPa). Table 4.1 presents the reinforcement-grade...
strengths, and Table 4.2 (a) and (b) presents geometrical properties of the various sizes of bars.

Welded wire fabric is increasingly used for slabs because of the ease of placing the fabric sheets, control of reinforcement spacing, and better bond. The fabric reinforcement is made of smooth or deformed wires that run in perpendicular directions and are welded together at intersections. Table 4.3 presents geometrical properties of some standard wire reinforcement.

4.3 BAR SPACING AND CONCRETE COVER FOR STEEL REINFORCEMENT

It is necessary to guard against honeycombing and ensure that the wet concrete mix passes through the reinforcing steel without separation. Since the graded aggregate size in structural concrete often contains 1-in. (19-mm diameter) coarse aggregate, minimum

<table>
<thead>
<tr>
<th>1982 Standard Type</th>
<th>Minimum Yield Point or Yield Strength, $f_y$ (psi)</th>
<th>Ultimate Strength, $f_u$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bilex steel (ASTM)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grade 40</td>
<td>40,000</td>
<td>70,000</td>
</tr>
<tr>
<td>Grade 60</td>
<td>60,000</td>
<td>90,000</td>
</tr>
<tr>
<td>Axile steel (ASTM)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Grade 40</td>
<td>40,000</td>
<td>70,000</td>
</tr>
<tr>
<td>Grade 60</td>
<td>60,000</td>
<td>90,000</td>
</tr>
<tr>
<td>Low-alloy steel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(A700) Grade 60</td>
<td>60,000</td>
<td>82,000</td>
</tr>
<tr>
<td>Deformed wire</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reinforced</td>
<td>75,000</td>
<td>85,000</td>
</tr>
<tr>
<td>Fabric</td>
<td>70,000</td>
<td>90,000</td>
</tr>
<tr>
<td>Smooth wire</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reinforced</td>
<td>70,000</td>
<td>80,000</td>
</tr>
<tr>
<td>Fabric</td>
<td>65,000, 56,000</td>
<td>75,000, 70,000</td>
</tr>
</tbody>
</table>
4.3 Bar Spacing and Concrete Cover for Steel Reinforcement

Table 4.2(a) Weight, Area, and Perimeter of Individual Bars

<table>
<thead>
<tr>
<th>Bar designation number</th>
<th>Weight per foot (lb)</th>
<th>Diameter, $d_b$ (in. [mm])</th>
<th>Cross-sectional area, $A_s$ (in.$^2$)</th>
<th>Perimeter (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5.376</td>
<td>0.375 (9)</td>
<td>0.11</td>
<td>1.178</td>
</tr>
<tr>
<td>4</td>
<td>0.688</td>
<td>0.500 (13)</td>
<td>0.20</td>
<td>1.571</td>
</tr>
<tr>
<td>5</td>
<td>1.045</td>
<td>0.625 (16)</td>
<td>0.31</td>
<td>1.963</td>
</tr>
<tr>
<td>6</td>
<td>1.502</td>
<td>0.750 (19)</td>
<td>0.44</td>
<td>2.356</td>
</tr>
<tr>
<td>7</td>
<td>2.044</td>
<td>0.875 (22)</td>
<td>0.60</td>
<td>2.749</td>
</tr>
<tr>
<td>8</td>
<td>2.670</td>
<td>1.000 (25)</td>
<td>0.79</td>
<td>3.142</td>
</tr>
<tr>
<td>9</td>
<td>3.400</td>
<td>1.128 (28)</td>
<td>1.00</td>
<td>3.544</td>
</tr>
<tr>
<td>10</td>
<td>4.303</td>
<td>1.270 (31)</td>
<td>1.27</td>
<td>3.990</td>
</tr>
<tr>
<td>11</td>
<td>5.313</td>
<td>1.410 (33)</td>
<td>1.56</td>
<td>4.450</td>
</tr>
<tr>
<td>14</td>
<td>7.15</td>
<td>1.694 (43)</td>
<td>2.25</td>
<td>5.32</td>
</tr>
<tr>
<td>18</td>
<td>12.61</td>
<td>2.257 (56)</td>
<td>4.00</td>
<td>7.09</td>
</tr>
</tbody>
</table>

allowable bar spacing and minimum required cover are needed. Additionally, to protect the reinforcement from corrosion, a clear cover of at least 2.5 in. (50 mm) is required. Some of the major requirements of ACI Code 318 are:

1. Clear distance between parallel bars in a layer must not be less than the bar diameter, $d_b$, or 1 in. (25 mm).
2. Clear distance between longitudinal bars in columns must not be less than $1.5d_b$, or 1.5 in. (38.1 mm).
3. Minimum clear cover in cast-in-place concrete beams and columns should not be less than 1.5 in. (38.1 mm) when there is no exposure to weather or contact with the ground; this same cover requirement also applies to stirrups, ties, and spirals.

Table 4.2(b) ASTM Standard Metric Reinforcing Bars

<table>
<thead>
<tr>
<th>Bar size designation (No.)</th>
<th>Weight (kg/m)</th>
<th>Diameter (mm)</th>
<th>Area (mm$^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 M</td>
<td>0.783</td>
<td>11.3</td>
<td>100</td>
</tr>
<tr>
<td>15 M</td>
<td>1.570</td>
<td>16.0</td>
<td>200</td>
</tr>
<tr>
<td>20 M</td>
<td>2.339</td>
<td>19.5</td>
<td>300</td>
</tr>
<tr>
<td>25 M</td>
<td>3.925</td>
<td>23.3</td>
<td>300</td>
</tr>
<tr>
<td>30 M</td>
<td>5.493</td>
<td>29.9</td>
<td>300</td>
</tr>
<tr>
<td>35 M</td>
<td>7.830</td>
<td>35.7</td>
<td>1000</td>
</tr>
<tr>
<td>45 M</td>
<td>11.775</td>
<td>43.7</td>
<td>1500</td>
</tr>
<tr>
<td>55 M</td>
<td>19.625</td>
<td>56.4</td>
<td>2500</td>
</tr>
</tbody>
</table>

ASTM A615M Grade 300 is limited to size No. 5, 19 M through No. 20 M, otherwise grades 40 or 50 MPa for all the sizes. Check availability with local suppliers for No. 45 M and 55 M.
<table>
<thead>
<tr>
<th>Smooth</th>
<th>Deformed</th>
<th>U.S. Customary</th>
<th>Nominal Diameter (in.)</th>
<th>Nominal Area (in.²)</th>
<th>Nominal Weight (lb/1000 ft)</th>
<th>Center-to-center spacing (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>W31</td>
<td>D31</td>
<td>0.628</td>
<td>0.310</td>
<td>1.054</td>
<td>1.86</td>
<td>1.24</td>
</tr>
<tr>
<td>W30</td>
<td>D30</td>
<td>0.618</td>
<td>0.300</td>
<td>1.020</td>
<td>1.83</td>
<td>1.20</td>
</tr>
<tr>
<td>W28</td>
<td>D28</td>
<td>0.597</td>
<td>0.280</td>
<td>0.952</td>
<td>1.56</td>
<td>1.12</td>
</tr>
<tr>
<td>W26</td>
<td>D26</td>
<td>0.576</td>
<td>0.260</td>
<td>0.934</td>
<td>1.54</td>
<td>1.10</td>
</tr>
<tr>
<td>W24</td>
<td>D24</td>
<td>0.553</td>
<td>0.240</td>
<td>0.816</td>
<td>1.44</td>
<td>0.96</td>
</tr>
<tr>
<td>W22</td>
<td>D22</td>
<td>0.529</td>
<td>0.220</td>
<td>0.748</td>
<td>1.32</td>
<td>0.88</td>
</tr>
<tr>
<td>W20</td>
<td>D20</td>
<td>0.504</td>
<td>0.200</td>
<td>0.680</td>
<td>1.20</td>
<td>0.80</td>
</tr>
<tr>
<td>W18</td>
<td>D18</td>
<td>0.478</td>
<td>0.180</td>
<td>0.612</td>
<td>1.08</td>
<td>0.72</td>
</tr>
<tr>
<td>W16</td>
<td>D16</td>
<td>0.451</td>
<td>0.160</td>
<td>0.544</td>
<td>0.96</td>
<td>0.64</td>
</tr>
<tr>
<td>W14</td>
<td>D14</td>
<td>0.422</td>
<td>0.140</td>
<td>0.476</td>
<td>0.84</td>
<td>0.56</td>
</tr>
<tr>
<td>W12</td>
<td>D12</td>
<td>0.390</td>
<td>0.120</td>
<td>0.408</td>
<td>0.72</td>
<td>0.48</td>
</tr>
<tr>
<td>W11</td>
<td>D11</td>
<td>0.374</td>
<td>0.110</td>
<td>0.374</td>
<td>0.66</td>
<td>0.44</td>
</tr>
<tr>
<td>W10.5</td>
<td>D10.5</td>
<td>0.366</td>
<td>0.105</td>
<td>0.357</td>
<td>0.63</td>
<td>0.42</td>
</tr>
<tr>
<td>W10</td>
<td>D10</td>
<td>0.356</td>
<td>0.100</td>
<td>0.330</td>
<td>0.60</td>
<td>0.40</td>
</tr>
<tr>
<td>W9.5</td>
<td>D9.5</td>
<td>0.348</td>
<td>0.095</td>
<td>0.323</td>
<td>0.57</td>
<td>0.38</td>
</tr>
<tr>
<td>W9</td>
<td>D9</td>
<td>0.338</td>
<td>0.090</td>
<td>0.306</td>
<td>0.54</td>
<td>0.36</td>
</tr>
<tr>
<td>W8.5</td>
<td>D8.5</td>
<td>0.329</td>
<td>0.085</td>
<td>0.289</td>
<td>0.51</td>
<td>0.34</td>
</tr>
<tr>
<td>W8</td>
<td>D8</td>
<td>0.319</td>
<td>0.080</td>
<td>0.272</td>
<td>0.48</td>
<td>0.32</td>
</tr>
<tr>
<td>W7.5</td>
<td>D7.5</td>
<td>0.309</td>
<td>0.075</td>
<td>0.255</td>
<td>0.45</td>
<td>0.30</td>
</tr>
<tr>
<td>W7</td>
<td>D7</td>
<td>0.296</td>
<td>0.070</td>
<td>0.238</td>
<td>0.42</td>
<td>0.28</td>
</tr>
<tr>
<td>W6.5</td>
<td>D6.5</td>
<td>0.288</td>
<td>0.065</td>
<td>0.221</td>
<td>0.39</td>
<td>0.26</td>
</tr>
<tr>
<td>W6</td>
<td>D6</td>
<td>0.276</td>
<td>0.060</td>
<td>0.204</td>
<td>0.36</td>
<td>0.24</td>
</tr>
<tr>
<td>W5.5</td>
<td>D5.5</td>
<td>0.264</td>
<td>0.055</td>
<td>0.187</td>
<td>0.33</td>
<td>0.22</td>
</tr>
<tr>
<td>W5</td>
<td>D5</td>
<td>0.252</td>
<td>0.050</td>
<td>0.170</td>
<td>0.30</td>
<td>0.20</td>
</tr>
<tr>
<td>W4.5</td>
<td>D4.5</td>
<td>0.240</td>
<td>0.045</td>
<td>0.153</td>
<td>0.27</td>
<td>0.18</td>
</tr>
<tr>
<td>W4</td>
<td>D4</td>
<td>0.225</td>
<td>0.040</td>
<td>0.136</td>
<td>0.24</td>
<td>0.16</td>
</tr>
<tr>
<td>W3.5</td>
<td>D3.5</td>
<td>0.211</td>
<td>0.035</td>
<td>0.119</td>
<td>0.21</td>
<td>0.14</td>
</tr>
<tr>
<td>W3</td>
<td>D3</td>
<td>0.195</td>
<td>0.030</td>
<td>0.102</td>
<td>0.18</td>
<td>0.12</td>
</tr>
<tr>
<td>W2.9</td>
<td>D2.9</td>
<td>0.192</td>
<td>0.029</td>
<td>0.098</td>
<td>0.174</td>
<td>0.116</td>
</tr>
<tr>
<td>W2.5</td>
<td>D2.5</td>
<td>0.178</td>
<td>0.025</td>
<td>0.085</td>
<td>0.15</td>
<td>0.10</td>
</tr>
<tr>
<td>W2</td>
<td>D2</td>
<td>0.159</td>
<td>0.020</td>
<td>0.068</td>
<td>0.12</td>
<td>0.08</td>
</tr>
<tr>
<td>W1.4</td>
<td>D1.4</td>
<td>0.135</td>
<td>0.014</td>
<td>0.049</td>
<td>0.084</td>
<td>0.056</td>
</tr>
</tbody>
</table>
4.4 Concrete Structural Systems

In the case of slabs, plates, shells, and folded plates, where concrete is not exposed to a severe environment and where the reinforcement size does not exceed a No. 11 bar diameter (85.8 mm), the clear cover should not be less than 1 in. (19 mm). Detailed requirements as to thickness of cover for various conditions can be found in various codes of practice, such as the Underwriters' National Building Code and the ACI 318 Code.

4.4 CONCRETE STRUCTURAL SYSTEMS

Every structure is proportioned as to both architecture and engineering to serve a particular function. Form and function go hand in hand, and the best structural system is the one that fulfills most of the needs of the user while being serviceable, attractive, and economically cost efficient. Although most structures are designed for a life span of 50 years, the durability performance record indicates that properly proportioned concrete structures have generally had longer useful lives.

Numerous concrete landmarks can be cited where major credit is due to the art and science of structural design applied with ingenuity, logic, and imagination. Such concrete structural systems as the TWA Terminal, New York; the Newark Terminal, New Jersey; Symphony Hall, Melbourne, Australia; Chicago's Marina Towers and Water Tower Place; the Dallas Super Dome; Two Union Square Towers, Seattle; Trump Tower, New York; the Borgata Tower and Casino complex in Atlantic City; and many others are a testimony to the marriage of form and function with superior engineering judgment. Photographs of several such landmarks appear throughout this book.

Such concrete systems are composed of a variety of concrete structural elements that, when synthesized, produce a total system. The components can be broadly classified into (1) floor slabs, (2) beams, (3) columns, (4) walls, and (5) foundations.

4.4.1 Floor Slabs

Floor slabs are the main horizontal elements that transmit the moving live loads as well as the stationary dead loads to the vertical framing supports of a structure. They can be slabs on beams, as in Fig. 4.3, or waffle slabs, slabs without beams (flat plates) resting directly on columns, or composite slabs on joists. They can be proportioned such that they act in one direction (one-way slabs) or proportioned so that they act in two perpendicular directions (two-way slabs and flat plates). A detailed discussion of the analysis and design of such floor systems is given in subsequent chapters.

4.4.2 Beams

Beams are the structural elements that transmit the tributary loads from floor slabs to vertical supporting columns. They are normally cast monolithically with the slabs and are structurally reinforced on one face, the lower tension side, or both the top and bottom faces. As they are cast monolithically with the slab, they form a T-beam section for interior beams or an L beam at the building exterior, as seen in Fig. 4.3. The plan dimensions of a slab panel determine whether the floor slab behaves essentially as a one-way or two-way slab.

4.4.3 Columns

The vertical elements support the structural floor system. They are compression members subjected in most cases to both bending and axial load and are of major importance in the safety considerations of any structure. If a structural system is also composed of horizontal compression members, such members would be considered as beam-columns.
4.4.4 Walls

Walls are the vertical members for building frames. They are not usually or necessarily made of concrete but of any material that esthetically fulfills the form and functional needs of the structural system. Additionally, structural concrete walls are often necessary as foundation walls, stairwell walls, and shear walls that resist horizontal wind loads and earthquake-induced loads.

4.4.5 Foundations

Foundations are the structural concrete elements that transmit the weight of the superstructure to the supporting soil. They could be in many forms, the simplest being the isolated footing shown in Fig. 4.3. It can be viewed as an inverted slab transmitting a distributed load from the soil to the column. Other forms of foundations are piles driven to rock, combined footings supporting more than one column, mat foundations, and rafts, which are basically inverted slab and beam construction.

The results of the analysis and design process of a structure have to be presented in concise and standardized form, which the constructor can use for building the entire system. Hence knowledge and easy reading of working drawings is important. A typical layout drawing of a multistory parking garage structure is shown in Fig. 4.4. The ACI Manual of Detailing gives an adequate coverage of typical working drawings for various structural systems and of the layout and detailing of reinforcement.

4.5 RELIABILITY AND STRUCTURAL SAFETY OF CONCRETE COMPONENTS

Three developments in recent decades have had a major influence on present and future design procedures. They are the vast increase in experimental and analytical evolution of concrete elements, the probabilistic approach to the interpretation of behavior, and the digital computational tools available for rapid analysis of the safety and reliability of systems.
Until recently, most safety factors in design have had an empirical background based on local experience over an extended period of time. As additional experience is accumulated and more knowledge is gained from failures as well as familiarity with the properties of concrete, factors of safety are adjusted and in most cases lowered by the codifying bodies.

A. L. L. Baker in 1956 (Ref. 4.6) proposed a simplified method of safety factor determination, as shown in Table 4.4, based on probabilistic evaluation. This method expects the design engineer to make critical choices regarding the magnitudes of safety margins in a design. The method takes into consideration that different weights should be assigned to the various factors affecting a design. The weighted failure effects $W_i$ for the various factors of workmanship, loading conditions, results of failure, and resistance capacity are tabulated in Table 4.4.

The safety factor against failure is

$$ SF = 1.0 + \frac{\sum W_i}{10} \quad (4.1) $$

where the maximum total weighted value $\sum W_i$ of all parameters affecting performance equals 10. In other words, for the worst combination of conditions affecting structural performance, the safety factor $SF = 2.0$.

This method assumes adequate information on prior performance data similar to a design in progress. Such data in many instances are not readily available for determining safe weighted values $W_i$ in Eq. 4.1. Additionally, if the weighted factors are numerous, a probabilistic determination of them is more difficult to codify. Hence an undue value-judgment burden is probably placed on the design engineer if the full economic benefit of the approach is to be achieved.

Another method with a smaller number of probabilistic parameters deals primarily with loads and resistances. Its approach for both steel and concrete structures is generally similar; both the load and resistance factor design methods (LRFD) and first-order second-moment method (FOSM) propose general reliability procedures for evaluating probability-based factored load design criteria, as in Refs. 4.7, 4.8, 4.10, and 4.14. They

<table>
<thead>
<tr>
<th>Weighted Failure Effect</th>
<th>Maximum $W_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Results of failure: 1.0 to 4.0</td>
<td></td>
</tr>
<tr>
<td>Serious, either human or economic</td>
<td>4.0</td>
</tr>
<tr>
<td>Less serious, only the exposure of non-damnable material</td>
<td>1.0</td>
</tr>
<tr>
<td>2. Workmanship: 0.5 to 2.0</td>
<td></td>
</tr>
<tr>
<td>Cast in place</td>
<td>0.5</td>
</tr>
<tr>
<td>Precast “factory manufactured”</td>
<td>2.0</td>
</tr>
<tr>
<td>3. Load conditions: 1.0 to 2.0</td>
<td></td>
</tr>
<tr>
<td>(high for simple spans and overload possibilities; low for load combinations such as live loads and wind)</td>
<td>2.0</td>
</tr>
<tr>
<td>4. Importance of number in structure: (beams may use lower value than columns)</td>
<td>0.5</td>
</tr>
<tr>
<td>5. Warming of failure</td>
<td>1.0</td>
</tr>
<tr>
<td>6. Depreciation of strength</td>
<td>0.5</td>
</tr>
</tbody>
</table>

$$ SF = 1.0 + \frac{\sum W_i}{10} $$

$$ \sum W_i = 10.0 $$
are intended for use in proportioning structural members on the basis of load types such that the resisting strength levels are greater than the factored load or moment distributions. As these approaches are basically load oriented, they reduce the number of individual variables that have to be considered, such as those listed in Table 4.4.

Assume that $\delta_i$ represents the resistance factors of a concrete element and that $\gamma_i$ represents the load factors for the various types of load. If $R_n$ is the nominal resistance of the concrete element and $W_i$ represents the load effect of various types of superimposed load,

$$\delta_i R_n \geq \gamma_i W_i$$  \hspace{1cm} (4.2)

where $i$ represents the various types of load, such as dead, live, wind, earthquake, or time-dependent effects.

Figure 4.5a and b shows a plot of the separate frequency distributions of the actual load $W$ and the resistance $R$ with mean values $\bar{R}$ and $\bar{W}$. Figure 4.5c gives the two distributions superimposed and intersecting at point $C$.

Figure 4.5 Frequency distribution of loads versus resistance.
It can be recognized that safety and reliable integrity of the structure can be expected to exist if the load effect $W$ falls at a point to the left of intersection $C$ on the $W$ curve, and to the right of intersection $C$ on the resistance curve $R$. Failure, on the other hand, would be expected to occur if the load effect or the resistance falls within the shaded area in Fig. 4.5c. If $\beta$ is a safety index, then

$$\beta = \frac{R - W}{\sqrt{\sigma_R^2 + \sigma_W^2}}$$

(4.3)

where $\sigma_R$ and $\sigma_W$ are the standard deviations of the resistance and the load, respectively.

A plot of the safety index $\beta$ for a hypothetical structural system is shown in Fig. 4.6 against the probability of failure of the system. One can observe that such a probability is reduced as the difference between the mean resistance $R$ and load effect $W$ is increased, or the variability of resistance and load effect as measured by their standard deviations $\sigma_R$ and $\sigma_W$ is decreased, thereby reducing the shaded area under intersection $C$ in Fig. 4.5c.

The extent of increasing the $R - W$ difference or decreasing the degree of scatter of $\sigma_R$ or $\sigma_W$ is naturally dictated by economic considerations. It is economically unreasonable to design a structure for zero failure, particularly since types of risk other than load are an accepted matter, such as the risks of severe earthquake, hurricane, volcanic eruption, or fire. Safety factors and corresponding load factors would thus have to disregard those types or levels of load, stress, and overstress whose probability of occurrence is very low. In spite of this, it is still possible to achieve reliable safety conditions by choosing such a safety index value $\beta$ through a proper choice of $R$ and $W$, values using the appropriate resistance factors $A$ and load factors $V$ in Eq. 4.2. A safety index $\beta$ having the value 1.75 to 3.2 for concrete structures is suggested in Ref. 4.8, where the lower value accounts for load contributions from wind and earthquake.

![Figure 4.6 Probability of failure versus safety index $\beta$.]
4.6 ACI Load Factors and Safety Margins

If the factored external load is expressed as \( U_e \), then \( \sum y_i W_i = U_e \) for the different loading combinations.

In cases where other load combinations, such as snow or lateral pressure are not present, a typical \( U_e \) value recommended in the ASCE-7 Standard (Ref. 4.14) and IBC 2000 (Ref. 4.17) for maximum \( U_e \) to be used in Equation 4.2, that is, \( \phi, R_s \geq \gamma, \gamma_i W_i \geq U_{min} \),

\[
U = \phi, R_s \text{ Maximum } [1.2D + 1.5L]
\]  

(4.4)

As more substantive records of performance are compiled with time, the details of the foregoing approach to reliability, safety, and reserve strength evaluation of structural components can be more universally accepted and extended beyond treatment of the component elements to the treatment of the total structural system, such as described in Table 4.4.

4.6 ACI LOAD FACTORS AND SAFETY MARGINS

4.6.1 General Principles

The general concept of safety and reliability of performance presented in the preceding section is inherent in a more simplified but less accurate fashion in the ACI 318 Code. The \( \gamma \) load factors and the \( \beta \) strength reduction factors give an overall safety factor based on load types such that

\[
S_F = \frac{\gamma_i D + \gamma_i L}{D_L} \times \frac{1}{\phi}
\]  

(4.5)

where \( \phi \) is the strength reduction factor and \( \gamma_i \) and \( \gamma_i \) are the respective load factors for the dead load \( D \) and the live load \( L \). Basically, a single common factor is used for dead load and another for live load. Variation in resistance capacity is considered in the \( \phi \) reduction factor. Hence, the method is a simplified empirical approach to the safety and reliability of structural performance that is not economically efficient for every case and not fully adequate in other instances, such as combinations of dead and wind loads.

The ACI factors are termed load factors, because they restrict the estimation of reserve strength to the loads only as compared to the other parameters listed in Table 4.4. The estimated service or working loads are modified by the coefficients, such as a coefficient of 1.2 for dead loads and 1.5 for live load, with the basic combination of vertical loads combining dead load plus live load. The dead load, which constitutes the weight of the structure and other relatively permanent features, can be estimated more accurately than the live load. The live load is estimated using the weight of nonpermanent loads, such as people and furniture. The transient nature of live loads makes them difficult to estimate more accurately. Therefore, a higher load factor is normally used for live loads than for dead loads.

It should also be noted that the philosophy used for combining the various load components for earthquake loading is essentially the same as that used for wind loading.

4.6.2 ACI Load Factors \( U \)

The ACI 318 Building Code for concrete structures is an international code. As such, it has to conform to the International Building Code, IBC 2000, IBC 2003 (Ref. 4.17) and be consistent with the ASCE-7 Standard on Minimum Design Loads for Buildings and Other Structures (Ref. 4.14). These two standards contain the same probabilistic values
for the expected safety resistance factors $R_0$, where $\phi$ is a strength reduction factor, depending on the type of stress being considered in the design, namely flexure or shear or compression, etc.

Thus, the new ACI design loads $U$ (factored loads) have to be at least equal to the values obtained from Equations 4.6(a) through 4.6(g). The effect of one or more loads not acting simultaneously has to be investigated.

\[
\begin{align*}
U &= 1.4(D + F) \\
U &= 1.2(D + F + T) + 1.6(L_0 + H) + 0.5(L_0 or S or R) \\
U &= 1.2D + 1.6L_0 or S or R + (1.0L or 0.8W) \\
U &= 1.2D + 1.6W + 0.5L_0 + 1.0(L_0 or S or R) \\
U &= 1.2D + 1.6E + 1.6L_0 + 0.25 \\
U &= 0.9D + 1.6W + 1.6H \\
U &= 0.9D + 1.6E + 1.6H
\end{align*}
\]

where

- $D$ = dead load; $E$ = earthquake load; $F$ = lateral fluid pressure load & maximum height.
- $H$ = load due to the weight and lateral pressure of soil and water in soil.
- $L$ = live load; $L_0$ = roof load; $K$ = rain load; $S$ = snow load;
- $T$ = self-straining force such as creep, shrinkage, and temperature effects;
- $W$ = wind load.

**Exceptions to the values in these expressions**

(a) The load factor on $L$ in Eq. 4.6(c) to 4.6(c) is allowed to be reduced to 0.5 except for garages, areas occupied as places of public assembly, and all areas where the live load $L$ is greater than 100 lb/ft$^2$.

(b) Where wind load $W$ has not been reduced by a directionality factor, the code permits to use 1.3W in place of 1.6W in Eq. 4.6(d) and 4.6(f).

(c) Where earthquake load $E$ is based on service-level seismic forces, 1.4E shall be used in place of 1.0E in Eq. 4.6(d) and 4.6(g).

(d) The load factor on $H$ is to be set equal to zero in Eq. 4.6(f) and Eq. 4.6(g) if the structural action due to $H$ counteracts that due to $W$ or $E$. Where lateral earth pressure provides resistance to structural actions from other forces, it should not be included in $H$ but shall be included in the design resistance.

Due regard has to be given to sign in determining $U$ for combinations of loadings, as one type of loading may produce effects of opposite sense to that produced by another type. The load combinations with 0.9D are specifically included for the case where a higher dead load reduces the effects of other loads. Consideration has also to be given to various combinations of loading to determine the most critical design condition, particularly when strength is dependent on more than one load effect, such as strength from combined flexure and axial load or shear strength in members with axial load. In cases where special circumstances require greater reliance on the strength of particular members than encountered in usual practice, the ACI Code allows some reduction in the stipulated strength reduction factors $\phi$, or an increase in the stipulated load factors $U$. 


4.6.3 Reduction in Live Loads

For large areas, it is reasonable to assume that the full intensity of live load does not cover the entire floor area. Hence, members having an influence area of 400 ft² (37.2 m²) or more can be designed for a reduced live load from the following equation:

\[ L = L_o \left(0.25 + \frac{15}{\sqrt{A}}\right) \]  \hspace{1cm} (4.7)

where

- \( L \) = Reduced design live load per square foot of area supported by the member,
- \( L_o \) = Unreduced design live load per square foot of area.
- \( A \) = Influence area. For other than cantilevered construction, \( A \) is 4 times the tributary area for a column, 2 times tributary area for beams, or equal area for a two-way slab (Ref. 4.17).

In SI units, Equation 4.7 becomes

\[ L = L_o \left(0.25 + \frac{4.57}{\sqrt{A}}\right) \]  \hspace{1cm} (4.8)

where \( L \), \( L_o \), and \( A \) are in square meters of area.

The reduced design live load cannot be less than 50% of the unit live load \( L_o \) for member supporting one floor or less than 40% of the unit live load \( L_o \) for members supporting two or more floors. For live loads of 100 lb/ft² (4.79 kN/m²) or less, no reduction can be made for areas used as places of public assembly, except that in the case of garages for passenger cars a reduction of up to 20% can be made. Live loads in all other cases not stipulated by the code cannot be reduced except as accepted by the jurisdictional authority.

4.7 DESIGN STRENGTH VERSUS NOMINAL STRENGTH: STRENGTH REDUCTION FACTOR \( \phi \)

The strength of a particular structural unit calculated using the current established procedures is termed nominal strength. For example, in the case of a beam, the resisting moment capacity of the section calculated using the equations of equilibrium and the properties of concrete and steel is called the nominal resisting moment capacity \( M_n \) of the section. This nominal strength is reduced using a strength reduction factor \( \phi \) to account for inaccuracies in construction, such as in the dimensions or position of reinforcement or variations in properties. The reduced strength of the member is defined as the design strength of the member.

For a beam, the design moment strength \( \phi M_n \) should be at least equal to or slightly greater than the external factored moment \( M_f \) for the worst condition of factored load \( U \).

The factor \( \phi \) varies for the different types of behavior and for the different types of structural elements. For beams in flexure, for instance, the reduction factor is 0.9.

For tied columns that carry dominant compressive loads, the \( \phi \) factor equals 0.65.

The smaller strength reduction factor used for columns is due to the structural importance of the columns in supporting the total structure compared to other members and to guard against progressive collapse and brittle failure with no advance warning of collapse. Beams, on the other hand, are designed to undergo excessive deflections before failure. Hence the inherent capability of the beam for advanced warning of failure permits the use of a higher strength reduction factor or resistance factor.
Table 4.5 Resistance or Strength Reduction Factor $\phi$

<table>
<thead>
<tr>
<th>Structural Element</th>
<th>Factor $\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beam or slab: bending or flexure</td>
<td>0.9</td>
</tr>
<tr>
<td>Columns with ties</td>
<td>0.65</td>
</tr>
<tr>
<td>Columns with spirals</td>
<td>0.70</td>
</tr>
<tr>
<td>Columns carrying very small axial loads (refer to Chapter 9 for more details)</td>
<td>0.65-0.9, 0.70-0.9</td>
</tr>
<tr>
<td>Beam: shear and torsion</td>
<td>0.75</td>
</tr>
<tr>
<td>Bearing except for strut-and-tie</td>
<td>0.65</td>
</tr>
<tr>
<td>Bearing areas in strut-and-tie</td>
<td>0.75</td>
</tr>
</tbody>
</table>

Table 4.5 summarizes the resistance factors $\phi$ for various structural elements as given in the ACI Code. A comparison of these values to those given in Ref. 4.8 indicates that the $\phi$ values in this table, as well as the load factors of Eq. 4.8, are in some cases more conservative than they should be. In cases of earthquakes, wind, and shear forces, the probability of load magnitude and reliability of performance is subject to higher randomness and hence a higher coefficient of variation than the other types of loading.

4.8 QUALITY CONTROL AND QUALITY ASSURANCE

Quality control assures the reliability of performance of the designed system in accordance with assumed and expected reserve strengths in the design. To exercise "quality control" and achieve "quality assurance" encompasses monitoring the roles and performance of all participants: the client or owner, the designer, the concrete producer, the laboratory tester, the constructor, and the user.

Most of the different phases of the total process of construction are affected by complex standards and regulations of the various codifying agencies. Also, in contrast to mechanized production such as in the case of machines, building construction does not follow the moving-belt or chain-production process, where the products move but the workers are relatively stationary; the contrary is true. Consequently, complications are more profound in constructed systems such as those made with concrete. This is due partially to the fact that concrete is a nonhomogeneous material with properties dependent on many variables, requiring extra effort in quality control due to the greater effect of the human factor on the quality of the finished product.

![Quality Assurance System](image-url)

*Figure 4.7: Components of a quality assurance system. (From Ref. 4.12)*
The reliability of the performance of personnel involved in the various stages of creating a concrete structural system from conception through design, construction, and use depends on knowledge, training, and communication at all levels. A smooth flow of correct information among all participants and a shared systematic understanding of the developing problems lead to increased motivation toward improved solutions and hence improved quality control and a resulting high level of quality assurance. In summary, a quality assurance system needs to be provided based on the exercise of quality control at the various phases and interacting parameters of a total system, as shown in Fig. 4.7.

4.8.1 The User

Construction of a designed system is governed basically by five primary tasks: planning, design, materials selection, construction, and use (including maintenance). Figure 4.8 presents schematically the sequence of these enumerated tasks and the respective divisions of responsibility. As seen from this diagram, the process starts with the user, since the principal aim of a project is to satisfy the user’s needs, and it culminates with the user as the primary beneficiary of the final product.

Quality assurance is necessary to satisfy the user’s needs and rights. It ensures that the activities influencing the final quality of a concrete structure are:

1. Based on clearly defined fundamental requirements that satisfy the operational, environmental, and boundary conditions set for the project at the outset.
2. Properly presented in accurate, well-dimensioned engineering working drawings based on optimal design procedures.
3. Correctly and efficiently carried out by competent personnel in accordance with predetermined plans and working drawings well supervised during the design stage.
4. Systematically executed in accordance with detailed specifications that conform to the applicable codes and local regulations.

To achieve these aims, the expertise denoted by the other components of the polyhedral in Fig. 4.8 are called upon, starting with the planner and designer and culminating with the constructor.

Photo 4.2 Flexural behavior of a reinforced concrete beam. (Test by Nusay et al.)
4.3.2 Planning

In order to plan the successful execution of a proposed construction system, all main and subactivities must be clearly defined. This is accomplished through dividing the total project into a network plan of separately defined activities, relating each activity with time, analyzing the input control and the resulting output control, and expressing those conditions in the form of a checklist. In such a manner, the successful decision-making process concerning performance requirements becomes easier to accomplish. Such a process usually entails decisions on what function needs to be accomplished in a construction project, where and when that function will be executed, how the system will be constructed, and what the user will be. Correct determination of these factors leads to a decision as to the level of quality control needed and the degree of quality assurance that is expected to result.

4.3.3 Design

Quality control in design aims at verifying that the designed system has the safety, serviceability, and durability required for the use to which the system is intended as required by the applicable codes, and that such a design is correctly presented in the working drawings and the accompanying specifications. The degree of quality control depends on the type of system to be constructed. The more important the system, the more control that is required.

As a minimum, a design must always be checked by an engineer other than the originating design engineer. Usually, one of three types of verifications is used, depending on the practice of the designing agency: (1) total direct checking, in which all computations are verified; (2) total parallel checking, in which calculations are independently made and the two sets of calculations compared; and (3) partial checking, in which selected parts are checked in both direct and parallel checking.
4.8 Quality Control and Quality Assurance

Quality control of the design calculations can generally be achieved through assuring that:

1. A clear understanding exists of the structural concept that applies to the particular system.
2. There is knowledge and compliance with the relevant fundamental requirements of the design and the environmental, operational, and boundary conditions.
3. Where possible, applicable computational models utilizing available computer programs are used for checks.
4. No discrepancies exist between the different phases or parts of the total design computations.
5. All expected load cases and load combinations as described in Section 4.6 are considered.
6. The appropriate safety factors are adopted and the required reliability levels verified.
7. Verifiable computer programs are used in the design, and the experienced designer is well acquainted with the programming steps and background of the programs, particularly when total computer-aided designs are used.

Since engineering working drawings are the primary link between the design process and the construction process, they should be a major object of design quality assurance. Consequently, the student has to be well acquainted with reading and interpreting working drawings and must be able to produce clear sketches that accurately express the design details of the constructed system to reflect the actual design. Figure 4.4, as well as Figs. 10.11 to 10.21, are intended to give general guidance on the systematic detailing necessary for composing sets of logical engineering working drawings.

Quality control of working drawings normally encompasses a verification of whether the following parameters are included in the project set of drawings:

1. General definition of the structure
2. Consistency among the working drawings
3. Compliance with the site boundary conditions, including soil test boring requirements
4. Listing of the type, grade, quality, and structural strength of the various construction materials involved, such as cement, concrete mixture proportions and strength, reinforcing steel, and formwork
5. No ambiguity and risk of misunderstanding of the details in the drawings
6. Compliance with the design computation results and correct dimensioning
7. Adequate cross-sectional and construction details, as well as explicit dimensional tolerances
8. Sequence of formwork placing and removal

4.8.4 Materials Selection

It has to be emphasized at the outset that the quality of materials such as reinforced concrete is not determined only by compressive or tensile strength tests. As seen from previous sections of this book, many other factors affect the quality of the finished product, such as water/cement ratio, cement content, creep and shrinkage characteristics, freeze and thaw properties, and other durability aspects and conditions.
Two types of quality are involved: (1) required quality, which is the specific contractual requirement for the material, and (2) usage quality, which is the ability of the material to satisfy the needs of the user (Ref. 4.12).

Required quality of the material, such as ready-mix concrete, is assured by production control. Such a process involves:

1. General organization of the production staffing and operation.
2. Production sequence and supply line of the constituent materials, such as stone, fine aggregate, cement, and additives.
3. Internal control, involving frequency of verifications and tests, analysis of test results, the recording and observation methods used, and the procedures applied in dealing with discrepancies and deviations.
4. Use of statistical control charts to classify the specified requirements of quality levels into measurable main variables and nonmeasurable variables, selection of the main variables to be controlled by the control charts, and the preparation of a mean chart and a range chart for each variable selected. The measurable variables are to be controlled using $\bar{X}$ and $R$ charts, while the nonmeasurable variables are controlled by $\bar{X}$ and $C$ charts, denoting the averages of means and variations. An action limit and a warning limit with an upper and a lower boundary need to be specified.
5. Classification of defects as a measure of the nondefinable variables.

Usage quality is determined by the compliance control set in the specifications. These are prescribed by the client for the mixture proportioning and design and the constituent materials of the concrete, as well as the quality of the reinforcement, whether normal or prestressing reinforcement. A projection of the expected quality control record and hence quality assurance of the concrete, for instance, can normally be made if the concrete producer has maintained good statistical quality control of strength test results over a lengthy period of operation. Hence compliance control levels can vary depending on the reliability and confidence in the effectiveness of quality management of the material producer to conform with the specifications of the delivered lots.

4.8.5 Construction

Construction is the execution stage of a project, which can be used to satisfy all design and specification requirements within prescribed time limits at minimum cost. To achieve the desired quality assurance, the construction phase has to be preceded by an elaborate and correct preparation stage, which can be part of the design phase. The preparation or planning phase is very critical since it gives an overall clear view of the various activities.

![Figure 4.9 Operations sequence network for concrete structures.](image-url)
involved and the possible problems that could arise at the various phases of execution. The use of computers in the planning phase is essential today for large projects in order that relevant input as to product quality, output, time scheduling, and costs can be charted and monitored.

The human factor is of major significance at the construction phase. In most instances, the major site activities involve labor-force use and path scheduling of its utilization. An improved information flow system, clear delineation of the chain of command, and reward for superior performance increase motivation and lead to overall improvement in the entire quality assurance system and an optimization of the efficiency/cost ratio of a project. The steps to be carried out to achieve quality assurance, as proposed in Ref. 4.12, are summarized briefly next and are represented graphically in Fig. 4.9.

1. Organizational preparation covering planning, time scheduling, contract details, and definition and assignment of duties
2. On-site preparation, involving access roads and trailers and offices, energy provisions, amenities, and so on
3. Formwork acquisition, type, and preparation for installation
4. Reinforcement procurement, fabrication, and planning
5. Concrete mixture proportioning, laboratory mixture designing, and coordination with design engineers
6. Concrete delivery, placing in the forms, and field slump testing
7. Curing and surface treatment of the hardened concrete
8. Quality control tests of the concrete at 7- and 28-day intervals
9. Removal of formwork, sequential removal of shoring supports, then reshoring

The probability of errors in execution for quality control can normally be expected. The extent and importance of such errors depend on a variety of factors described in previous sections. To apply corrective measures, a logical sequence of steps has to be followed for the detection and analysis of the undesired occurrence. A quantitative analysis of the error impact can often be made provided that the probabilities of occurrence of all the basic events are known to the investigators.

The flowchart in Fig. 4.10 depicts the cause-effect sequence that can be followed in identifying an undesired event in a quality assurance program. γ in the chart is the safety margin factor available in the original design.

In the case of massive projects where huge quantities of concrete and extensive construction activities are involved, special procedures have to be undertaken in order to facilitate rapid flow of information between the participants, even through the Internet. Figure 4.11 gives a flowchart of the sequence of operations for achieving the quality control and quality assurance needed for such massive projects, as well as maintaining an electronic database available on daily basis through the Internet to all participants in such projects.

In summary, the brief discussion in Section 4.8 should provide the reader with an introduction to a continuously evolving topic that has a profound impact on the strength and durability of constructed systems, that is, quality control and quality assurance. It should complement Section 4.5 on reliability and structural safety and Sections 4.6 and 4.7 on load factors and design strengths.
Figure 4.10 Cause–effect flowchart. (From Ref. 4.12.)
Figure 4.11 Flowchart for Quality Control-Quality Assurance Operations Using Internet for Data Control and Transfer (Refs. 4.18, 4.72)

SELECTED REFERENCES


4.11. ACI Committee 318, *Building Code Requirements for Structural Concrete (ACI 318-05)*, and the Commentary (ACI 318R-05), American Concrete Institute, Farmington Hills, MI, 2005, 442 pp.


5.1 INTRODUCTION

Loads acting on a structure, be they live gravity loads or other types, such as horizontal wind loads, earthquakes, or those due to shrinkage and temperature, result in bending and deformation of the constituent structural elements. The bending of the beam elements is the result of the deformational strain caused by the flexural stresses due to the external load.

As the load is increased, the beam sustains additional strain and deflection, leading to development of flexural cracks along the span of the beam. Continuous increases in the level of the load lead to failure of the structural elements when the external load reaches the capacity of the element. Such a load level is termed the limit state of failure in flexure. Consequently, the designer has to design the cross section of the element or beam such that it would not develop excessive cracking at service load levels and have adequate safety and reserve strength to withstand the applied loads or stresses without failure.

Flexural stresses are a result of the external bending moments. They control in most cases the selection of the geometrical dimensions of a reinforced concrete section. The design process through the selection and analysis of a section is usually started by satisfying the flexural (bending) requirements, except for special components such as

Photo 5.1 Priory Church, St. Louis, Missouri. (Courtesy Portland Cement Association.)
footings. Thereafter, other factors, such as shear capacity, deflection, cracking, and bond development of the reinforcement, are analyzed and satisfied.

While the input data for the analysis of sections differ from the data needed for design, every design is essentially an analysis. One assumes the geometrical properties of a section in a design and proceeds to analyze such a section to determine if it can safely carry the required external loads. Hence a good understanding of the fundamental principles in the analysis procedure significantly simplifies the task of designing sections. The basic mechanics of materials principles of equilibrium of internal couples have to be adhered to at all stages of loading.

If a beam is made up of homogeneous, isotropic, and linearly elastic material, the maximum bending stress can be obtained using the well-known beam flexure formula \( f = M / I \). At ultimate load, the reinforced concrete beam is neither homogeneous nor elastic, thereby making that expression inapplicable for evaluating the stresses. However, the basic principles of the theory of bending can still be used to analyze reinforced concrete beam cross sections. Figure 5.1 shows a typical continuous reinforced concrete beam. If the beam is so proportioned that all its constituent materials attain their capacity prior to failure, both the concrete and the steel fail simultaneously at midspan when the ultimate strength of the beam is reached. The corresponding strain and stress diagrams are shown in Figure 5.2.
5.1 Introduction

![Diagram of a reinforced concrete beam with labels for shear and moment reinforcement.]

Figure 5.1 Typical reinforced concrete beam: (a) elevation; (b) section A-A.

The following assumptions are made in defining the behavior of the section:

1. Strain distribution is assumed to be linear. This assumption is based on Bernoulli's hypothesis that plane sections before bending remain plane and perpendicular to the neutral axis after bending.
2. Strain in the steel and the surrounding concrete is the same prior to cracking of the concrete or yielding of the steel.
3. Concrete is weak in tension. It cracks at an early stage of loading at about 10% of its limit compressive strength. Consequently, concrete in the tension zone of the section is neglected in the flexural analysis and design computations, and the tension reinforcement is assumed to take the total tensile force.

To satisfy the equilibrium of the horizontal forces, the compressive force $C$ in the concrete and the tensile force $T$ in the steel should balance each other, that is,

$$ C = T \tag{5.1} $$

The terms in Figure 5.2 are defined as follows:

- $b$ = width of the beam at the compression side
- $d$ = depth of the beam measured from the extreme compression fiber to the centroid of steel area
- $t$ = total depth of the beam
- $A_s$ = area of the tension steel
- $A_c$ = area of the compression steel
- $e_s$ = strain in extreme compression fiber
- $e_t$ = strain at the level of tension steel
- $f'_c$ = compressive strength of the concrete
- $f'_s$ = stress in the tension steel
- $f_y$ = yield strength of the tension reinforcement
- $c$ = depth of the neutral axis measured from extreme compression fibers

Photo 5.3 Simply supported beam in flexural failure. (Tests by Nazi.)
Figure 5.2 Stress and strain distribution across beam depth: (a) beam cross-section; (b) strains; (c) actual stress block; (d) assumed equivalent stress block.
5.2 THE EQUIVALENT RECTANGULAR BLOCK

The actual distribution of the compressive stress in a section has the form of a rising parabola, as shown in Figure 5.2c. It is time-consuming to evaluate the volume of the compressive stress block if it has a parabolic shape. An equivalent rectangular stress block due to Whitney can be used with ease and without loss of accuracy to calculate the compressive force and hence the flexural moment strength of the section. This equivalent stress block has a depth $d$ and an average compressive strength $0.85f'_c$. As seen from Figure 5.2d, the value of $d = eta c$ is determined using a coefficient $\beta_c$ such that the area of the equivalent rectangular block is approximately the same as that of the parabolic compressive block, resulting in a compressive force $C$ of essentially the same value in both cases.

The $0.85f'_c$ value for the average stress of the equivalent compressive block is based on the core test results of concrete in the structure at a minimum age of 28 days. Based on exhaustive experimental tests, a maximum allowable strain of 0.003 in./in. was adopted by the ACI as a safe limiting value. Even though several forms of stress blocks including trapezoidal have been proposed to date, the simplified equivalent rectangular block is accepted as the standard in the analysis and design of reinforced concrete. The behavior of the steel is assumed to be elasto-plastic, as shown in Figure 5.3a.

![Figure 5.3](image)

Figure 5.3: Strain distribution across depth; (a) idealized stress–strain diagram of the reinforcement; (b) strain distribution for various modes of flexural failure.
Using all the preceding assumptions, the stress distribution diagram shown in Fig. 5.2c can be redrawn as shown in Figure 5.2d. One can easily deduce that the compression force \( C \) can be written as \( 0.85f'_{ck}a \), that is, the volume of the compressive block at or near the ultimate when the tension steel has yielded, \( \sigma_s > \sigma_0 \). The tensile force \( T \) can be written as \( A_f \). Thus equilibrium Eq. 5.1 can be rewritten as

\[
0.85f'_{ck}a = A_f \sigma
\]

or

\[
a = \frac{A_f \sigma}{0.85f'_{ck}}
\]

The moment of resistance of the section, that is, the nominal strength \( M_n \), can be expressed as

\[
M_n = (A_f \sigma)\frac{d}{2} \quad \text{or} \quad M_n = (0.85f'_{ck}a)\frac{d}{2}
\]

where \( \frac{d}{2} \) is the lever arm, denoting the distance between the compression and tensile forces or the internal resisting couple. Using the simplified equivalent rectangular stress block from Figure 5.2d, the lever arm is

\[
\frac{d}{2} = \frac{a}{2}
\]

Hence the nominal moment of resistance becomes

\[
M_n = A_f \left( \frac{d}{2} - \frac{a}{2} \right)
\]

Since \( C = T \), the moment equation can also be written as

\[
M_n = 0.85f'_{ck}a \left( \frac{d}{2} - \frac{a}{2} \right)
\]

If the reinforcement ratio \( \rho = A_s/a \), Eq. 5.3 can be rewritten as

\[
a = \frac{\rho f'_{ck}}{0.85f'_{ck}}
\]

If \( r = \frac{b}{h} \), Eq. 5.4 becomes
5.2 The Equivalent Rectangular Block

\[ M_e = \rho d\sigma_y (d - \frac{2b}{1.7f'/c}) \]  
\[ M_e = \rho f'/c (1 - 0.39a)d' \]

where \( \rho = \frac{2}{f'/c} \). Equation 5.5b is sometimes expressed as

\[ M_e = Rh' \]

where

\[ R = \frac{f'/c}{1 - 0.39a} \]

Equations 5.5 and 5.6 are useful for the development of charts. A plot of the \( R \) value for singly reinforced beams is shown in Figure 5.4.

Depending on the type of failure, namely, yielding of the steel or crushing of the concrete, analysis of the strain state in the tension reinforcement becomes the determinant of the measure of ductility of the reinforced or prestressed concrete element. The percentage of the tension reinforcement would, therefore, determine the magnitude of

![Figure 5.4 Strength–R curves for singly reinforced beams.](image-url)
strain, and whether failure develops by initial yielding of steel (a ductile type of failure) or by initial crushing of the concrete (a brittle mode of failure). If failure precipitates by simultaneous yielding of the tensile reinforcement and the crushing of the concrete extreme compression fibers, such a mode of failure is termed as balanced failure. In such a case, the corresponding limit strain, $\varepsilon_L$, in the tensile reinforcement is reached at the same time that the limit strain, $\varepsilon_L$, in concrete (0.003 in./in.)

In order to prevent such a state of behavior in flexural members, a strain greater than $\varepsilon_L$ in the extreme tensile reinforcement has to be required in design. For example, if 60 ksi grade steel is used as reinforcement, the yield strain $\varepsilon_y = f_y/E_s = 60,000 / 29 \times 10^6 = 0.002$ in./in. The design has to be based on $\varepsilon_y$ (termed $\varepsilon_y$ at the level of the extreme tensile reinforcement layer) sufficiently larger than 0.002 in./in. in flexural members to ensure ductile performance. To achieve this result, the percentage of reinforcement $\rho = A_s / b h$ should be in the range of 30 to 60% of the percentage needed for the limit balanced behavior. Such a lower percentage of reinforcement would also prevent congestion of the
5.2 The Equivalent Rectangular Block

Reinforcement in the concrete section. However, a minimum percentage of reinforcement has also to be maintained, so that the reinforced concrete element does not behave as a plain concrete section.

The neutral axis depth, \(c\), can be expressed from Figure 5.2 as

\[
\frac{c}{d} = 0.003 \left( \frac{d_e}{c} - 1 \right) \quad \text{or} \quad 0.03 \left( \frac{d_e}{c} - 1 \right)
\]  

(5.7a)

For the limit balanced strain \(\varepsilon_e = 0.002\) in/in. at the extreme tensile reinforcement fibers,

\[
\frac{c_e}{d} = \frac{0.03}{0.003 + f_t/E_s}
\]  

(5.7b)

where \(c_e = \) balanced neutral axis depth at the limit strain \(\varepsilon_e = 0.002\) in/in. for 60 ksi steel
\(d_e = \) effective depth to the extreme tensile reinforcement layer

If the modulus of mild steel reinforcement, \(E_s\), is taken as 29 x 10^6 psi. Equation 5.7(b) becomes

\[
\frac{c_e}{d_e} = \frac{87,000}{87,000 + f_t}
\]  

for 60 ksi steel

(5.7c)

and

\[
f_t = \varepsilon_s E_s \left( \frac{d_e}{c_e} - 1 \right) = 87,000 \left( \frac{d_e}{c_e} - 1 \right) \leq f_t
\]  

(5.7d)

for the stress in the extreme tensile reinforcement.

The relationship between the depth \(d\) of the equivalent rectangular stress block and the depth \(c\) of neutral axis is

\[
a = \beta c
\]  

(5.8)

The value of the stress block depth factor \(\beta\) is

\[
\beta_c = \begin{cases} 
0.85 & \text{for } 2500 < f_t \leq 4000 \text{ psi} \\
0.85 - 0.05 \frac{f_t - 4000}{4000} & \text{for } 4000 \text{ psi} < f_t \leq 8000 \text{ psi} \\
0.65 & \text{for } f_t > 8000 \text{ psi}
\end{cases}
\]  

(5.9)

The code also stipulates the minimum steel requirement as

\[
A_{s,\text{min}} = \frac{3 \sqrt{f_t}}{f_y} b_d \leq \frac{200 b_d}{f_y}
\]  

(5.10a)

and for statically determinate T section with the flange in tension, or for cantilever,

\[
A_{s,\text{min}} = \frac{6 \sqrt{f_t}}{f_y} b_d \leq \frac{200 b_d}{f_y}
\]  

(5.10b)

but not greater than that calculated by Eq. 5.10a with \(b_d\) set equal to the width of the flange. (Equation 5.10b resulted from having to compute the minimum \(A_s\) over a width of flange = 2 \(b_d\).) Both Equations 5.10a and \(b_d\) need not be applied to each section, provided that \(A_s\) is at least one third greater than required by analysis.
5.3 STRAIN LIMITS METHOD FOR ANALYSIS AND DESIGN

5.3.1 General Principles

In this approach, sometimes referred to as the "unified method," being maintained as applicable to flexural analysis of prestressed concrete elements, the nominal flexural strength of a concrete member is reached when the net compressive strain in the extreme compression fibers reaches the ACI code-allowed limit 0.003 in./in. It also stipulates that when the net tensile strain in the extreme tension steel, \( \varepsilon_t \), is sufficiently large, as discussed in the previous section, at a value equal or greater than 0.005 in./in., the behavior is fully ductile. The concrete beam section is characterized as tension-controlled, with ample warning of failure as denoted by excessive cracking and deflection.

If the net tensile strain in the extreme tension fibers, \( \varepsilon_t \), is small, such as in compression members, being equal or less than a compression-controlled strain limit, a brittle mode of failure is expected, with little warning of such an impending failure. Flexural members are usually tension-controlled. Compression members are usually compression-controlled. However, some sections, such as those subjected to small axial loads, but large bending moments, the net tensile strain, \( \varepsilon_t \), in the extreme tensile fibers, will have an intermediate or transitional value between the two strain limit states, namely, between the compression-controlled strain limit \( \varepsilon_t = \frac{f_y}{E_y} = 0.003 \times 10^6 = 0.002 \) in./in., and the tension-controlled strain limit \( \varepsilon_t = 0.005 \) in./in. Figure 5.5 delineates these three zones as well as the variation in the strength reduction factors applicable to the total range of behavior. See also Figure 5.6.

For the tension-controlled state, the strain limit \( \varepsilon_t = 0.005 \) corresponds to reinforcement ratio \( \rho_s = 0.63 \), where \( \rho_s \) is the balanced reinforcement ratio for the balanced strain \( \varepsilon_s = 0.002 \) in the extreme tensile reinforcement. The net tensile strain \( \varepsilon_t = 0.005 \) for a tension-controlled state is a single value that applies to all types of reinforcement regardless whether mild steel or prestressing steel. High reinforcement ratios that produce a net tensile strain

![Figure 5.5 Strain Limit Zones and Variation of Strength Reduction Factor \( \phi \) with \( \varepsilon_t \)]
5.3 Strain Limits Method for Analysis and Design

less than 0.005 result in a d-factor value lower than 0.90, resulting in less economical sections. Therefore, it is more efficient to add compression reinforcement if necessary or deepen the section in order to make the strain in the extreme tension reinforcement, \( \epsilon_n \geq 0.005 \).

**Variation of \( \phi \) as a Function of Strain.** Variation of the \( \phi \) value for the range of strain between \( \epsilon_n = 0.002 \) and \( \epsilon_n = 0.005 \) can be linearly interpolated to give the following expressions.

**Tied Sections:**
\[
0.65 \approx \left[ \phi = 0.65 + (\epsilon_n - 0.002) \left( \frac{250}{3} \right) \right] \leq 0.90
\]  \hspace{1cm} (5.11a)

**Spirally-reinforced sections:**
\[
0.70 \approx \left[ \phi = 0.7 + (\epsilon_n - 0.002) \left( \frac{200}{3} \right) \right] \leq 0.90
\]  \hspace{1cm} (5.11b)

**Variation of \( \phi \) as a Function of Neutral Axis Depth Ratio \( c/d \).** Equations 5.7a and b can be expressed in terms of the ratio of the neutral axis depth \( c \) to the effective \( d \) of the layer of reinforcement closest to the tensile face of the section as follows:

**Tied Sections:**
\[
0.65 \approx \left[ \phi = 0.65 + 0.25 \left( \frac{1 - \epsilon_n}{\epsilon_n} - \frac{5}{3} \right) \right] \leq 0.90
\]  \hspace{1cm} (5.12a)

**Spirally-reinforced sections:**
\[
0.70 \approx \left[ \phi = 0.7 + 0.2 \left( \frac{1 - \epsilon_n}{\epsilon_n} - \frac{5}{3} \right) \right] \leq 0.90
\]  \hspace{1cm} (5.12b)

For rectangular beams, it is easy to determine whether the tension reinforcement has yielded or not, namely, if \( f_t = f_{y} \) from the following expression, with

Limit strain \( \epsilon_y = 0.002: \)
\[
\frac{\phi}{\epsilon_y} = \left( \frac{87,000}{87,000 + f_{y}} \right)
\]  \hspace{1cm} (5.13a)

and comparing this ratio to \( \phi/d \), of the beam being analyzed. Alternatively, at a straia of 0.004, which is close to the balanced strain condition, the corresponding reinforcement percentage is

\[
\rho_t = 0.85 \frac{f_{y}}{f_t} \left( \frac{\epsilon_y}{\epsilon_t} - 0.004 \right)
\]  \hspace{1cm} (5.13b)

For ductile behavior such that the beam is well into the tension controlled zone, a reinforcement percentage, \( \rho_t \), should be chosen in the range of 40 to 60 percent of the \( \rho_t \) value

![Figure 5.6: Strain Limits (a) Tension Controlled, (b) Compression Controlled.](attachment:image.png)
Chapter 5 Flexure in Beams

In Eq. 5.13 b. For one layer of tension reinforcement in singly-reinforced beam sections, Eq. 5.6 in conjunction with Figure 5.4 can be used for proportioning the geometry of the section on the basis of a reasonably assumed strength $R$ value.

It should be remembered that for flexural members with axial load less than $0.10 f'_c A_s$ and a strain less than 0.004 at nominal moment strength, the resulting $\phi$ value can become significantly lower than 0.9 for flexure since the section geometry would have to be modified or reinforcement percentage increased to accommodate the required nominal moment strength with ductile behavior; hence the code strain limiting value of 0.004 for non-prestressed members and for those with axial load less than $0.10 f'_c A_s$.

In summary, when the net tensile strain in the extreme tension reinforcement is sufficiently large (equal to or greater than 0.005), the section is defined as tension-controlled where ample warning of failure with extensive deflection and cracking can occur. When the net tensile strain in the extreme tension reinforcement is small (less than or equal to the compression-controlled strain limit), a brittle failure condition is expected to develop, with little warning of impending failure. Flexural members are usually tension-controlled.

Some sections, such as those with small axial load and large bending moment, will have net tensile strain between the above limits in the tension reinforcement closest to the extreme tensile fibers of a concrete section. These sections are in a transition region between compression- and tension-controlled sections.

In such cases for non-prestressed members and with axial load equal or less than $0.10 f'_c A_s$, the net tensile strain $\varepsilon_t$ at the extreme tension steel should not be less than 0.004 at nominal moment strength. Otherwise, the resulting $\phi$ value can become so low that additional reinforcement would be needed to produce the required nominal strength and with reduced ductility.

A balanced strain condition develops at a section when the maximum strain at the extreme compression fibers just reaches 0.003 in./in. simultaneously with the first yield strain $\varepsilon_y = f'_c / E_s$ in the tension reinforcement corresponding to a net tensile strain in the tension reinforcement set in this method at a value $\varepsilon_{ty} = 0.002$ in./in. This condition cannot be used in the flexural design of beams not subjected to compression. In such members, a strain $\varepsilon_t$ in the extreme tensile reinforcement need not considerably exceed 0.0075 in./in. for practical consideration of section size.

As a rule of thumb, for the first trial in the design of a beam for flexure, a $\phi_{ty}$ ratio of 75% of the limit $\phi_{ty} = 0.375$ would ensure a ductile behavior, with $\phi_{ty} > 0.005$.

When using the $\phi$ values obtained from Equations 5.11 or 5.12 for cases where the compression member is in the tensile failure zone of an interaction diagram, somewhat larger moment strength values are obtained in this method than might be available. This is due to the fact that columns are primarily loaded in compression, and $\varepsilon_t$ values will be less than $f'_c / E_s$ if not actually less than zero.

5.3.2 Negative Moment Redistribution in Continuous Beams

The code permits decreasing the negative moments at the supports for continuous members by not more than 1000 $\varepsilon_t$% with a maximum of 20%. The reason is that for ductile members, plastic hinges develop at points of maximum moment and cause a shift in the elastic moment diagram. The result is a reduction of the negative moment and a corresponding increase in the positive moment. The redistribution of the negative moment as permitted by the code can only be used when $\varepsilon_t$ is equal to or greater than 0.0075 in/in at the section at which the moment is reduced. This redistribution is logically inapplicable to working stress design or to slab systems designed by the direct design method (DDM).

Figure 5.7 shows the permissible moment redistribution for minimum rotational capacity. It should be emphasized that the procedure presented in Sec. 5.3.1 does not in any way alter the strength computations for non-prestressed concrete sections. For prestressed concrete sections, when the reinforcement ratio for combined prestressed and
Figure 5.7 Allowable redistribution percentage for minimum rotational capacity.

Anchorage reinforcement has a reinforcement index of not exceeding 0.36, upper limit allowed by the ACI 318 Code. A modified computational procedure has to be followed.

The ACI 318 Code, as in Equations 5.12, stipulates a maximum strength reduction factor \( \phi = 0.96 \) for pure bending, to be used in computing the design strength of flexural members. This corresponds to neutral axis depth ratio \( \alpha_d = 0.375 \) or lower. For a useful redistribution of moment in continuous members, this neutral axis depth ratio should be considerably lower, so that the net tensile strain is within the range of \( \varepsilon_t = 0.0075 \text{ in/in.} \), giving a redistribution factor of 7.5%, and a limit \( \varepsilon_t = 0.020 \), giving 20% redistribution, as shown in Figure 5.7.

As an example, if \( d_t = 20 \text{ in.} \) and neutral axis depth \( c = 5.1 \text{ in.} \),
\[
\varepsilon_t = 0.003 \left( \frac{d_t}{c} - 1 \right) = 0.003 \left( \frac{20}{5.1} - 1 \right) = 0.0088 \text{ in/in.} \geq 0.0075 \text{ in/in.} \text{ minimum value for inclusion redistribution to be applied.}
\]

In this case, the maximum allowable moment redistribution = 1000 \( \varepsilon_t = 13% \).

This gives a net reduction in negative moment value = \( (100 - 8.8) = 91.2% \).

Photo 5.7 Beam subjected to combined axial load and bending. The neutral axis is at 70% of depth. (Tests by Nassy et al.)
5.4 ANALYSIS OF SINGLY REINFORCED RECTANGULAR BEAMS FOR FLEXURE

The sequence of calculations presented in the flowchart of Figure 5.8 can be used for the analysis of a given beam for both longhand and computers. The flowchart was developed using the method of analysis presented in Section 5.2. The following examples illustrate typical analysis calculations following the flowchart logic in Figure 5.8.

---

Figure 5.8  Flowchart for analysis of singly reinforced rectangular beams in bending
5.4.1 Example 5.1. Flexural Analysis of a Singly Reinforced Beam (Tension Reinforcement Only)

A singly reinforced concrete beam has the cross-section shown in Figure 5.9. Determine if the beam is tension- or compression-controlled. Given $f'_c = 4000$ psi (27.6 MPa), determine if the beam satisfies the ACI Code if:

(a) $f_s = 60,000$ psi; (b) $f_s = 40,000$ psi

$p = \frac{6.0}{10 \times 15} = 0.033$

From Eq. 5.10a,

minimum allowable reinforcement ratio $\rho_{min} = \frac{3V}{f'_c A_c} = \frac{3 \times 4000}{40,000} = 0.033$

![Diagram of beam section and stress-strain distribution](image)

**Figure 5.9** Stress and strain distribution in a typical singly reinforced rectangular section: (a) cross-section; (b) strains; (c) stresses.
Chapter 5  Flexure in Beams

Photo 5.9  Crushing of concrete at compression side of beam subjected to flexure.

Solution:  (a) $f_c = 60,000$ psi

$$ a = \frac{A_{ci}}{A_{fl}} = \frac{6.0 \times 60,000}{0.85 \times 4,000 \times 10} = 10.59 \text{ in.} \quad c = 12.46 \text{ in.} $$

$$ e = \frac{12.46}{18.0} = 0.69 > 0.60 \text{ from Figure 5.5} $$

Hence, $A_i$ did not yield and the strain is smaller than 0.002 in./in. Brittle behavior results as the section is compression-controlled and does not satisfy the ACI Code requirements for flexural beams.

Solution:  (b) $f_c = 40,000$ psi

$$ \beta_i = 0.65 $$

Assume $f_i = f_c$.

$$ p_{res} = 2 \sqrt{4000} = 0.0047 \times p_{res} = 200 = 0.005 \text{ (controls)} $$

$$ \alpha = \frac{0.85 \times 4,000 \times 10}{7.06 \text{ in.}} = 8.31 \text{ in.} $$

$$ e_i = 0.002 \left( \frac{d - c}{c} \right) = 0.003 \left( \frac{18 - 8.31}{8.31} \right) = 0.0035 \text{ in./in.} < 0.005 $$

$$ e_i = \frac{f_i}{E_i} = \frac{40,000}{59 \times 10^6} = 0.00137 \text{ in./in.} < 0.0035 $$

$$ c = \frac{8.31}{15.0} = 0.46 > 0.375 < 6.0 $$

Hence, the beam is in the transition zone, tension steel yielded. But $e_i < 0.005$, hence a reduced $\phi$ for calculating $M_{el}$ can be used.

Therefore, ACI requirements for flexure with $\phi$ in the transition zone is satisfied. However, as previously discussed using a $\phi$ value less than 0.60 makes the section uneconomical. Thus this section is uneconomical. To improve the design decrease $A_i$ or increase depth.

5.4.2 Example 5.2: Nominal Resisting Moment in a Singly Reinforced Beam

For the beam cross section shown in Figure 5.10, calculate the nominal moment strength if $f_c$ is 60,000 psi (413.4 MPa) and $f_i$ is (a) 3000 psi (20.7 MPa); (b) 5000 psi (34.5 MPa); (c) 9000 psi (62.1 MPa).
Figure 5.10 Beam cross-section strain and stress diagrams. Ex. 5.2: (a) cross section; (b) strains; (c) stresses.

Solution:

- $b = 10$ in. (254.0 mm)
- $d = 18$ in. (457.2 mm)
- $A_s = 4$ in.$^2$ (2560 mm$^2$)
- $f_s = 60,000$ psi

Note that $f_s$ should be in psi units in the $\rho_{sw}$ expression.

(a) $f' = 3000$ psi (21.7 MPa)

$\rho_{sw} = \frac{3 \sqrt{f'}}{f_s} = \frac{3 \sqrt{3000}}{60000} = 0.0027$

or

$\rho_{sw} = \frac{200}{f_s} = \frac{200}{60000} = 0.0033$ (controls)

$\rho = \frac{A_s}{bd} = \frac{4}{10 \times 18} = 0.0222 > 0.0033$ O.K.

$\beta_1 = 0.85$

Since the reinforcement is in one layer,

- $c = d = 18$ in.
- $C = 0.85f_s/\rho = 0.85 \times 3000 \times 10 \times 10 = 25,500$ kips
- $T = A_s f_s = 4.0 \times 60000 = 240,000$ kips
- $C = T$  
  $a = \frac{240,000}{25,500} = 9.4$ kips
- $\beta_2 = \frac{9.4}{0.85} = 11.1$ in.

\[ \varepsilon = \frac{a}{\beta_2} = \frac{9.4}{11.1} = 0.85 \quad \text{in.} \]

\[ \varepsilon = \frac{11.1}{18.0} = 0.62 > 0.60, \text{ compression-controlled} \]

Hence the beam is not ductile and does not satisfy the ACI 318 Code.
Solution: (b)

\[ f'_{c} = 5,000 \text{ psi}, \quad \beta = 0.80 \]

\[ \beta_{cr} = \frac{3\sqrt{f'_{c}}}{60,000} = 0.0035 \]

Actual \( \beta = \frac{4.0}{10 \times 18} = 0.022 \). O.K.

\[ C = 0.85 \times 5,000 \times 10 \times a = 42,500 \text{ in.} \]

\[ T = 240,000 \times a = 240,000 \times \frac{42,500}{42,500} = 5.65 \text{ in.} \]

\[ c = \frac{a}{B} = \frac{5.65}{18} = 0.32 \text{ in.} \]

\[ \epsilon_{c} = \frac{c}{d} = \frac{0.32}{8} = 0.04 > 0.0375 < 0.60 \]

Hence the beam is ductile, but in the transition zone with \( \epsilon_c < 0.90 \).

\[ \epsilon_{c} = \epsilon \]

\[ M_{e} = A_{f} \left( d - \frac{a}{2} \right) = 4.0 \times 60,000 \left( 18 - \frac{5.65}{2} \right) \]

\[ = 3,642,000 \text{ in.-lb (411 kN-m)} \]

Solution: (c)

\[ f'_{c} = 9,000 \text{ psi}, \quad \beta = 0.65 \]

\[ \beta_{cr} = \frac{3\sqrt{f'_{c}}}{60,000} \]

\[ = \frac{0.0047 < 0.022}{\text{ok}} \]

\[ a = \frac{4.0 \times 90,000}{0.85 \times 90,000} = 3.14 \text{ in.} \]

\[ c = \frac{a}{B} = \frac{3.14}{8} = 0.39 \text{ in.} \]

\[ \epsilon_{c} = 0.001 \left( \frac{d - c}{c} \right) \]

\[ = 0.001 \left( \frac{8 - 0.39}{0.39} \right) = 0.0082 \text{ in./in.} \]

\[ >> 0.005 > 0.0075, \text{ hence, tension-controlled, namely, ductile behavior, } \phi = 0.60 \]

\[ M_{e} = 4.0 \times 60,000 \left( 18 - \frac{3.14}{2} \right) = 3,843,200 \text{ in.-lb (446 kN-m)} \]

5.5 TRIAL-AND-ADJUSTMENT PROCEDURES FOR THE DESIGN OF SINGLELY REINFORCED BEAMS

In Ex. 5.2, the geometrical properties of the beam, that is, \( b, d, \) and \( A_{f} \), were given. In a design example, an assumption of width \( b \) (or the ratio \( b \) to \( d \)) and the level of reinforcement ratio \( \rho \) have to be made. The ratio \( b/d \) varies between 0.3 and 0.6 in usual practice. Although the ACI Code permits a tensile reinforcement ratio \( \rho \) at a limit strain of 0.005, it is advisable to use a higher strain value, such as \( \epsilon_c = 0.0075 \) in order to prevent conges-
tion of steel, secure a good bond between the reinforcement and the adjacent concrete, and provide good deflection control. Studies on cost optimum design indicate that cost-effective sections can be obtained using a minimum practical $b/d$ ratio and a maximum practical reinforcement ratio $p$ within the above-stated limitations. Hence one could use the following steps to design the beam cross section following the flowchart logic of Figure 5.11.

1. Calculate the external factored moment. To obtain the beam self-weight, an assumption has to be made for the value of $d$. The minimum thickness for deflection specified in the ACI Code can be used as a guide. Assume a $b/d$ ratio $r$ between 0.3 and 0.6 and calculate $b = rd$. A first trial assumption $d = d_2$ is recommended.

2. (a) Select a value of moment factor $R$ based on an assumed $p$ value based on $e_i = 0.005$ or higher. or $d/i = 0.375$. Assuming that $b = d_2$, calculate $d$ for $M_o = Rbd^2$ and proceed to analyze the section.

Start

Given: loading, span, end condition, $f_c', f_y$

Assume section depth as function of the span, using ACI deflection and control guidelines. Assume $r = 0.6$ between 0.30-0.60, say 0.60. Compute the factored moment $M_o$ and the corresponding $M_o$

Select trial reinforcement percentage, $p$, from Figure 5.4, as $M_o = Rbd^2$

Assume:
1. $d/d_c < 0.375$, say 0.30
2. $r > 0.005$, say 0.007
3. $b > 0.5d_c$, where $b_c$ corresponds to limit strain state $c = 0.002$ in/in.

From $M_o$ and the trial $p$, compute $a = A_y/f_y(0.85 f_c' b)$

No

Deduct $c = assumed c$

Yes

Select and refine final selection

End

Figure 5.11 Flowchart for sequence of operations for flexural design of singly-reinforced beams.
(b) Alternatively, choose \( d \) on the basis of minimum deflection requirement. Choose a width \( b \) as in 2(a). Assume a moment arm \( d_0 = 0.85d \) to 0.90d. Calculate \( A_t \) as a first trial, then analyze the sections using \( b = d_0 \).

3. Assume the neutral axis depth ratio \( c/d \) less than 0.375.
4. Equate forces \( C = T \) to get \( A_t \), then check final \( e \) to verify that its value \( > 0.005 \).

The process of arriving at the final section is highly convergent even by hand-method computations in that it should not require more than three trial cycles. The use of computers enormously simplifies the design-analysis process and permits the student or engineer to proportion sections at a fraction of the time needed when using handbooks, charts, or hand-method computations, easy as these other means can be.

For designers who prefer charts, Eq. 5.6 \( (M_r = Rb^2) \) can be used for the first trial in design. The value of \( R \) can be obtained from charts (see Figure 5.4) for various values of \( p \), \( f_0 \), and \( f_0 \) available in handbooks.

5.5.1 Example 5.3: Design of a Simply Reinforced Simply Supported Beam for Flexure

A reinforced concrete simply supported beam has a span of 30 ft (9.14 m) and is subjected to a service uniform live load \( W_l = 1600 \) lb/ft (24.1 kN/m), as shown in Figure 5.9. Design a beam section to resist the factored external bending load. Given:

\[

g_1 = 4000 \text{ psi (27.6 MPa)} \\

f_y = 60,000 \text{ psi (414 MPa)}
\]

Solution: Assume a minimum thickness from the ACI Code deflection table:

\[

dl \frac{b}{16} = \frac{30 \times 12}{16} = 22.5 \text{ in.}
\]

Try a section with \( b = 12 \) in., \( d = 23 \) in., and \( h = 26 \) in. \((r = bd = 0.5)\)

Self-weight = \( \frac{12 \times 26}{144} = 26 \text{ lb/ft} \)

Factored, \( w_r = 1.2 \times 325 + 1.6 \times 1650 = 3030 \text{ lb/ft} \)

Factored moment, \( M_r = \frac{3030(30)^2}{8} \times 12 = 4,390,500 \text{ in.-lb} \)

Required resisting moment, \( M_r = \frac{4,090,500}{0.9} = 4,545,000 \text{ in.-lb} \)

Try maximum area of tension reinforcement to satisfy depth \( c \) to the neutral axis \( = 0.5c \).

\( d_0 = 23 \) in.

Limit bending strain state:

\[
c_s = 23 \left( \frac{87,000}{87,000 + 64,000} \right) = 13.61 \text{ in.}
\]

Try \( e = 0.5c = 6.8 \text{ in.} \), or 5.7 in.

or, alternatively,

\[
\text{Try } \frac{c}{d_0} = 0.30, \quad 0.30 \times 23 = 6.9 \text{ in.}
\]
Figure 5.12 Simply supported reinforced concrete uniformly loaded beam: (a) elevation; (b) cross section; (c) strains; (d) stresses.

\( a = 3.81 \times 60,000 \times 0.89 \times 4,000 = 5.60 \text{ in.}, \ c = 6.59 \text{ in.} \)

\( \sigma = 0.001 \left( \frac{d - c}{c} \right) = 0.001 \left( \frac{23 - 6.59}{6.59} \right) = 0.0067 \text{ in./in.} > 0.005 \text{ in./in.} \), O.K.

\( f_0 = 4.54 \text{ in.} \), O.K. Adopt the design.
5.5.2 Arrangement of Reinforcement

Figure 5.13 shows the cross-section of the beam at midspan. In arranging the reinforcing bars, one should satisfy the minimum cover requirements explained in Section 4.3. The required clear cover for beams is 1.5 in. (38 mm).

The stirrups shown in Figure 5.13 should be designed to satisfy the shear requirements of the beam explained in Chapter 6. Two bars called "June" are placed on the compression side to support the stirrups. Reinforcement detailing provisions and bar development length requirements are discussed in Chapter 10.

5.6 ONE-WAY SLABS

One-way slabs are concrete structural floor panels for which the ratio of the long span to the short span equals or exceeds a value of 2.0. When this ratio is less than 2.0, the floor panel becomes a two-way slab or plate, as discussed in Chapter 11. A one-way slab is designed as a singly reinforced 12-in. (304.8-mm) wide beam strip using the same design and analysis procedure discussed earlier for singly reinforced beams. Figure 5.14 shows a one-way slab floor system.

Loading for slabs is normally specified in pounds per square foot (psf). One has to distribute the reinforcement over the 12-in. strip and specify the center-to-center spacing of the reinforcing bars. In slab design, a thickness is normally assumed, and the reinforcement is calculated using a trial lever arm \( d = a/2 \) or 0.96d.

Supported slabs, that is, slabs not on grade, do not normally require shear reinforcement for typical loads. Transverse reinforcement has to be provided perpendicular to the direction of bending in order to resist shrinkage and temperature stresses. Shrinkage and temperature reinforcement should not be less than 0.002 times the gross area for grade 40 bars and 0.0018 for grade 60 steel and welded wire fabric. For structural slabs and footings of uniform thickness, the maximum spacing of tension reinforcement should not exceed five times the thickness or 18 in.

5.6.1 Example 5.4: Design of a One-way Slab for Flexure

A one-way single-span reinforced concrete slab has a simple span of 10 ft (3.05 m) and carries a live load of 140 psf (5.75 kPa) and a dead load of 20 psf (0.96 kPa) in addition to its self-weight. Design the slab and the size and spacing of the reinforcement at midspan assuming a simple support moment. Given:
Figure 5.14  Isometric view of four-span continuous one-way-slab floor system.

\( f' = 4000 \text{ psi (27.5 MPa), normal-weight concrete} \)
\( f_c = 60,000 \text{ psi (413.4 MPa)} \)

Minimum thickness for deflection = \( \frac{1}{20} \)

Solution: Minimum depth for deflection, \( b = \frac{1}{20} \times 10 \times 12 = 6 \text{ in. (152 mm)} \)
Assume for flexure an effective depth \( d = 5 \text{ in. (127 mm)} \).

Self-weight of a 12-in. strip = \( \frac{6 \times 12}{44} \times 150 = 75 \text{ lb/ft (3.59 kN/m)} \)

Therefore,

factored external load \( w_{e} = 1.2(20 + 75) + 1.6 \times 140 = 338 \text{ lb/ft} \)

factored external moment \( M_{e} = \frac{338 \times 10^{2}}{8} \times 12 \text{ in.-lb} \)

\[ M_{e} = 50,700 \text{ in.-lb (5.7 N-m)} \]

required nominal moment strength \( M_{n} = \frac{M_{e}}{0.9} = 56,334 \text{ in.-lb} \)

Assume moment arm \( d = 0.90 \times 5 = 4.5 \text{ in.} \)

\[ M_{n} = T_{d}d = A_{f}f_{d}d \]

or \[ 56,334 = A_{f} \times 60,000 \times 4.5 \]

\[ A_{f} = 0.21 \text{ in.}^{2}/\text{in. strip} \]

\[ a = \frac{A_{f}f_{d}}{0.85b} = 0.21 \times 60,000 \]

Recalculate \( A_{f} \) using the correct moment arm:

\[ 56,334 = A_{f} \times 60,000 \left( s - \frac{0.31}{2} \right) \]
to give $A_e = 0.199 \text{ in.}^2$ per 12-in. slab strip. Use #4 bars at 12-in. center-to-center spacing.

Check strain $\varepsilon$:

\[
\varepsilon = \frac{\sigma}{E} = \frac{0.31}{0.85} = 0.37 \text{ in.}
\]

\[
\kappa = 0.003 \left( \frac{d - c}{c} \right) = 0.003 \left( \frac{5 - 0.37}{0.37} \right) = 0.036 \text{ in./in.} > 0.005 \text{ in./in.}
\]

hence, section is tension-controlled. $\sigma = 0.90$.

**Check Minimum Reinforcement**

Actual $\rho = \frac{0.20}{5.0 \times 12} = 0.0033$

\[
\rho_{min} = \frac{200}{f_y} = \frac{200}{60,000} = 0.0033 \text{ (controls)}
\]

\[
3\sqrt{\frac{f_y}{f_c}} = \frac{3\sqrt{4,000}}{60,000} = 0.0031
\]

Accept the design.

**Shrinkage and temperature reinforcement**

Min. $A_e = 0.0018 \text{ lb/in.}^2$

area of steel = $0.0018 \times 6 \times 12 = 0.12 \text{ in.}^2 = 40 \text{ bars at 18 in. c-c}$

Provide No. 4 bars at 18-in. center to center (maximum allowable spacing = 5h = 5 x 6 = 30 in. for temperature and shrinkage).

Hence this design can be adopted with slab thickness $h = 6$ in. (152.4 mm) and effective depth $d = 6.0 - (0.75 + 0.25) = 5$ in. (127.0 mm) to satisfy the 1-in. minimum concrete cover requirement. Use for main reinforcement No. 4 bars at 12 in. center to center and for temperature reinforcement No. 4 bars at 18 in. center to center, as shown in Figure 5.15.

![Diagram](image)

**Figure 5.15** Reinforcement details of the one-way slab in Ex. 5.4: (a) sectional view; (b) reinforcement details.
5.7 DOUBLY REINFORCED SECTIONS

Doubly reinforced sections contain reinforcement both at the tension and at the compression face, usually at the support section only. They become necessary when either architectural limitations restrict the beam web depth at midspan, or the midspan section dimensions are not adequate to carry the support negative moment even when the tensile steel at the support is sufficiently increased. In such cases, about one-third to one-half of the bottom bars at midspan are extended and well anchored at the support to act as compression reinforcement. The bar development length has to be well established and the compressive and tensile steel at the support section well tied with close stirrups to prevent buckling of the compressive bars at the support.

In analysis or design of beams with compression reinforcement $A'_c$, the analysis is so divided that the section is theoretically split into two parts, as shown in Figure 5.16. The two parts of the solution comprise (1) the singly reinforced part involving the equivalent rectangular block, as discussed in Section 5.2, with the area of tension reinforcement being $(A_t - A'_t)$; and (2) the two areas of equivalent steel $A'_t$ at both the tension and compression sides to form the couple $T_1$ and $C_1$ as the second part of the solution.

It can be seen from Figure 5.16 that the total nominal resisting moment $M_n = M_{nt} + M_{nc}$, that is, the summation of the moments for parts 1 and 2 of the solution.

**Part 1.** The tension force $T_1 = A'_t f'_t = C_1$. But $A'_t = (A_t - A'_t)$ since equilibrium requires that $A'_t$ at the tension side be balanced by an equivalent $A'_t$ at the compression side. Hence the nominal resisting moment

$$M_{nt} = A'_t f'_t \left( \frac{d - \frac{a}{2}}{2} \right)$$

or

$$M_{nt} = (A_t - A'_t) f'_t \left( \frac{d - \frac{a}{2}}{2} \right) \quad (5.14a)$$

where

$$a = \frac{A_c f_c}{0.8 f'_t b} = \frac{(A_t - A'_t) f'_t}{0.8 f'_t b}$$

**Part 2.**

$$A'_c = A'_t = (A_t - A_t)$$

$$T_2 = C_2 = A'_c f'_c$$

---

**Figure 5.16** Doubly reinforced beam design: (a) cross-section; (b) strains; (c) part 1 of the solution, singly reinforced part; (d) part 2 of the solution, contribution of compression reinforcement.
Taking the moment about the tension reinforcement, we have

\[ M_{1d} = A_d f_c (d - d') \quad (5.14b) \]

Adding the moments for parts 1 and 2 yields

\[ M_i = M_{1d} + M_{2d} - (A_i - A'_i) f_c \left( d - \frac{a}{2} \right) + A'_i f_c (d - d') \quad (5.15a) \]

The design moment strength \( \phi M_i \) must be equal to or greater than the external factored moment \( M_o \) such that

\[ M_o = \phi \left[ (A_i - A'_i) f_c \left( d - \frac{a}{2} \right) + A'_i f_c (d - d') \right] \quad (5.15b) \]

This equation is valid only if \( A'_i \) yields. Otherwise, the beam has to be treated as a singly reinforced beam neglecting the compression steel, or one has to find the actual strain \( \varepsilon'_i \) in the compression reinforcement \( A'_i \) and use the actual force in the moment equilibrium equation.

**Strain-Compatibility Check.** It is always necessary to verify that the strains across the depth of the section follow the linear distribution indicated in Fig. 5.16. In other words, a check is necessary to ensure that strains are compatible across the depth at the strength design levels. Such a verification is called a strain-compatibility check.

In order to ensure tension-controlled behavior, the ratio \( \frac{e_d}{d} \approx 0.375 \), preferably 0.30. In this manner, the strain \( e_i \) in the tensile reinforcement is greater than 0.005. Find \( e_i = 0.003 (dfc = 1) \). Once the strain \( e_i \) is verified to be higher than 0.005, say 0.005-0.006, the nominal moment strength is computed as in Equation 5.22.

For \( A'_i \) to yield, the strain \( \varepsilon'_i \) in the compression steel should be greater than or equal to the yield strain of reinforcing steel, which is \( f'_c / E' \). The strain \( \varepsilon'_i \) can be calculated from the following formulas. Referring to Fig. 5.14b,

\[ \varepsilon'_i = \frac{0.003 (c - d')}{c} \]

or

\[ \varepsilon'_i = 0.003 \left( 1 - \frac{d'}{c} \right) \]

Since

\[ e = \frac{a}{\beta_i} = \frac{(A_i - A'_i) f_c}{\beta_i \times 0.85 f'_c b} = \frac{(a - a') f_c d}{\beta_i \times 0.85 c} \]

![Photo 5.10 Flexural cracking in heavily reinforced beam. (Tests by Novy, Poznacky, et al.)](image-url)
\[ \epsilon_i = 0.003 \left[ 1 - \frac{0.85b f_y d_y}{(p - p_y d_y)} \right] \]  

(5.16)

As mentioned earlier, for compression steel to yield, the following condition must be satisfied:

\[ \epsilon_i' = \frac{f_c}{E_c} \text{ or } \epsilon_i' = \frac{f_y}{29 \times 10^6} \]

The compression steel yields if

\[ 0.003 \left[ 1 - \frac{0.85b f_y d_y}{(p - p_y d_y)} \right] \geq \frac{f_c}{29 \times 10^6} \]

or

\[ \frac{0.85b f_y d_y}{(p - p_y d_y)} \geq \frac{f_c - 87,000}{87,000} \]  

(5.17)

or

\[ p - p_y = \frac{0.85b f_y d_y}{f_c} \geq \frac{87,000 - f_c}{87,000} \]  

(5.18)

If \( \epsilon_i' \) is less than \( \epsilon_i \), the stress in the compression steel, \( f_c \), can be calculated as

\[ f_c = E_c \epsilon_i' = 29 \times 10^6 \epsilon_i' \]  

(5.19)

Using Eqs. 5.16 and 5.19 yields

\[ f_c = 29 \times 10^6 \times 0.003 \left[ 1 - \frac{0.85b f_y d_y}{(p - p_y d_y)} \right] \]  

(5.20)

This value of \( f_c \) can be used as a first approximation in the strain-compatibility check.

In this discussion, adjustment for the concrete area replaced by the compression reinforcement is disregarded as being insignificant for practical design purposes. Note that in cases where the compression reinforcement \( A_y \), did not yield the depth of the rectangular compressive block should be calculated using the actual stress in the compression steel from the calculated strain value \( \epsilon_i' \) at the compression reinforcement level so that

\[ a = \frac{A f_y - A_y f_y'}{0.85f_y b} \]  

(5.21)
Equation 5.20 can be used for the $f'_t$ value in the first trial in order to obtain an "u" value and hence the first trial neutral axis depth value $c$. Once $c$ is known, $e'$ can be evaluated from similar triangles in Figure 5.10b. Thereby, obtaining the first approximation of $f'_t$ to be used in recalculating a more refined value. More that one or two additional trials for calculating $f'_t$ are justified since undue refinement has negligible practical effect on the true value of the nominal moment strength $M_u$.

The nominal moment strength in Eq. 5.15 becomes in this case:

$$M_u = (A_s f' - A_s f'_t) \left( d - \frac{c}{2} \right) + A_s f'_t (d - d')$$  \hspace{1cm} (5.22)

The flowchart in Fig. 5.17 can be used for the sequence of calculations in the analysis of doubly reinforced beams. Examples 5.5 and 5.6 illustrate the analysis and design of doubly reinforced sections.

---

Figure 5.17 Flowchart for the analysis of doubly reinforced rectangular beams.
5.7 Doubly Reinforced Sections

5.7.1 Example 5.5: Analysis of a Doubly Reinforced Beam for Flexure

Calculate the nominal moment strength $M_n$ of the doubly reinforced section shown in Figure 5.18. Given:

- $f'_c = 5000$ psi (34.5 MPa), normal-weight concrete
- $f_s = 60,000$ psi (413 MPa)
- $d' = 2.5$ in. (64 mm)
- $d = 21$ in.
- $A_r = 4$ No. 10 bars
- $A_s' = 2$ No. 7 bars

Solution:

- $A_s = 5.08$ in.$^2$, $p = \frac{A_s}{bd} = \frac{5.08}{14 \times 21} = 0.0173$
- $A_s' = 1.2$ in.$^2$, $p' = \frac{A_s'}{bd} = \frac{1.2}{14 \times 21} = 0.0041$
- $A_r = A_s = 5.08 - 1.2 = 3.88$ in.$^2$

To check if the compression steel yielded, using Eq. 5.18.

Alternatively, assume that compression steel yielded, to be subsequently verified.

To check whether the compression steel has yielded, use Eq. 5.18:

- $p - p' \leq \frac{0.858 f_s' d'}{f_c} \frac{87,000}{87,000 - f_c} = \frac{0.85 \times 0.93 \times 5000 \times 2.5}{60,000 \times 21} \frac{87,000}{87,000 - 60,000} \geq 0.0217$

Figure 5.18 Doubly reinforced cross-section geometry and stress and strain distribution: (a) cross section; (b) strains; (c) part 1 section; (d) part 1 forces; (e) part 2 forces.
The actual \((a - p') = 6.032 < 0.0217\). Therefore, the compression steel did not yield and \(f'_c\) is
less than \(f_c\). For the first trial in cases where the compression steel did not yield
\[
f'_c = 87,000 \left(1 - \frac{0.859f_c}{(a - p')f_c} \frac{d'}{d}ight)
\]
\[
= 87,000 \left(1 - \frac{0.85 \times 8.8 \times 5000}{0.032 \times 5000 \times 2.5} \times \frac{21}{21} \right) = 42,538 \text{ psi}
\]
\[
\sigma = \frac{A_f'f_c - A_f'(d - d')}{0.85f_c} = \frac{5.08 \times 60,000 - 1.2 \times 42,538}{0.85 \times 5000 \times 14} = 4.26 \text{ in. (108 mm)}
\]
neutral-axis depth \(c = \frac{4.26}{0.80} = 5.325 \text{ in.}
\]
By trial and adjustment, from similar triangles in Fig. 5.38b, the strain \(e_c\) at the compression
steel level is \(0.0059 \text{ in./in.},\) giving \(f'_c = 0.0059 \times 29 \times 10^6 = 46,110 \text{ psi. An additional trial cycle for a more refined value of } \sigma = 4.21 \text{ in. (108 mm)}\
\]
\[
e_c = 0.003 \left(\frac{4 - c}{c}ight) = 0.003 \left(\frac{21 - 5.26}{5.26}\right) = 0.003 \times 0.008 \text{ in./in.}
\]
hence, tensile steel reinforcement yielded, tension-controlled, with \( \phi = 0.90 \).
\[
M_t = (A_f'f_c(d - c') + A_f'(d - d'))
\]
\[
= (3.08 \times 60,000 - 1.2 \times 45,650)(21.0 - 4.21)
\]
\[
+ 1.2 \times 45,650(21.0 - 2.5) = 5,738,808 \text{ in.-lb (648 kN-m)}
\]
From Equation 5.12a,
\[
\phi = 0.23 + 0.25(c/d) = 0.23 + \frac{0.25}{5.26/21} = 1.23 > 0.90, \text{ hence use } \phi = 0.90
\]
\[
M_e = 6M_t = 5,738,808 \times 0.9 = 5,164,927 \text{ in.-lb (583 kN-m)}
\]
Note that if \( e_c < 0.005, \) compare \( \phi \) from Eq. 5.11a or 5.12a, find \( f_c \) for the tension reinforcement and multiply the moment
\[
M_t = (A_f'f_c(d - c') + A_f'(d - d'))
\]
by the new \( \phi \) value to get the design moment \( M_e \).

5.7.2 Trial-and-Adjustment Procedure for the Design of Doubly Reinforced Sections for Flexure

1. Midspan section. The trial-and-adjustment procedure described in Section 5.5 is fol-
lowed in order to design the section at midspan if it is a rectangular section; other-
wise, follow the same procedure as that for the design of T beams and L beams
(Section 5.10).

2. Support section. The width \( b \) and the effective depth \( d \) are already known from part
1 together with the value of the external negative factored moment \( M_e \).
(a) Find the strength \( M_{d_1} \) of a singly reinforced section using the already estab-
lished \( b \) and \( d \) dimensions of the section at midspan and a reinforcement area
to give \( e_c < 0.005 \).
(b) From step (a), find \( M_s = M_c = M_b \) and determine the resulting \( A_s = A'_s \). The total steel area at the (tension side) would be \( A_s = A_s + A'_{s} \).

(e) Alternatively, determine how many bars are attached from the midspan to the support to give the \( A'_s \) to be used in calculating \( M_s \).

(d) From step (c), find the value of \( M_s = M_c - M_b \). Calculate \( A_s \) for a singly reinforced beam as the first part of the solution. Then determine total \( A_s = A_s + A'_s \). Verify that \( A_s \) does not give \( \sigma \) < 0.065 if it is revised in the solution.

(e) Check for the compatibility of strain in both alternatives to verify whether the compression steel yielded or not and use the corresponding stress in the steel for calculating the forces and moments.

(g) Select the appropriate bar sizes.

If it is necessary to design a doubly reinforced rectangular prestressed continuous beam, alternative method 2(a) or 2(b) of Section 5.5 for singly reinforced beams can be followed. An assumption is made of an \( R \) value higher than the \( R \) value that is used for singly reinforced beams for selection of the first trial section. Since it is not advisable to use an \( A'_s \) value larger than \( 1A_s \) to \( 1A_s \), assume that \( R' = 1.3R \) to 1.5\( R \).

5.7.3 Example 5.6: Design of a Doubly Reinforced Beam for Flexure

A doubly reinforced concrete beam section has a maximum effective depth \( d = 25 \) in. (635 mm) and is subjected to a total factored moment \( M' = 9.4 \times 10^6 \) in.-lb (1062 kN·m), including its self-weight. Design the section and select the appropriate reinforcement at the tension and the compression faces to carry the required load. Given:

\[
\begin{align*}
f_c & = 4000 \text{ psi (27.58 MPa)} \\
f_y & = 60,000 \text{ psi (413.4 MPa)}
\end{align*}
\]

Minimum effective cover \( d' = 2.5 \) in. (63.5 mm)

Solution: Assume \( b = h = 14 \) in.; also assume \( c = 0.5c_b \) for the singly-reinforced part of the solution, where \( c_b \) = neutral axis depth for balanced strain (=6.005)

\[
A_s = 25 + \frac{2}{14} = 25.14 \text{ in}.^2 \quad (d' - d') = 25 - 2.5 = 22.5 \text{ in.}
\]

Alternatively, assume \( c = 0.32 \)

\[
\text{hence, } c = 0.32 \times 25.5 = 8.16 \text{ in.}
\]

\[
a = \frac{b}{c} = 0.85 \times 8.16 = 6.84 \text{ in.}
\]

Use \( a = 6.50 \) in. as sufficiently accurate for computing \( A_{sb} \)

\[
C = 0.85f_ybf = 0.85 \times 4000 \times 14 \times 6.94 = 330,344 \text{ lb} = T \text{ for the singly-reinforced part of the solution.}
\]

\[
T = 0.85f_ybf = 60,000A_s
\]

\[
A_s = 5.50 \text{ in.}^2
\]

\[
M_{sd} = A_s f_y\left(d' - \frac{d''}{2}\right) = 5.50 \times 60,000\left(25 - \frac{0.54}{2}\right) = 7,104,000 \text{ in.-lb}
\]

required \( M_s = \frac{9.4 \times 10^6}{0.90} = 10.44 \times 10^6 \text{ in.-lb} \)

\[
> M_{sd}, \text{ hence a doubly-reinforced section is needed.}
\]
Figure 5.19  Reinforcing details of the doubly reinforced beam in Ex. 5.6.

\[ M_{ci} = 10,440,000 \text{ in.-lb} - 7,104,900 = 3,335,100 \]

\[ \epsilon_c = 0.003 \left( \frac{d - d'}{d} \right) = 0.003 \left( \frac{8.16 - 2.5}{8.16} \right) = 0.00236 \text{ in./in.} \]

\[ \epsilon_s = \frac{f_s}{E_s} = \frac{60,000}{29 \times 10^6} = 0.00207 < \epsilon_c \]

hence the compression steel yielded.

Note that in cases where the strain is less than 0.005, namely, the section is in the transition zone of Fig. 5.5, a value of \( \phi \) lower than 0.30 for flexure has to be used for the final design moment, with a strain no less than 0.004 as a limit.

For the second part of the solution:

\[ M_{ci} = 3,335,100 = A' \times f'(d - d') \]

or

\[ 3,335,100 = A' \times 60,000(25 - 2.5) \]

\[ A' = \frac{3,335,100}{60,000 \times 22.5} = 2.47 \text{ in.}^2 \]

Total tension steel = 5.90 + 2.47 = 7.37 in.²

Use eight #9 bars in two layers = 8.00 in.² and two No. 6 and two No. 8 bars as compression steel = 2.46 in.²

Alternate Check of Stress in the Compression Reinforcement:

\[ \rho = \frac{8.00}{14 \times 25} = 0.0230 \]

\[ \rho' = \frac{2.46}{14 \times 25} = 0.0070 \]

Actual \( \rho = \rho' = 0.0230 - 0.0070 = 0.0160 \)

\[ (\rho - \rho') = \frac{0.037 f_d d}{f'_d d} \times \frac{87,000}{87,000 - f_d} \]

\[ = 0.037 \times 4.00 \times 4.00 \times 2.5 \times \frac{87,000 - 60,000}{60,000 \times 25.5} \times \frac{87,000}{87,000 - 60,000} = 0.0152 \]
5.8 Nonrectangular Sections

Actual \( \rho = \rho' > 0.0157 \), hence, compression reinforcement yielded, \( f' = f' = f_{cd} \)

\[
\sigma = \frac{(A_0 - A')d}{0.85 f' b} = \frac{(8.0 - 2.46) \times 60.000}{0.85 \times 4000 \times 14} = 6.98 \text{ in.}
\]

\[ e = \frac{a}{b} = \frac{6.98}{0.85} = 8.21 \text{ in.} \]

\[ c/d = \frac{0.21}{25.5} = 0.321 < 0.375 \text{, hence } \phi = 0.90 \text{ to give the required design moment } M_d = 9.4 \times 10^6 \text{ in.-lb.} \]

Alternatively,

\[ e = 0.003 \left( \frac{d - c}{c} \right) = 0.003 \left( \frac{25.5 - 7.92}{7.92} \right) = 0.007 > 0.005 \text{ in./in.} \]

hence, the section is tension-controlled and the tensile-side reinforcement \( (A_A = A_A') \) has yielded, \( \phi = 0.90 \).

Check for Minimum Reinforcement

\[ \rho_{min} = \frac{200}{400} = 0.500 = 0.0033 < \rho_{act} \]

or

\[ \rho_{min} = \frac{3V}{L} = \frac{3 \times 4000}{68.000} = 0.0031 < \rho_{act} \text{ O.K.} \]

Hence, adopt Section.

5.8 NONRECTANGULAR SECTIONS

T beams and L beams are the most commonly used flanged sections. Because slabs are cast monolithically with the beams as shown in Figure 5.20, additional stiffness or strength is added to the rectangular beam section from participation of the slab. Based on extensive tests and longstanding engineering practice, a segment of the slab can be considered to act as a monolithic part of the beam across the beam flange. In the case of composite sections, if the beam and slab are continuously shored during construction

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**Figure 5.20** T- and L-beams as part of a slab beam floor system (cross-section at beam midspan).
(supported continuously), the slab and beam can be assumed to act together in supporting all loads, including their self-weight. However, if the beam is not shored, the beam must carry its weight plus the weight of the slab while it hardens. After the slab has hardened, the two together will support the additional loads.

The flange width accepted for inclusion with the beam in forming the flanged section has to satisfy the following requirements:

**T Beams:**
- Effective overhang > \(8h_f\)
- Overhang width on each side > the clear distance to the face of the next web (> \(h_f\))
- Flange width \(b\) > \(1\) of supporting beam span = \(L\)

**Spandrel or Edge Beams (beams with a slab on one side only):**
- The effective overhang > \(6h_f\) or > \(\frac{h}{3}\) the clear distance to the next web (> \(h_f\)) or \(\frac{h}{3}\) the span length of the beam.
- Beams with overhang on one side are called \(L\)-beams.

### 5.9 ANALYSIS OF T AND L BEAMS

#### 5.9.1 General Principles

Flanged beams are considered primarily for use as sections at midspans, as shown in Figure 5.20. This is because the flange is in compression at midspan and can contribute to the moment strength of the midspan section. At the support, the flange is in tension; consequently, it is disregarded in the flexural strength computations of the support section. In other words, the support section would be an inverted doubly reinforced section having the compressive steel \(A'\) at the bottom fibers and tensile steel \(A_t\) at the top fibers.

Figure 5.21 shows an elevation of a continuous beam with sections taken at midspan and at the supports to illustrate this discussion.
The basic principles used for the design of rectangular beams are also valid for the flanged beams. The major difference between the rectangular and flanged sections is in the calculation of compressive force $C$. Depending on the depth of the neutral axis, $c$, the following cases can be identified.

**Case 1: Depth of Neutral Axis $c$ Less than Flange Thickness $h$ (Figure 5.22)**

This case can be treated similarly to the standard rectangular section provided that the depth $a$ of the equivalent rectangular block is less than the flange thickness. The flange width $b$ of the compression side should be used as the beam width in the analysis.

**Figure 5.22** T-beam section with neutral axis within the flange ($c < h$): (a) cross section; (b) strains; (c) stresses.
Referring to Figure 5.22 for force equilibrium, where $C$ is equal to $T$, gives

$$0.85f_{cb} = A_f$$

or

$$a = \frac{A_f}{0.85f_{cb}}$$

The nominal moment strength would thus be $M_f = A_f f$ ($d - a/2$). This expression is the same as that of Eq. 5.4 for the rectangular section. Since the force contribution of concrete in the tension zone is neglected, it does not matter whether part of the flange is in the tension zone.

**Case 2: Depth of Neutral Axis $c$ Larger than Flange Thickness $h_f$ (Fig. 5.23).** In this case, $c > h_f$, the depth of the equivalent rectangular stress block $a$ could be smaller or larger than the flange thickness $h_f$. If $c$ is greater than $h_f$ and $a$ is less than $h_f$, the beam could still be considered as a rectangular beam for design purposes. Hence the design procedure explained for case 1 is applicable to this case.

If both $c$ and $a$ are greater than $h_f$, the section has to be considered as a T-section. This type of T-beam ($a > h_f$) can be treated in a manner similar to that for a doubly reinforced rectangular cross section (Figure 5.23). The contribution of the flange overhang to the compression force is considered analogous to the contribution of imaginary compressive reinforcement. In Figure 5.23, the compressive force $C_c$ is equal to the average concrete strength $f$ multiplied by the cross-sectional area of the flange overhangs.

Thus $C_c = 0.85f_{cb} = 0.85f_c (b - b_c) h_f$, where $b_c$ is the overhang length on each side of the web. The compressive force $C_c$ is equal to a tensile force $T$, for equilibrium such that $T = (A_f 	imes f)$, where $A_f$, an imaginary compressive steel area whose force capacity is equivalent to the force capacity of the compression flange overhang. Consequently, an equivalent area $A_f$ of compression reinforcement to develop the overhang flanges would have a value of

$$A_f = \frac{0.85f_{cb}(b - b_c)h_f}{f_c}$$

(5.23)

For a beam to be considered as a real T-beam, the tension force $A_f f$ generated by the steel should be greater than the compression force capacity of the total flange area $0.85f_{cb}b_h$. Hence

$$a = \frac{A_f f}{0.85f_{cb}} > h_f$$

(5.24a)

Figure 5.23 Stress and strain distribution in flanged sections design (T-beam transfer):
(a) cross section; (b) strains; (c) transformed section; (d) part-1 forces; (e) part-2 forces.
or
\[ h_t < \frac{1.18 d}{f} \]  \hspace{1cm} (5.24b)

where \( \bar{u} = \frac{A_r}{b_t d} (f_t f_r) \).

The concrete stress block is, in reality, parabolic and extends to the neutral-axis depth \( c \). Consequently, from a theoretical viewpoint, if one were using a parabolic stress block, Eq. 5.24b for a T-beam can also be written as
\[ h_t < \frac{1.18 \bar{u} d}{b_t} \]  \hspace{1cm} (5.24c)

In order to ensure tension-controlled behavior, the ratio \( c/d \) is 3.375, and preferably less, such as a value of 3.0. In this manner, the strain \( \varepsilon \) in the tensile reinforcement is greater than 0.005 in./in. Find \( \varepsilon_t = 0.003 (d/r - 1) \). Once the strain \( \varepsilon_t \) is verified to be higher than 0.005, the nominal moment strength is computed as in Equation 5.27.

A strain-compatibility check is not needed since the imaginary steel area \( A_r \) is assumed to yield in all cases. To satisfy the requirement of minimum reinforcement so that the beam does not behave as nonreinforced, for positive reinforcement,
\[ \varepsilon_r = \frac{A_r}{b_t d} = \frac{201}{f_r} \leq \frac{3 \sqrt{f_r}}{f} \]  \hspace{1cm} (5.25a)

---

Photo 5.13 The Trump Towers under construction, Fifth Avenue, New York City. (Courtesy of Concrete Industry Board.)
For negative reinforcement and T sections with flanges in tension,

\[ \rho_{\text{min}} = \frac{\delta \sqrt{f}}{f} \geq \frac{200}{f} \]  

(5.25b)

It is to be noted that \( b_s \) is used in Eq. 5.25(a) instead of width \( b \), which is used in the case of singly or doubly reinforced beams.

As in the case of design and analysis of doubly reinforced sections, the reinforcement at the tension side is considered to be composed of two areas: \( A_s \), to balance the rectangular block compressive force on area \( b_s \), and \( A_r \), to balance the imaginary steel area \( A_s \). Consequently, the total nominal moment strength for parts 1 and 2 of the solution is

\[ M_n = M_{n1} + M_{n2} \]

(5.26a)

\[ M_{n1} = A_s f f \left( \frac{d - \frac{a}{2}}{2} \right) + A_r f f \left( \frac{d - \frac{a}{2}}{2} \right) \]

(5.26b)

\[ M_{n2} = A_s f f \left( \frac{d - \frac{b_s}{2}}{2} \right) + A_r f f \left( \frac{d - \frac{b_s}{2}}{2} \right) \]

(5.26c)

The design moment strength \( \phi M_n \), which has to be at least equal to the external factored moment \( M_n \), becomes

\[ M_n = \phi M_n = \phi \left[ \frac{A_s}{A_0} \frac{f f}{f} \left( \frac{d - \frac{a}{2}}{2} \right) + A_r f f \left( \frac{d - \frac{b_s}{2}}{2} \right) \right] \]

(5.27)

The flowchart in Figure 5.24 presents the sequence of calculations for the analysis of the T-beam. The following analysis example illustrates the nominal moment strength calculations for a typical precast T beam.

5.9.2 Example 5.7: Analysis of a T Beam for Moment Capacity

Calculate the nominal moment strength and the design ultimate moment of the precast T-beam shown in Figure 5.24 if the beam span is 30 ft (9.14 m). Given:

\[ f_s = 4000 \text{ psi (27.6 MPa)} \]

(41.4 MPa)

\[ f_c = 60,000 \text{ psi (413.4 MPa)} \]

Reinforcement area at the tension side:

(a) \( A_s = 4.0 \text{ in.}^2 \) (2580 mm²)

(b) \( A_s = 6.0 \text{ in.}^2 \) (3870 mm²)

Solution (a):

\[ A_s = 4.0 \text{ in.}^2 \]

\[ \rho_{\text{min}} = \frac{200}{60,000} = 0.0033 \]

\[ \rho = \frac{A_s}{b_s f_s} = \frac{4.0}{10 \times 18} = 0.022 > \rho_{\text{min}} \text{ O.K.} \]

\[ \mu = \frac{A_s f_s}{b_s f_s} = \frac{4.0}{18} \left( \frac{60,000}{4000} \right) = 0.083 \]

\[ e = \frac{1.14 A_s f_s}{b_s f_s} = \frac{1.14 \times 0.083 \times 18}{0.65} = 2.10 \text{ in.} \]

\[ < h_s = 2.5 \text{ in.} \]
Figure 5.24 Flowchart for the analysis of T- and L-beams.

Therefore, the beam can be analyzed as a rectangular beam using \( b, d, \) and \( A_c \).

\[
\sigma = \frac{A_c f_c}{0.85 \times \text{area}} = \frac{4.0 \times 60,000}{0.85 \times 4 \times 100 \times 40} = 1.76 \text{ ksi.}
\]

\[
\epsilon = \frac{1.76}{0.85} = 2.08 \text{ in.}
\]

\[
\epsilon_0 = 0.003 \left( \frac{\epsilon}{c} - 1 \right) = 0.003 \left( 2.08 - 1 \right) = 0.009 \text{ in.}
\]

\[
\epsilon = 0.015, \text{ hence tension-controlled and } f_c = f' = 0.90
\]
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Figure 5.25  Geometry, strain, and force distributions in the T-beam of Ex. 5.7:
(a) cross section; (b) strains; (c) stresses.

\[ M_s = A_s f \left( \frac{d}{2} - \frac{d}{2} \right) = 4 \times 60,000 \left( 18 - \frac{1.765}{2} \right) = 4,108,200 \text{ in.-lb} \]

\[ M_s = aM_s = 0.90 \times 4,108,200 = 3,697,380 \text{ in.-lb} (418 \text{ kN-m}) \]

Solution (b): \[ A_s = 6.0 \text{ in.}^2 \]

\[ \rho = \frac{6.0}{6} = 0.33 \gg \rho_{cr} = 0.003 \]

\[ \sigma = \frac{6.0 \times 60,000}{48 \times 18} = 0.125 \]

\[ c = \frac{1.18i^2d}{\beta_i} = \frac{1.18 \times 0.125 \times 18}{0.85} = 3.124 > (u_f = 2.5) \]

Therefore, the neutral axis is below the flange. The beam has to be treated as a T-beam or equivalent doubly-reinforced beam with imaginary compressive area \( A_{c'} \).

\[ A_{c'} = \frac{0.85(6 - 0.125)}{f_f} = \frac{0.85 \times 4,000(40 - 10)}{2.5} = \frac{60,000}{4.25} \text{ in.}^2 \]

\[ A = \frac{(A_s - A_{c'})f_f}{A} = \frac{(6.0 - 4.25) \times 60,000}{0.85} = 3.09 \text{ in.} \]

\[ c = \frac{3.09 \times 0.85}{3.64} = 3.64 \text{ in.} \]

\[ \epsilon_i = 0.003 \left( \frac{d - c}{c} \right) = 0.003 \left( \frac{18 - 3.64}{3.64} \right) = 0.12 \text{ in./in.} \]

\( >> 0.005 \), hence tensile-controlled plastic behavior, \( A_i = 0.90 \).
5.10 Trial-and-Adjustment Procedure for the Design of Flanged Sections

\[ M_e = (A_i - A_e) f_e \left( \frac{d - s}{2} \right) \]
\[ = (6.00 - 2.50) 60.000 \left( \frac{18 - \frac{3.00}{2}}{2} \right) = 1.727.780 \text{ in.-lb} \]
\[ M_e = A_e \left( \frac{b}{2} \right) - 4.25 \times 60.000 \left( \frac{18 - \frac{2.5}{2}}{2} \right) = 4.271.220 \text{ in.-lb} \]

Total moment due to both parts of the solution is \( M_e = M_{EC} + M_{AC} = 1.727.780 + 4.271.220 = 6.000 \text{ in.-lb} \)

\[ M_e = 6.000 \text{ in.-lb} = 6.000 \text{ in.-kip} \]
\[ M_e - \delta M_e = 5.9 \times 6.000 \text{ in.-lb} = 5.400 \text{ in.-lb} (610 \text{ kN-m}) \]

5.10 TRIAL-AND-ADJUSTMENT PROCEDURE FOR THE DESIGN OF FLANGED SECTIONS

The slab thickness \( h_s \) of the flange overhang is known at the outset since the slab is designed first. Also available is the external factored moment \( M_e \) at midspan. The trial-and-adjustment steps for proportioning the web of the beam section can be summarized as follows.

1. Choose a singly reinforced beam section that can resist the external factored moment \( M_e \) and the moment due to self-weight. Remember that a T-section or an L-section would have a smaller size or depth than a singly reinforced section.
2. Check whether the span/depth ratio is reasonable, between 2 and 18. If not, adjust the preliminary section.
3. Calculate the flange width on the basis of the criteria in Section 5.8.
4. Choose \( cd \) = 0.375 to ensure that \( e_i > 0.005 \text{ in.} \).
5. Determine if the neutral axis is within or outside the flange, where the neutral-axis depth \( e \leq b_y / b \) for rectangular singly reinforced sections.
   (a) If \( e < b_y \), the beam has to be treated as a singly reinforced beam with a width \( b \) equivalent to the flange width determined in step 3.
   (b) If \( e > b_y \), and the equivalent block depth \( a \) is also \( > b_y \), design as a T-beam or an L-beam, as the case may be.
6. Find the equivalent compressive steel area \( A_{seq} \) for the flange overhang and analyze the assumed section as in Ex. 5.7(b). Calculate the nominal resisting capacities \( M_{Ed} \) and \( M_{Ed} \).
7. Repeat steps 4 and 5 until the calculated \( 6M_e = \delta (M_{Ed} + M_{Ed}) \) is close in value to the factored moment \( M_e \), and verify that the assumed self-weight of the web is correct.
8. Alternatively, the first trial section can be chosen using a moment factor \( R^* \geq R \) in step 3(a) of Section 5.5 for singly reinforced beams such that \( R^* = 1.35R - 1.50R \).
   Select a trial section depth from \( M_e = Rd_b^2 \) and proceed to analyze the section.

5.10.1 Example 5.8: Design of an End-span L-Beam

A roof-garden floor is composed of a monolithic one-way slab system on beams as in Figure 5.26. The effective beam span is 35 ft (10.67 m) and all beams are spaced at 7 ft 6 in. (2.29 m) clear span. The floor supports a 6-ft 4-in. (1.92 m) depth of soil in addition to its self-weight. Assume also that the slab edges support a 12-in.-wide 7.5-ft-span wall weighing 660 lb per linear foot. Design the midspan section of the edge spanned L beam AB assuming that the moist soil weighs 128 lb/ft (2.56 kN/m²). Given:

\[ f'_c = 3000 \text{ psi} (20.7 \text{ MPa}) \text{, normal-weight concrete} \]
\[ f' = 60.000 \text{ psi} (413.7 \text{ MPa}) \]
Given:

- Effective span = 35 ft
- $f'_c = 3,000$ psi, normal weight
- $f'_t = 60,000$ psi
- Soil weight = 125 lb/ft$^2$

Solution:

**Slab Design**

Effective slab span = 7.5 ft

Weight of slab = 6.33 x 7.5 = 91, say 800 psf (38.3 kPa)

No appreciable live load is assumed, as the structure supports a roof garden with deep soil fill.

Assume a slab thickness $t = 4$ in. (101 mm)

$$ t = \frac{4}{.625} \times 150 = 50 \text{ psf} $$
\[ d = \frac{1}{2} \text{ in. cover} + \frac{1}{2} \text{ dia. of No. 4 bars} \]
\[ = 0.30 + 1.0 = 3.0 \text{ in.} \]

Factored load intensity

\[ w_c = 1.2(200 + 90) = 390 \text{ lb/ft}^2 (49 \text{ kPa}) \]

(assuming the live load relatively insignificant.)

From the ACI Code, the negative moment for the interior support of a continuous slab is

\[ -M_{r} = -\frac{w_c L^2}{12} = -\frac{102(7.5)^2}{12} = 62.591 \text{ in.-lb} \]

The required slab negative moment strength

\[ -M_s = \frac{62.591}{c} = 69.346 \text{ in.-lb} \]

Assume \( A_i \) yield to be subsequently verified

\[ M_s = A_i f_c \left( d - \frac{a}{2} \right) \]

Assume that \( (d - a/2) = 0.9d = 0.9 \times 3.0 = 2.7 \text{ in.} \)

\[ A_i = \frac{69.346}{60,000 \times 3.7} = 0.429 \text{ in.}^2 \text{ on a I-cm. strip} \]

Try No. 5 bars at 7.5 in center to center. \( A = 0.50 \text{ in.}^2 \).

Actual \( d = \frac{0.50}{12 \times 3} = 0.0039 \)

\[ \rho_{cr} = \frac{1}{\sqrt{f_t}} = 0.0027 \]

also \( \frac{200}{f_t} = 0.0033 < 0.0139 \), hence, O.K.

\[ a = \frac{A_i f_c}{0.65 f_t} = \frac{0.50 \times 60,000}{6.5 \times 3,000 \times 12} = 0.98 \text{ in.} \]

\[ c = \frac{\rho_{cr}}{f_t} = 0.0027 \times 3.7 = 1.15 \text{ in.} \]

\[ \varepsilon = \frac{\rho_{cr}}{\rho_{cr}} = \frac{(d - a/2)}{c} = \frac{0.0039 \times (3.0 - 1.15)}{0.98} = 0.003 \text{ in./in.} \]

hence, tension-controlled and \( f = f \)

\[ M_s = A_i f_c \left( d - \frac{a}{2} \right) = 0.50 \times 60,000 \left( 3.0 - \frac{0.98}{2} \right) \]

\[ = 75,300 \text{ in.-lb} > \text{ required } M_s = 69.346 \text{ in.-lb} \text{ O.K.} \]

Similarly, for the positive moment. \( +M_r = w_c L^2 / 16 \), requiring No. 5 bars at 10 in. c-c. Use No. 5 bars at 7 in. center-to-center main negative reinforcement (15.0-mm diameter at 190.5-mm spacing) and No. 5 bars at 10 in. center-to-center for main positive reinforcement.

Temperature steel \( = 0.0018 \times 12 \times 4.0 = 0.0864 \text{ in.}^2 \)

\[ Temperature \ text{ steel } = 0.0018 \times 12 \times 4.0 = 0.0864 \text{ in.}^2 \]
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Maximum allowable spacing = 3.0 \times 5 = 20 in. or 18 in. max.

Use #3 bars at 15 in. (0.088 in² for 0.53 mm diameter at 380 mm spacing)

**Beam Web Design**

Figure 5.27 gives the forces acting on the flanged section.

In order to choose a trial web section, assume that $h = l/16$ for deflection, or

$$h = \frac{35.0 \times 12}{18} = 23.33 \text{ in.}$$

Assume that $h = 26$ in. (660 mm), $a = 22.8$ (572 mm) and $b = 14$ in. (356 mm).

Load area on L-beam AB = $7.5\frac{2}{3} + 14\frac{12}{12} = 4.92$ ft.

Superimposed working load

$$\Delta_{w} = (4.92 - 1.0) \times 800 = 3136 \text{ lb/ft}$$

Slab weight:

$$\Delta_{s} = \frac{4.92}{12} \times 150 \times 4.92 = 246 \text{ lb/ft}$$

Weight of beam web:

$$\Delta_{b} = \frac{4(7.5 - 4)}{144} \times 150 = 321 \text{ lb/ft}$$

7-ft wall weight = 840 lb/ft.

Total service dead load = $3136 + 246 + 321 + 840 = 4543$ lb/ft

Factored load:

$$\Delta_{w} = 1.2 \times 4543 = 5452 \text{ lb/ft}$$

Use the ACI Code positive moment factor for unrestrained discontinuous end for this case, namely, $M_{n} = \Delta_{w} Q/11$

Apply a factored moment:

$$M_{n} = \frac{\Delta_{w} Q}{11} = \frac{5452(35.0)}{11} \times 12 = 7.385,851 \text{ in.-lb}$$

assuming reinforcement $A_{s}$ to have yielded, to be subsequently verified, $d = 0.9W$.

$$M_{n} = \frac{M_{n}}{d} = 7,385,851 \times 0.9 = 8,095,384 \text{ in.-lb} (915 \text{ kN-m})$$

To determine whether the beam is an actual L-beam or not, it is necessary to find if the neutral axis falls outside the flange. Consequently, the area of the tension steel $A_{t}$ has to

---

**Figure 5.27** Forces and stresses in L-beams: (a) cross section; (b) strain diagram; (c) transformed section; (d) part-1 forces; (e) part-2 forces.
be assumed. If rectangular section is initially assumed with an appropriate moment arm
\(j / d = 0.85d\)

\[ M_b = A_f / jd \quad \text{or} \quad 0.095394 = A_f \times 60,000 \times 19.3 \]
\[ A_f = \frac{0.095394}{60,000 \times 19.3} = 6.99 \text{ in.}^2 \]

Assume seven 7/3 bars in two layers = 7.0 in.7 (4515 mm7)

\[ \rho = \frac{A_f}{b_d} \]
\[ b = b_r + \frac{l_f}{12} = 14 + \frac{35 \times 12}{12} = 49 \text{ in.} \]
\[ b = b_r + 6h_r = 14 + 6 \times 4.0 = 38 \text{ in. (985 mm)}, \text{ Controls.} \]
\[ \rho = \frac{38}{22.5} = 0.0082 \]
\[ \bar{w} = \rho \bar{f} = 0.0082 \times \frac{60,000}{3,000} = 0.164 \]

Depth of the neutral axis

\[ e = \frac{1.18\bar{w}d}{\rho} = \frac{1.18 \times 0.164 \times 12.5}{0.85} \]

or

\[ e = 5.12 \text{ in.} > 4.0 \text{ in.} \]
\[ a = \rho e - 0.85 \times 5.12 = 4.35 > 40 \]

Hence, the section is an L-beam since the neutral axis is below the flange, as shown in Figure 5.27.
Figure 5.2b  Midspan section flexural reinforcement details for beam AB of Ex. 5.8.

\[ M_e = 0.9 \times 8,475,996 = 7,628,006\text{ in} \cdot \text{lb} \]

Actual factored moment:

\[ M_e = 7,205,855\text{ in} \cdot \text{lb} < 7,586,006\text{ in} \cdot \text{lb} \]

Adopt the design. Flexural reinforcement details for the L-beam AB are shown in Figure 5.2b

5.10.2 Example 5.9: Design of an Interior Continuous Floor Beam for Flexure

Design a rectangular interior beam having a clear span of 25.1 (6.4 m) and carrying a working live load of 4000 lb per lineal foot (62 kN/m) in addition to its self-weight and slab weight, as shown in Figure 5.2a. Assume the beam to have a 4 in. (10 cm) slab cast monolithically with it.
5.10 Trial-and-Adjustment Procedure for the Design of Flanged Sections

\( f_c = 4000 \text{ psi (27.6 MPa)}, \text{ normal-weight concrete} \)
\( f_y = 60,000 \text{ psi (413.4 MPa)} \)
\( d'' = 2.5 \text{ in.} \)

Assume no wind or earthquake.

Solution: Assume that the web self-weight = 400 lb/ft.

Factored load \( w_n = 1.2 \times 400 + 1.6 \times 9000 \)

\( = 14,400 \text{ lb/ft} \)

The positive factored moment \( M_e \) for interior midspan lower fibers (ACI) is

\( + M_e = \frac{w_n l^2}{16} + \frac{(14,400 \times (25.0))^2}{16} \times 12 = 6,750,000 \text{ in.-lb (763 kN-m)} \)

The negative factored moment \( M_e \) as support (tension at top fibers) is

\( -M_e = \frac{14,400 (25.0)^2}{11} \times 12 = 9,818,182 \text{ in.-lb (1190 kN-m)} \)

Section at Midspan (T-Beams)

Assume that \( b_e = 14 \text{ in. (0.3556 m)} \).

\( b \times 16 \times 4 + 14 = 78 \text{ in.} \)

\( \frac{25 \times 12}{4} = 75 \text{ in.} \)

Therefore, overhang \( > \) half clear distance to next web not known.

In order to obtain a reasonable area of steel at the tension side and also to determine if the beam section is flanged, find for a first trial an \( A_s \) for a rectangular section that can resist \( M_e = 7,500,000 \text{ in.-lb} \).

For deflection purposes assume that interior partitions would be damaged by excessive deflection; hence use

\( d'' = \frac{f}{12} = 25.0 \times 12 = 25 \text{ in. (635 mm)} \)

For a self-weight of 400 lb/ft, try \( b_e = 14 \text{ in. (254 mm)} \) and \( h = 28.0 \text{ in. (711 mm)} \).

\( \text{self-weight} = \frac{14 \times 28}{144} \times 150 = 460 \text{ lb/ft}; \text{ no need to revisit the moments.} \)

Midspan section \( d = 28 = \left( d'' + \text{shirrup} + \frac{1}{2} \text{ bar dia.} \right) \)

\( 28 = (15.5 + 0.5 + 1.27/2) \)

\( 25.3 \text{ in.} - d'' \)

Support section \( d = 25.3 - \frac{1.37}{3} = 24.67 \text{ in.} \)
Chapter 5  Flexure in Beams

From before, required \( M_e = 7,500,000 \text{ in.-lb} \)

Assume that moment arm \( d = 0.9d = 0.9 \times 25.3 = 22.7 \text{ in.} \)

\[
A_e = \frac{M_e}{f_e \times 0.9d} = \frac{7,500,000}{60,000 \times 22.7} = 5.51 \text{ in.}^2
\]

Assuming four No. 10 bars at midspan. \( A_e = 4 \times 1.27 = 5.08 \text{ in.}^2 \) to be verified

\[
p = \frac{5.08}{5.08} = 1.00 = 0.027
\]

Check for the neutral-axis position

\[
\frac{E}{F} = \frac{p}{F} \times \frac{0.027 \times 60,000}{4000} = 0.040
\]

\[
c = \frac{1.1NF}{P} = 1.18 \times 0.040 = 0.85
\]

\[
p_c = \frac{1.42}{25.3} = 1.76 \text{ in.}, \text{ tensile steel}
\]

\[
e = \frac{\beta_c}{1.42} = 0.58 \times 1.42 = 2.1 \text{ in.}
\]

\[
f = 0.89 < 0.375, \text{ hence, tension-controlled and } e = 0.99
\]

The nominal resisting moment \( M_c = A_{c,f}(1 - \alpha f) \)

\[
M_c = 5.08 \times 60,000 \left( 25.3 - \frac{321}{3} \right) = 7,527,036 \text{ in.-lb (850 kN-m)}
\]

The actual \( M_c \) is larger than the required \( M_e = 7,500,000 \text{ in.-lb} \).

\[
A_r = \frac{5.08}{14 \times 25.3} = 0.0143 > \frac{700}{F} = 0.01033 \text{ and } \frac{1600}{F} = 0.01333 \text{, hence O.K.}
\]

Adopt a midspan section with \( b_e = 14 \text{ in.} (355.6 \text{ mm}), h = 28 \text{ in.} (686 \text{ mm}), d = 250 \text{ in.} \\
(635 \text{ mm}), \text{ and } A_e = 4 \text{ bars (diameter 32 mm)}.

Section at support (doubly reinforced rectangular section)

This section is subjected to moments similar to the moments acting on the section in 
Ex. 5.6 and has the same cross-sectional dimensions. The required nominal moment of resis-
tance \( M_e = 10,909,091 \text{ in.-lb (1212 kN-m)} \). Assume that two No. 10 bars extend from the 
midspan to the support and \( 2.54 \text{ in.} \\
M_{cs} = A_{c,f}(1 - \alpha f) \), assuming that \( A_{e} \) has yielded since the area is so close to \( 2.48 \text{ in.}^2 \) in Ex. 5.6, to be subsequently verified, or .

Support section \( d = 24.67 \text{ in.} \) from before.

\[
M_{cs} = 2.54 \times 60,000(24.67 - 2.5) = 3,378,708 \text{ in.-lb}
\]

\[
M_e = 10,909,091 - 3,378,708 = 7,530,383 \text{ in.-lb}
\]

Assume that moment arm \( d = 0.85d = 0.85 \times 24.67 = 21.0 \text{ in.} \)

\[
\text{trial } A_e = \frac{7,530,383}{60,000 \times 21.0} = 5.98 \text{ in.}^2
\]

\[
\text{total } A_e = A_{e1} + A_{e2} = 5.98 + 2.54 = 8.52 \text{ in.}^2
\]

Try seven No. 10 bars (50.8 mm) in two layers. \( A_{e1} = 8.89 \text{ in.}^2 \)

\[
A_{e1} = 8.89 - 2.54 = 6.35 \text{ in.}^2
\]

\[
p = \frac{6.35}{14 \times 24.67} = 0.0184 > 0.0155
\]

from Ex. 5.6; hence the assumption that \( A_{e} \) has yielded is valid.

\[
a = \frac{(A_e - A_{c,f})d}{0.85} = \frac{6.35 \times 60,000}{0.85 \times 4000 \times 14} = 8.00 \text{ in.}
5.10 Trial-and-Adjustment Procedure for the Design of Flanged Sections

\[ c = \frac{\sigma}{f'} = \frac{8.00}{0.85} = 9.41 \text{ in.} \]

\[ \frac{c}{d} = \frac{9.41}{23.3} = 0.372 < 0.375, \text{ O.K.} \]

Hence

\[ \phi = 0.90 \]

\[ M_{re} = (A_e - A_d) \left( \frac{d - \frac{c}{2}}{2} \right) = 6.35 \times 60.000 \left( \frac{24.67 - \frac{8.00}{2}}{2} \right) = 7,875,270 \text{ in.-lb} \]

Available \[ M_e = M_{re} + M_{ce} = 7,875,270 + 3,378,708 = 11,253,978 \text{ in.-lb} > \text{required} - M_e = 9,009,601 \text{ in.-lb} \]

Hence adopt the design.

Design \[ M_e = 0.90 \times M_e = 0.90 \times 11,253,978 = 10,125,680 \text{ in.-lb} \]

Therefore, use seven No. 10 bars on top at the support in two layers and two No. 10 bars at the bottom. Stirrups of the section at the support. Provide closed stirrups to tie the tension and the compression steel at the support. It is to be noted that bar sizes larger than No. 11 should be avoided where possible in superstructure beams, because they are difficult to cut and less efficient for crack control.

For the design to be complete, diagonal tension capacity, serviceability and bar development checks have to be made, as discussed in Chapters 6, 8, and 10. Details of the reinforcement over the span are shown in Figure 5.30.

![Diagram of Reinforcement Arrangement](image)

Figure 5.30 Reinforcement arrangement for the continuous beam in Ex. 5.9: (a) sectional elevation (not to scale); (b) midspan section B-B; (c) support section A-A.
5.11 CONCRETE JOIST CONSTRUCTION

This type of construction comprises closely spaced reinforced concrete joists which are noncohesively built with thin concrete slabs. Standard pan forms for joist construction have 20 or 30 in. clear space between the ribs at the lower fibers as shown in Figure 5.31. The depth of the rib varies between 10 and 20 in. in 2-in. increments.

An advantage of such construction systems is their effectiveness in spanning longer openings and in reducing the dead loads by essentially eliminating concrete in tension in the space between the ribs below the neutral axis. While the joint ribs are essentially beam ribs, some distinction is made in the ACI Code. The differentiation is because of the openness of the joint ribs in a floor system resulting into a good redistribution of local overloads to adjacent members. Hence, a higher shear capacity and a less stringent concrete cover requirement are allowed, provided that the dimensional requirements of Figure 5.31 are adhered to.

The minimum concrete cover for the joints if not exposed to weather or in contact with the ground is 1 in. for No. 11 size bars and smaller unless fire requirements govern. A 1-in. cover may be adequate for a 1-h rating and 1-in. cover for a 2-h rating. Hence, the design engineer should apply the local fire building code requirements for determining the cover thickness to be used in the design.

5.11.1 Example 5.10: Design of Reinforced Concrete Joist Sections

Compute the negative and positive nominal moment strengths \(M_c\) of the joist section shown in Figure 3.32. Given

\[
\begin{align*}
\sigma_f &= 4,000 \text{ psi} (27.6 \text{ MPa}) \\
\sigma_y &= 60,000 \text{ psi} (413 \text{ MPa}) \\
A_t &= 2 \times 5 \text{ at bottom and } 6 \times 4 \text{ in the flange} \\
A_c &= 2 \times 0.31 = 0.62 \text{ in}^2 (400 \text{ mm}^2)
\end{align*}
\]

Assume cover \(d = 1.25 \text{ in. to center of reinforcement both at top and bottom.}\)

Solution: (a) Positive moment strength \(-M_c\):

\[
\begin{align*}
d &= 16.0 + 3.5 - 1.25 = 18.25 \text{ in. (464 mm)} \\
\delta_t &= 2 \times 0.31 = 0.62 \text{ in.} (16 \text{ mm})
\end{align*}
\]
5.12 SI Expressions and Example for Flexural Design of Beams

\[ d = \frac{A_f}{0.85f_{\text{c}}} = \frac{0.62 \times 60,000}{0.85 \times 4,000 \times 36} = 0.304 \text{ in.} < h_y = 3.3 \text{ in. (89 mm)} \]

\[ c = \frac{D}{B} = 0.35 \text{ in.} \]

The neutral axis is inside the flange, hence consider it a singly-reinforced beam.

\[ e_n = 0.900 \left( \frac{d - c}{c} \right) \]

\[ = 0.900 \left( \frac{18.25 - 0.35}{0.35} \right) \]

\[ = 0.15 > 0.005 \]

hence, tension-controlled

\[ +M_e = A_f \left( d - \frac{d}{2} \right) \]

\[ = 0.62 \times 60,000 \left( 18.25 - \frac{0.35}{2} \right) = 673,264 \text{ in.-lb} = 56 \text{ ft.-kip (76 kN-m)} \]

(b) Negative moment strength, \(-M_e\).

Compression side width \(b = 6.0 \text{ in.}\)

\[ A_c = 6 \times 0.20 = 1.20 \text{ in.}^2 (774 \text{ mm}^2) \]

\[ a = 1.20 \times 60,000 \]

\[ c = a \left( \frac{0.35}{0.35} \right) = 4.15 \text{ in.} \]

\[ a = 3.33 \left( \frac{0.33}{18.25} \right) = 0.192 > 0.075 \]

hence, section tension-controlled

\[ -M_e = 1.20 \times 60,000 \left( 18.25 - \frac{3.33}{2} \right) = 1,186,920 \text{ in.-lb} = 99.0 \text{ ft.-kip (134 kN-m)} \]

5.12 SI EXPRESSIONS AND EXAMPLE FOR FLEXURAL DESIGN OF BEAMS

\[ E_c = w_c^{0.043} \sqrt{f_c} \text{ MPa} \]

where \( w_c = 1500 \text{ to } 2500 \text{ kg/m}^3 \) (90 to 155 lb/ft\(^3\)). For standard, normal-weight concrete, \( w_c = 2400 \text{ kg/m}^3 \) to give \( E_c = 29,700 \text{ MPa} \).
\( E_r = 206,000 \text{ MPa} \)

modulus of rupture \( f_r = 0.7 \sqrt{f} \)

\[ A_{n,\text{free}} = \sqrt{f_r^2 b d / 4f_r} \]

where \( f_r \) is in MPa units.

For cantilevers and negative moment zone,

\[ A_{n,\text{free}} = \frac{\sqrt{f_r^2}}{2f_r} b d \]

\[ \beta = 0.85 - 0.008 (f_r - 30) \]

The value of \( \beta \) for strengths above 30 MPa should be reduced continuously at the rate of 0.008 for each 1 MPa of strength in excess of 30 MPa, but \( \beta \) cannot be less than 0.65.

Spacing of reinforcement for structural slabs and footings in the direction of the span should not exceed 3 × slab thickness or 450 mm.

For singly reinforced beams, from Eq. 5.4 or 5.5,

\[ M_u = A_p f_r^2 d/2 \]

or \[ w = \frac{0.85}{f_r^2} \left( 0.003 E_r - 0.003 E_r + f_r \right) \]

where \( f_r \), \( f_r^2 \), and \( E_r \) are in MPa units and \( r = b d \)

For doubly reinforced beams, from Eq. 5.22,

\[ M_u = (A_p f_r - A_p f_r) \left( d - \frac{a}{2} \right) + A_p f_r (d - d') \]

or

\[ M_u = 0.85 f_r b d \left( d - \frac{a}{2} \right) + A_p f_r (d - d') \]

\[ f'_r = 0.003 E_r \left[ 1 - \frac{0.85 f_r f'_r d'}{(p - p') d} \right] \]

For flanged sections, from Eq. 5.27

\[ M_u = (A_p - A_p) f_r \left( d - \frac{a}{2} \right) + A_p f_r \left( d - \frac{d'}{2} \right) \]

where

\[ A_p = \frac{0.85 f'_r (b - b_d) y_d}{f_r} \]

and

\[ a = \frac{A_p - A_p}{0.85 f'_r b_d} \]
5.12 SI Expressions and Example for Flexural Design of Beams

For tension-controlled state,

\[ \sigma_0 = \left( \frac{600}{600 + f_e} \right) \text{where } f_e \, (\text{MPa}) \]
\[ \phi = 0.36 + \frac{200}{c/d} \]
\[ 0.65 \leq \phi \leq 0.48 + 83 \, \varepsilon \leq 0.90 \]
\[ \varepsilon = 0.003 \left( \frac{d - c}{c} \right) \]
\[ \varepsilon' = 0.003 \left( \frac{d - d'}{c} \right) \]

5.12.1 Example 5.11

Solve Ex. 5.3 using SI units and the strain limit approach.

**Solution:**

\[ f_e' = 27.6 \, \text{MPa} \]
\[ f_e = 414 \, \text{MPa} \]
\[ w_e = 24.1 \, \text{kN/m} \]
\[ l_e = 9.14 \, \text{m} = 9140 \, \text{mm} \]

Concrete unit weight = 23.6 kN/m\(^3\)
\[ = 23.6 \times 10^{-3} \, \text{N/mm}^3 \]

\[ P_s = \text{N/m}^2 \]

MPa = N/mm\(^2\)

Assume a minimum thickness from the ACI deflection table
\[ \frac{l_e}{16} = \frac{9.14}{16} = 0.571 \, \text{m} = 571 \, \text{mm} \]

For the purpose of estimating the preliminary self-weight, assume a total thickness \( t = 700 \, \text{mm} \), effective depth \( d = 600 \, \text{mm} \), and width of beam \( b = 300 \, \text{mm} \).

Beam self-weight
\[ = 300 \times 700(23.6 \times 10^{-3}) \, \text{kN/m} \]
\[ = 4956 \, \text{kN/m} = 49.56 \, \text{kN/m} \]

Factored load
\[ w_e = 1.2D + 1.67t = 1.2 \times 4.96 + 1.67 \times 24.1 \, \text{kN/m} = 44.5 \, \text{kN/m} \]

Required factored moment \( M_e = \frac{24439.14}{8} = 465 \, \text{kN.m} \)

\[ \text{Maximum moment strength} \]
\[ = 517 \, \text{kN.m} \]

Assuming it is tension-controlled with \( \phi = 0.90 \) to be subsequently verified.

\[ d_e = 600 \, \text{mm} = 60 \, \text{cm} \]
Chapter 5 Flexure in Beams

Assume $c = 0.27 < 0.375$, hence, tension-controlled.

\[ c = 0.27 \times 600 = 162 \text{ mm} \]

\[ a = b = c = 0.85 \times 162 = 138 \text{ mm} \]

\[ C = 0.85 \times a = 0.85 \times 162 \times 300 = 18 = 971,244 \text{ Newtons} \]

\[ \text{trial } A_t = 971,244/444 = 2,236 \text{ (2 No. 30M + One No. 23M bars = 2,400 mm$^2$)} \]

\[ s = 0.003 \left( \frac{d - c}{c} \right) = 0.003 \left( \frac{600 - 162}{162} \right) = 0.0081 > 0.003 \]

hence, the beam is tension-controlled; \( f_s = f, \) and \( \phi = 0.90. \)

\[ \rho = \frac{2400}{300 \times 600} = 0.0133 \]

\[ \rho_{\text{max}} = \frac{\sqrt{2}}{45} = \frac{\sqrt{2}}{45} \]

\[ \sigma = \frac{A_{fs}}{A} = \frac{2400 \times 414}{0.85 \times 22.6 \times 300} = 141 \text{ mm} \]

Therefore, the available nominal moment strength is:

\[ M_u = A_{fs} \left( d - \frac{d}{2} \right) = 2400 \times 414 \left( 600 - \frac{141}{2} \right) = 526 \times 10^3 \text{ N-mm} > \text{ required } M_u = 517 \text{ kN-m} \]

Adopt the section. Use two No. 30M and one No. 35M metric bars \( = 2 \times 700 + 1000 = 2400 \text{ mm}^2. \)

Note that the designed section resists a slightly larger moment than the required moment:

\[ \text{Percent Overdesign} = \frac{526 - 517}{517} = 1.7\% \]

SELECTED REFERENCES


5.7 AIT Committee 318, "Building Code Requirements for Structural Concrete (ACI 318-95) and Commentary (ACI 318R-95), American Concrete Institute, Farmington Hills, Ml, 2005, pp. 442.

5.8 ACI Committee 540, ACI Design Handbook, Special Publication SP-17 (97), American Concrete Institute, Farmington Hills, Ml, 1997, 482 pp.


5.1. For the beam cross-section shown in Fig. 5.33 determine whether the failure of the beam will be initiated by crushing of concrete or yielding of steel. Given:

- $f_c = 4000 \text{ psi (27.6 MPa)}$ for case (a), $A_s = 10 \text{ in.}^2$
- $f_y = 7000 \text{ psi (48.3 MPa)}$ for case (b), $A_s = 5 \text{ in.}^2$
- $f_y = 60,000 \text{ psi (414 MPa)}$

Also determine whether the section satisfies ACI Code requirements.

![Figure 5.33](image)

5.2. Calculate the nominal moment strength of the beam sections shown in Fig. 5.34. Given:

- $f_c = 3000 \text{ psi (20.7 MPa)}$ for case (a)
- $f_y = 6000 \text{ psi (41.4 MPa)}$ for case (b)
- $f_y = 60,000 \text{ psi (414 MPa)}$

![Figure 5.34](image)

5.3. Calculate the safe distributed load intensity that the beam shown in Fig. 5.35 can carry. Given:
5.4. Design a one-way slab to carry a live load of 100 psf and an external dead load of 50 psf. The slab is simply supported over a span of 12 ft. Given:

\( f_c = 4000 \text{ psi (27.6 MPa)}, \) normal-weight concrete
\( f_y = 60,000 \text{ psi (414 MPa)} \)

5.5. Design the simply supported beams shown in Fig. 5.36 as rectangular sections. Given:

\( f_c = 5000 \text{ psi (34.5 MPa)}, \) normal-weight concrete
\( f_y = 60,000 \text{ psi (414 MPa)} \)

5.6. Check whether the sections shown in Fig. 5.37 satisfy ACI 318 Code requirements for maximum
Problems for Solution

57. Compute the stresses in the compression steel, \( f_s \), for the cross section shown in Fig. 5.38. Also compute the neutral/moment strength for the section in part (b). Given:

- \( f'_c = 9000 \) psi (62 MPa), normal-weight concrete
- \( f_c = 6000 \) psi (41 MPa)

![Figure 5.37](image)

58. Calculate the ultimate moment capacity of the beam sections of Problem 5.2. Assume two No. 4 bars for compression reinforcement.

59. Solve Problem 5.3 if two No. 8 bars are added as compression reinforcement. Assume \( d' = 3.0 \) in.
5.10. At failure, determine whether the present sections shown in Fig. 5.39 will act similarly to rectangular sections or as flanged sections. Given:
\[ f' = 4000 \text{ psi (27.6 MPa)} \], normal-weight concrete
\[ f_c = 60,000 \text{ psi (414 MPa)} \]

![Diagram of beams and sections](image)

Figure 5.39

5.11. Check whether the sections of Problem 5.10 satisfy ACI Code requirements.
5.12. Calculate the nominal moment strength of the sections shown for Problem 5.10.
5.13. Repeat Problem 5.5 using a T-section instead of a rectangular section. Use a flange thickness of 3 in. (76.2 mm) and a flange width of 30 in. (762 mm).
5.14. Using the details of Problem 5.4, design a reinforced concrete T-beam for the slab floor system shown in Fig. 5.40. The floor area is 30 ft x 60 ft (9.14 m x 18.29 m) with an effective T-beam span of 30 ft (9.14 m).

![Diagram of T-beam](image)

Figure 5.40 Plan of one-way slab floor system.
6.1 INTRODUCTION

This chapter presents procedures for the analysis and design of reinforced concrete sections to resist the shear forces resulting from externally applied loads. Since the strength of concrete in tension is considerably lower than its strength in compression, design for shear is of major importance in concrete structures.

The behavior of reinforced concrete beams at failure in shear is distinctly different from their behavior in flexure. They fail abruptly without sufficient advanced warning, and the diagonal cracks that develop are considerably wider than the flexural cracks. The accompanying photographs show typical beam shear failure in diagonal tension as discussed in the subsequent sections. Because of the brittle nature of such failures, the designer has to design sections that are adequately strong to resist the external factored shear loads without reaching their shear strength capacity. Shear is also a significant parameter in the behavior of brackets, corbels, and deep beams. Consequently, the design of these elements is also discussed in detail.

Photo 6.1 Water Tower Place, Chicago. (Courtesy of Portland Cement Association.)

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Photo 6.3 Typical diagonal tension failure at rupture load level. (Test by Newy et al.)

Photo 6.3 Simply supported beam prior to developing diagonal tension crack
(From: Newy et al.)
6.2 Behavior of Homogeneous Beams

Consider the two infinitesimal elements $A_1$ and $A_2$ of a rectangular beam in Figure 6.1a made of homogeneous, isotropic, and linearly elastic material. Figure 6.1b shows the bending stress and shear stress distributions across the depth of the section. The tensile normal stress $f$ and the shear stress $\tau$ are the values in element $A_i$ across plane $x_i - y$, at a distance $y$ from the neutral axis. From the principles of classical mechanics, the normal stress $f$ and the shear stress $\tau$ for element $A_i$ can be written as

$$ f = \frac{M_y}{I} $$

and

$$ \tau = \frac{V A y}{I b} $$

(6.1)

(6.2)

where $M$ and $V$ = bending moment and shear force at section $a_i - a_j$,

$A$ = cross-sectional area of the section at the plane passing through the centroid of element $A$,

$y$ = distance from the element to the neutral axis

$v$ = distance from the centroid of $A$ to the neutral axis

$I$ = moment of inertia of the cross section

$b$ = width of the beam

Figure 6.2 shows the internal stresses acting on the infinitesimal elements $A_1$ and $A_2$. Using Mohr's circle in Fig. 6.2b, the principal stresses for element $A_1$ in the tensile zone below the neutral axis become

$$ f_{(\max)} = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + \nu^2} \quad \text{principal tension} $$

(6.3a)

$$ f_{(\min)} = \frac{f}{2} - \sqrt{\left(\frac{f}{2}\right)^2 + \nu^2} \quad \text{principal compression} $$

(6.3b)
Figure 6.1 Stress distribution for a typical homogeneous rectangular beam.

\[
\tan \theta_{\text{max}} = \frac{v}{f/2}
\]

**6.3 BEHAVIOR OF REINFORCED CONCRETE BEAMS AS NONHOMOGENEOUS SECTIONS**

The behavior of reinforced concrete beams differs in that the tensile strength of concrete is about one-tenth of its strength in compression. The compressive stress \( f \) in element \( A_2 \) of Fig. 6.2c above the neutral axis prevents cracking because the maximum principal stress in the element is in compression. For element \( A_1 \) below the neutral axis, the maximum principal stress is in tension; hence cracking ensues. As one moves toward the support, the bending moment and hence \( f \) decrease, accompanied by corresponding increase in the shear stress. The principal stress \( f_{\text{max}} \) in tension sets at an approximately 45° plane to the normal at sections close to the support, as seen in Fig. 6.3. Because of the low tensile strength of concrete, diagonal cracking develops along planes perpendicular to the planes of principal tensile stress; hence the term diagonal tension cracks. To prevent such cracks from opening, special "diagonal tension" reinforcement has to be provided.

If \( f \) close to the support in Fig. 6.3 is assumed equal to zero, the element becomes nearly in a state of pure shear and the principal tensile stress, using Eq. 6.3a, would be equal to the shear stress \( v \) on a 45° plane. It is this diagonal tension stress that causes the inclined cracks.
Figure 6.1 Stress state in elements A, and A2: (a) stress state in element A1; (b) Mohr's circle representation, element A1; (c) stress state in element A2; (d) Mohr's circle representation, element A2.
Chapter 6 Shear and Diagonal Tension in Beams

Definitive understanding of the correct shear mechanism in reinforced concrete is still incomplete. However, the approach of the ACI-ASCE Joint Committee 428 gives a systematic empirical correlation of the basic concepts developed from extensive test results.

6.4 REINFORCED CONCRETE BEAMS WITHOUT DIAGONAL TENSION REINFORCEMENT

In regions of large bending moments, cracks develop almost perpendicular to the axis of the beam. These cracks are called flexural cracks. In regions of high shear due to the diagonal tension, the inclined cracks develop as an extension of the flexural cracks and are termed flexure shear cracks. Figure 6.4 portrays the types of cracks expected in reinforced concrete beams with or without adequate diagonal tension reinforcement.

6.4.1 Modes of Failure of Beams without Diagonal Tension Reinforcement

The slenderness of the beam, that is, its shear span/depth ratio, determines the failure mode of the beam. Figure 6.5 demonstrates schematically the failure patterns. The shear span $a$ for concentrated loads is the distance between the point of application of the load and the face of support. For distributed loads, the shear span $l$, is the clear beam span. Fundamentally, three modes of failure or their combinations occur: (1) flexural failure, (2) diagonal tension failure, and (3) shear compression failure. The more slender the beam, the stronger the tendency toward flexural behavior, as seen from the following discussion.

---

Figure 6.3 Trajectories of principal stresses in a homogeneous isotropic beam. Solid lines: tensile trajectories; dashed lines: compressive trajectories.

Figure 6.4 Crack categories.
Figure 6.5 Failure patterns as a function of beam slenderness: (a) flexural failure; (b) diagonal tension failure; (c) shear compression failure.

6.4.2 Flexural Failure (F)

In this region, cracks are mainly vertical in the middle third of the beam span and perpendicular to the lines of principal stress. These cracks result from a very small shear stress $V$ and a dominantly flexural stress $f$ of a value close to an almost horizontal principal stress $f_{xx}$. In such a failure mode, a few very fine vertical cracks start to develop in the middepth area at about 50% of the failure load in flexure. As the external load increases, additional cracks develop in the central region of the span, and the initial cracks widen and extend deeper toward the neutral axis and beyond, with a marked increase in the deflection of the beam. If the beam is un-reinforced, failure occurs in a ductile manner by initial yielding of the main longitudinal flexural reinforcement. This type of behavior gives ample warning of the imminence of collapse of the beam. The shear span/depth ratio for this behavior exceeds a value of 5.5 in the case of concentrated loading and is in excess of 16 for distributed loading.
6.4.3 Diagonal Tension Failure (DT)

This failure precipitates if the strength of the beam in diagonal tension is lower than its strength in flexure. The shear span/depth ratio is of intermediate magnitude, with the ratio $a/d$ varying between 2.5 and 5.5 for the case of concentrated loading. Such beams can be considered of intermediate slenderness. Cracking starts with the development of a few fine vertical flexural cracks at midspan, followed by the destruction of the bond between the reinforcing steel and the surrounding concrete at the support. Thereafter, without ample warning of impending failure, two or three diagonal cracks develop at about 1/2 to 2/3 distance from the face of the support. As they stabilize, one of the diagonal cracks widens into a principal diagonal tension crack and extends to the top compression fibers of the beam, as seen in Figure 6.5b or 6.7. Notice that the flexural cracks do not propagate to the neutral axis in this essentially brittle failure mode, with relatively small deflection at failure.

6.4.4 Shear Compression Failure (SC)

These beams have a small shear span/depth ratio, $a/d$, of magnitude 1 to 2.5 for the case of concentrated loading and less than 5.0 for distributed loading. As in the diagonal tension case, few fine flexural cracks start to develop at midspan and stop propagating at destruction of the bond between the longitudinal bars and the surrounding concrete at the support region. Thereafter, an inclined crack steeper than in the diagonal tension case suddenly develops and proceeds to propagate toward the neutral axis. The rate of its progress is reduced with crushing of the concrete in the top compression fibers and redistribution of stresses within the top region. Sudden failure takes place as the principal inclined crack dynamically joints the crushed concrete zone, as illustrated in Fig. 6.9. This type of failure can be considered relatively less brittle than the diagonal tension failure due to the stress redistribution. Yet it is, in fact, a brittle type of failure with limited warning, and such a design should be avoided completely.

A reinforced concrete beam or element is not homogeneous, and the strength of the concrete throughout the span is subject to a normally distributed variation. Hence one cannot expect that a stabilized failure diagonal crack occurs at both ends of the beam. Also, because of these properties, overlapping combinations of flexure-diagonal tension failure and diagonal tension-shear compression failure can occur at overlapping shear span/depth ratios. If the appropriate amount of shear reinforcement is provided, brittle failure of horizontal members can be eliminated with little additional cost to the structure. Table 6.1 summarizes the effect of the slenderness values on the mode of failure.

<table>
<thead>
<tr>
<th>Beam Category</th>
<th>Failure Mode</th>
<th>Shear Span/Depth Ratio as a Measure of Slenderness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slender</td>
<td>Flexure (F)</td>
<td>Shear Span/Depth Ratio as a Measure of Slenderness</td>
</tr>
<tr>
<td>Intermediate</td>
<td>Diagonal tension (DT)</td>
<td>Concentrated Load, $a/d$</td>
</tr>
<tr>
<td>Deep</td>
<td>Shear compression (SC)</td>
<td>1-2.5</td>
</tr>
</tbody>
</table>

\[ a = \text{shear span for concentrated loads} \]
\[ l_i = \text{shear span for distributed loads} \]
\[ d = \text{effective depth of beam} \]
6.5 Diagonal Tension Analysis of Sler and Intermediate Beams

6.5 DIAGONAL TENSION ANALYSIS OF SLENDER AND INTERMEDIATE BEAMS

The occurrence of the first inclined crack determines the shear strength of a beam without web reinforcement. Because crack development is a function of the tensile strength of the concrete in the beam web, knowledge of the principal stress in the critical section is necessary, as discussed in Sections 6.2 and 6.3. The controlling principal stress in concrete is the result of the shearing stress \( \tau \) due to the external factored shear \( V' \), and the horizontal flexural stress \( f \) due to the external factored bending moment \( M \). The ACI Code provides an empirical model based on results of extensive tests to failure of a large number of beams without web reinforcement. The model is a regression solution to the basic equation for two-dimensional principal stress at a point

\[
f_{\text{max}} = f = \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + \tau^2}
\]

where \( f_{\text{max}} \) is the principal stress in tension and can be assumed to be equal to a constant multiplied by the tensile splitting strength \( f' \) of plain concrete. Since \( f' \) has been proven to be a function of \( \sqrt{f} \), Eq. 6.3b becomes

\[
\sqrt{f} = K_f \left[ \frac{f}{2} + \sqrt{\left(\frac{f}{2}\right)^2 + \tau^2} \right]
\]

(6.4)

where \( K_f \) is a constant.

The flexural stress \( f \) in the concrete is a function of the steel stress in the longitudinal reinforcement or the moment of resistance of the section, namely,

\[
f = \frac{E_s}{E_c} f_s = \frac{f M}{E_s A_f}
\]

But the reinforcement ratio \( \rho_s = A_f / b d \) at the tension side and \( E_s / E_c \) have a constant value. Hence

\[
f_s = F_2 \frac{M}{\rho_s b d}
\]

(6.5)

where \( F_2 \) is a constant to be determined by test and \( M \) is the nominal moment strength of a given section. The shear stress \( \nu \) at the specific cross-section \( b d \) due to the vertical external factored shear force \( V' \) is

\[
\nu = F_2 \frac{V'}{b d}
\]

(6.6)

where \( V' \) is the nominal shear resistance at the section under consideration and \( F_2 \) is the other constant, to be determined from the beam tests. Coefficients \( F_1 \) and \( F_2 \) both depend on several variables, including the geometry of the beam, type of loading, amount and arrangement of reinforcement, and the interaction between the steel reinforcement and the concrete.

Substituting \( f_s \) of Eq. 6.5 and \( \nu \) of Eq. 6.6, rearranging terms, and evaluating the constants \( K_f, F_1, \) and \( F_2 \) of the experimental model yields the following regression expression:

\[
V' = K_0 f' + V_{\text{cd}} M \sqrt{f'} \approx 3.5
\]

(6.7)

A plot of Eq. 6.7 is shown in Fig. 6.6. Note that \( M / V' \) is a def (see Figure 6.5); consequently, Eq. 6.7 accounts indirectly for the shear span/depth ratio, hence it accounts for the thickness of the member. If the nominal shear resistance of the plain concrete...
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Figure 6.6  Shear resistance of reinforced concrete beam webs.

web is termed $V_W$, $V_W$ in the left-hand side of Eq. 6.7 has to be expressed as $V_s$. Transforming Eq. 6.7 into a force format for evaluation of the nominal shear resistance of the web of a beam of normal concrete and having no diagonal tension steel gives

$$V_s = 1.9b_b d \sqrt{f_t} + 0.1b_b d \frac{V_{p,d}}{M_s} d \sqrt{f_t}$$

(6.8)

It is to be emphasized that the ratio $V_{p,d}/M_s$ or $V_{p,d}/M_s$ cannot exceed 1.0, where $V_s = V_f/d$ and $M_s = M_{f,d}$ as the values of shear and moment at the section for which $V_s$ is being evaluated. Also note that using nominal values (subscript n) results in a minor inaccuracy since $\phi$ is 0.3 for moment while it is 0.75 for shear.

The first critical values of $V_s$ and $M_s$ are taken as a distance $d$ from the face of the support since the stabilized (principal) diagonal tension cracks develop in that zone, as seen from Fig. 6.5b. As one moves toward the midspan of the beam, the values of $M_s$ and $V_s$ will change. The appropriate moments $M_s$ and shears $V_s$ have to be calculated for the particular section that is being analyzed for web steel reinforcement.

For simplicity of calculations, a more conservative ACI expression can be applied, particularly if the same beam section is not repetitively used in the structure:

$$V_s = f_0 \times 2.0 \sqrt{f_t} b_b, d$$

(6.9)

where $f_0$ is a factor dependent on the type of concrete, with values of 1.0 for normal-weight concrete, 0.85 for sand lightweight concrete, and 0.75 for all lightweight concrete. When axial compression also exists, $V_s$ in Eq. 6.9 becomes

$$V_s = 2f_0 \left( \frac{N_0}{2000A_k} \right)^{1/2} b_b, d$$

(6.10a)

When significant axial tension exists.
6.6 Web Steel Planar Truss Analogy

\[ V_c = 2A \left( 1 + \frac{N_e}{50A_e} \right) V_t \sqrt{f_t} \]

Equation (6.14b)

\( N_e/4A_e \) is expressed in psi, where \( N_e \) is the axial load on the member and \( A_e \) is the gross area of the section; \( N_e \) is negative in tension.

In the case of circular members, the area used to compute \( V_c \) should be taken as the product of the diameter and the effective depth of the concrete section. It is permitted to take the effective depth as 0.8 times the diameter of the concrete section. For the circular hoops that are used for such sections, \( A_e \) is taken as 2.5 times the area of the bar size of a circular hoop, i.e., or spiral at a spacing \( s \) and \( f_s/s \) as the specified yield strength of circular hoop, i.e., or spiral reinforcement.

6.6 WEB STEEL PLANAR TRUSS ANALOGY

As discussed previously, web reinforcement has to be provided to prevent failure due to diagonal tension. Theoretically, if the necessary steel bars in the form of the tensile stress trajectories shown in Fig. 6.5 are placed in the beam, no shear failures can occur. However, practical considerations eliminate such a solution, and other forms of reinforcing are used to neutralize the principal tensile stresses at the critical shear failure planes. The mode of failure in shear reduces the beam to a simulated arched section in compression at the top and tied at the bottom by the longitudinal beam tension bars, as shown in Fig. 6.7(a). If one isolates the main concrete compression element shown in Fig. 6.7(b), it can be considered as the compression member of a triangular truss, as shown in Fig. 6.7(c) with the polygon of forces \( C_t, T_u \), and \( T_r \), representing the forces acting on the truss members—hence the expression truss analogy. Force \( C_t \) is the compression in the simulated concrete strut, force \( T_u \) is the tensile force increment of the main longitudinal tension bar, and \( T_r \) is the force in the bent bar. Figure 6.7(b) shows the analogy truss for the case of using vertical stirrups instead of inclined bars with the forces polygon having a vertical tensile force \( T_r \) instead of the inclined one in Fig. 6.7(c).

As can be seen from the previous discussion, the shear reinforcement basically performs four main functions:

1. Carries a portion of the external factored shear force \( V_c \).
2. Restricst the growth of the diagonal cracks.
3. Yields the longitudinal main reinforcing bars in place so that they can provide the dowel capacity needed to carry the flexural load.
4. Provides some confinement to the concrete in the compression zone if the stirrups are in the form of closed ties.

This discussion, however, does not adequately account for the equilibrium role of additional longitudinal tensile reinforcement in enhancing the shear strength of a beam (Ref. 6.16 Sec. 5.2.1). To do so and hence maintain total equilibrium in beam shear carried by shear-bearing interaction, one has to consider the horizontal tensile component \( V_h \), or \( \theta \) of the vertical external nominal shear force \( V_c \). This component is considered as equally shared by the top compression bars (tensile compression chords) and the bottom longitudinal tensile bars (tensile ties), as shown in Fig. 6.8. While neglecting this tensile component is not significant when only shear is present, it has to be accounted for when torsion is also acting, as is discussed in Sec. 7.2. In such a case, the shear flow concept in a membrane element model should be applied in which both the longitudinal and transverse reinforcement have to be considered, as in Chapter 7, Fig. 7.10.
6.6.1 Web Steel Resistance

If $V_s$, the nominal shear resistance of the plain web concrete, is less than the nominal total vertical shearing force $V_{v/f} = V_e$, web reinforcement has to be provided to carry the difference in the two values; hence

$$V_v = V_e - V_s$$  \hspace{1cm} (6.11)

The nominal resisting shear $V_v$ can be calculated from Eq. 6.8 or 6.9, and $V_e$ can be determined from equilibrium analysis of the bar forces in the analogous triangular truss cell. From Figure 6.7A(c),

$$V_v = T_r \sin \alpha = C_t \sin \beta$$  \hspace{1cm} (6.12a)

where $T_r$ is the force resultant of all web stirrups across the diagonal crack plane and $n$ is the number of spacings $s$. If $s_f = s_n$ in the bottom tension chord of the analogous truss cell, then

$$s_f = j s (\cot \alpha + \cot \beta)$$  \hspace{1cm} (6.12b)
Assuming that moment arm $m = l$, the stress force per unit length from Eqs. 6.12a and 6.12b, where $s_y = n_z$, becomes

$$
\frac{T_y}{s_y} = \frac{T_z}{n_z} = \frac{V_y}{n_z} \frac{1}{\sin \theta (\cot \beta + \cot \alpha)}
$$

(6.12c)

If there are $v$ inclined struts within the $y$ length of the analogous true chord, and if $\lambda$, is the area of one inclined strut.
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The subscript b and t denote the bottom and top net force, respectively.

Figure 6.5  Shear-force interaction equilibrium.

\[ T_s = nA_f f_y \]  \hspace{1cm} (6.13a)

where \( f_y \) = strength of transverse reinforcement.

Hence

\[ nA_f = \frac{V_{pS}}{d \sin \alpha (\cot \beta + \cot \alpha)} \]  \hspace{1cm} (6.13b)

But assume that in the case of diagonal tension failure the compression diagonal makes an angle \( \beta = 45^\circ \) with the horizontal; Eq. 6.13b becomes

\[ V_c = \frac{A_f f_y d}{s} \left[ \sin \alpha (1 + \cot \alpha) \right] \]

to get

\[ V_c = \frac{A_f f_y d}{s} \left( \sin \alpha + \cot \alpha \right) \]  \hspace{1cm} (6.14a)

or

\[ s = \frac{A_f f_y d}{V_c - V_s} \left( \sin \alpha + \cot \alpha \right) \]  \hspace{1cm} (6.14b)

If the inclined web steel consists of a single bar or a single group of bars all bent at the same distance from the face of the support.

\[ V_c = A_f f_y \sin \alpha \approx 3.0 \sqrt{f_y} b_y d \]

If vertical stirrups are used, angle \( \alpha \) becomes 90°, giving

\[ V_c = \frac{A_f f_y d}{V_c} \]  \hspace{1cm} (6.15a)

or

\[ s = \frac{A_f f_y d}{(V_c/\phi) - V_c} = \frac{A_f f_y d}{V_s - \phi V_c} \]  \hspace{1cm} (6.15b)

6.6.2 Limitations on Size and Spacing of Stirrups

Equations 6.14 and 6.15 give inverse relationships between the spacing of the stirrups and the shear force or shear stress they resist, with the spacing \( s \) decreasing with the increase \( V_c \). In order for every possible diagonal crack to be resisted by a vertical stirrup as seen in Fig. 6.7A(c), minimum spacing should be determined to ensure that the forces can be resisted.
6.7 Web Reinforcement Design Procedure for Shear

1. \( V_c - V_\phi > 4 \sqrt{f'_c b_d d} \) \( t_{min} = d/4 = 12 \) in.
2. \( V_c - V_\phi \leq 4 \sqrt{f'_c b_d d} \) \( t_{min} = d/2 = 24 \) in.
3. \( V_c - V_\phi > 8 \sqrt{f'_c b_d d} \) enlarge section.

A minimum web steel area \( A_s \) is to be provided if the factored shear force \( V_c \) exceeds one-half of the shear strength \( \phi V' \) of the plain concrete web. This precaution is necessary to prevent brittle failure, that is, that both the stirrups and the beam compression zone continue carrying the increasing shear after the formation of the first inclined crack.

Minimum \( A_s = \frac{0.75 \sqrt{f'_c b_d d}}{f_p} \) or \( A_s = \frac{500 \frac{d}{f_p}}{2} \) whichever is larger \( (6.16) \)

where \( A_s \) is the area of all the vertical stirrup legs in the cross-section.

Summarizing, the minimum area of shear reinforcement, \( A_{s,min} \), should be provided in all reinforced and prestressed flange members where \( V_c \) exceed 0.5 \( \phi V' \).

6.7 WEB REINFORCEMENT DESIGN PROCEDURE FOR SHEAR

The following is a summary of the recommended sequence of design steps.

1. Determine the critical section and calculate the factored shear force \( V_c \). When the reaction, in the direction of applied shear, introduces compression into the end regions of a member, the critical section can be assumed at a distance of \( d \) from the support, provided that no concentrated load acts between the support face and distance \( d \) thereafter.

2. Check whether

\[ V_c = 0.75 \sqrt{f'_c b_d d} \]

where \( b_d \) is the web width or diameter of the circular section. If this condition is not satisfied, the cross section has to be enlarged.

3. Use minimum shear reinforcement \( A_s \) if \( V_c \) is larger than one-half \( \phi V' \) with the following exceptions:
   (a) Concrete joint construction
   (b) Slabs and footings
   (c) Small shallow beams of depth not exceeding 10 in. (254 mm) or 2\( \phi \) times the flange thickness.

Minimum \( A_s = 0.75 \sqrt{f'_c b_d d} \) or \( A_s = \frac{500 d}{f_p} \) whichever is larger

where \( f_p \) = strength of transverse reinforcement.

Good construction practice dictates that some stirrups always be used to facilitate proper handling of the reinforcement cage.

4. If \( V_c > \phi V' \), shear reinforcement must be provided such that \( V_c \leq \phi (V'_c + V_c) \), where

\[ V_c = \frac{A_{s,min} f_p d}{s} \quad \text{for vertical stirrups} \]

\[ \frac{A_{s,min} f_p d}{s} (\sin \alpha + \cos \alpha) \quad \text{for inclined stirrups} \]

5. Maximum spacing \( s \) must be \( s = d/2 \leq 24 \) in., except that in cases where

\[ V_c > 4 \sqrt{f'_c b_d d} \] the spacing then becomes \( s = d/4 \leq 12 \) in.

Figure 6.9 presents a flowchart for the performance of the sequence of calculations necessary for the design of vertical stirrups. Simple corresponding modifications of this chart can be made so that the chart can be used in the design of inclined web steel.
6.8 EXAMPLES OF THE DESIGN OF WEB STEEL FOR SHEAR

6.8.1 Example 6.1: Design of Web Stirrups

A rectangular isolated beam has an effective span of 25 ft (7.62 m) and carries a working live load of 500 lb per linear foot (110 kN/m) and no unusual dead load except its self-weight. Design the necessary shear reinforcement. Use the simplified form of Eq. 6.9 for calculating the capacity \( V_s \) of the plain concrete web. Given:

- \( f'_c = 4000 \text{ psi} (27.6 \text{ MPa}) \), normal-weight concrete
- \( f'_s = 60,000 \text{ psi} (414 \text{ MPa}) \)
- \( P = \frac{5}{2} \text{ in. (125 mm)} \)
- \( d = 22 \text{ in. (711 mm)} \)
- \( h = 75 \text{ in. (758 mm)} \)
- Longitudinal tension steel: six No. 9 bars (diameter 38.6 mm)
6.8 Examples of the Design of Web Steel for Shear

Solution:

**Factored shear force (Step 1)**

- Beam self-weight = \( \frac{1.2 \times 30}{144} \times 150 = 438 \text{ lb/ft} \)
- Total factored load = \( 1.2 \times 438 + 1.0 \times 7500 = 12325 \text{ lb/ft} \)

The factored shear force on the face of the support is

\[ V_c = \frac{25}{2} \times 12526 = 156572 \text{ lb} \]

The first critical section is at a distance \( d = 28 \text{ in.} \) from the face of the support of this beam (half-span = 150 in.).

\[ V_c = \frac{150 - 28}{150} \times 12526 = 127348 \text{ lb} \]

**Shear capacity (Step 2)**

The shear capacity of the plain concrete in the web from the simplified equation for normal weight concrete \((k = 1.0)\) is

\[ V_C = 2.0a \sqrt{f'_{c}} b_d = 2 \times 1.0 \sqrt{4000} \times 14 \times 28 = 49585 \text{ lb} \]

Check for adequacy of section for shear:

\[ (\phi + 2.0) \sqrt{f'_{c}} b_d = 1.0 \sqrt{f'_{c}} b_d = 247923 \text{ lb} \]

required \( V_c = \frac{V_C}{0.75} = 169797 \text{ lb} \) cross-section O.K.

\[ V_c > \frac{1}{2} V_C \text{ hence stirrups are necessary} \]

**Shear reinforcement (Step 3 to 5)**

Try No. 4 two-legged stirrups (area per 1 run = 0.20 in.²).

\[ A_s = 2 \times 0.2 = 0.40 \text{ in.}² \]

From Eq. 6.15e,

\[ s = \frac{A_f}{V_C / h} = \frac{0.4 \times 60,000 \times 28}{12526 - 49585} = 5.6 \text{ in.} (142 \text{ mm}) \]

Since \( V_c = 4 \sqrt{f'_{c}} b_d \), the maximum allowable spacing \( s = d/4 = 28/4 = 7 \text{ in.} \). At the critical section, \( d = 28 \text{ in.} \) from the face of the support, the maximum allowable spacing would in this case be 5.6 in.

The shear force for distributed load decreases linearly from the support to midspan of the beam. Hence the web reinforcement can be reduced accordingly after determining the zone where minimum reinforcement is necessary and the zone where no web reinforcement is needed. The same size and spacing of stirrups needed at the critical section \( d \) from face of support should be continued to the support. Figure 6.10 illustrates the various values being calculated:

**Critical phase x** (consider the midspan as the origin): \( V_c = 169993 \text{ lb} \) and from before:

\( r = 5.6 \text{ in.} \) from midspan point = 150 - 28 = 122 in.

**Phase x** at 8 = 1.0 maximum spacing:

\[ V_c = 4 \sqrt{f'_{c}} b_d = 4 \sqrt{4000} \times 14 \times 28 = 99160 \text{ lb} \]

\[ V_c = 99160 + 49585 = 148745 \text{ lb} \]

\( x \), from midspan point = \( (150 - 28) \times \frac{148745}{169797} = 106.9 \text{ in.} \)
Chapter 6 Shear and Diagonal Tension in Beams

Plane $x_2$ at $s = d/2$ maximum spacing:

$$ s = \frac{A_p f_p d}{V_c - V_t} \quad \text{or} \quad \frac{2s}{d} = \frac{0.4 \times 60,000 \times 28}{V_c} $$

or

$$ V_c = 48,000 \text{ lb} $$

$$ V_{tc} = 48,000 + 49,585 = 97,585 \text{ lb} $$

$$ x_2 \text{ from midspan point} = \frac{122 \times 169,797}{169,797} \approx 70.1 \text{ in.} $$

From Figure 6.1(a), the distance 36.73 in. is the transition zone from $s = 7$ in. to $s = 14$ in.; hence a stirrup spacing of 8 in. center to center is shown in Figure 6.1(b).

Plane $x_3$ at shear force $V_c$:

$$ V_c = 2\sqrt{f_p b_s d} = 49,585 \text{ lb} $$

$$ x_3 \text{ from midspan point} = \frac{122 \times 169,797}{169,797} = 15.6 \text{ in.} $$

Discontinue the stirrups at plane where $V_c = 1/4 V_c$.

Minimum web area: Test when $V_c = 1/4 V_c$, or $V_c > 4 V_c$

$$ V_c = 127,348 $$

$$ \frac{1}{4} V_c = \frac{1}{2} \times 49,585 = 24,793 \text{ lb} $$

$$ \text{minimum } A_r = \frac{500\sqrt{f_p}}{f_{ca}} = \frac{50 \times \frac{14}{14}}{60,000} = 0.16 \text{ in.}^2 $$

$$ < \text{ actual } A_r = 0.40 \text{ in.}^2 \quad \text{O.K.} $$

or

$$ \text{maximum allowed } s = \frac{A_p f_p}{50 V_c} = \frac{0.40 \times 60,000}{20 \times 14} = 34.3 \text{ in.} $$

$$ \text{versus maximum used } s = \frac{d}{2} = 14 \text{ in.} \quad \text{O.K.} $$

$$ x_3 = 122.0 \times 24,793 = 17.8 \text{ in. from midspan} $$

Proportion the spacing of the vertical stirrups accordingly.

The shaded area in Figure 6.1(a) is the shear force area for which stirrups must be provided. The spacing of the stirrups in Figure 6.1(b) is based on the practical consideration of the desirability of using whole spacing dimensions and varying the spacing as little as possible.

6.8.2 Example 6.2: Alternative Solution to Example 6.1

Find the force $V_c$ and the change in stirrup spacing for the beam in Fig. 6.1 if the more refined Eq. 6.8 is used where the separate contribution of the main longitudinal steel at the tension side is more accurately reflected.

Solution: The shear capacity of the plain concrete in the web is

$$ V_c = 1.94 \sqrt{f_p} b_s d + 2500 \sqrt{f_p} b_s d = 3.3 \sqrt{f_p} b_s d $$

- $t = 0.75$ is the longitudinal stress ratio in the web at the tension side only.
Figure 6.10 Stirrup arrangements for Ex. 6.1: (a) shear envelope and stirrup design segments; (b) vertical stirrups spacing.

\[
\nu_c = \frac{6.0}{14 \times 28} = 0.0153
\]

\[V_c \text{ at } d \text{ from support} = 127,348 \text{ lb (Ex. 6.1)}\]

\[V_c d = 127,348 \times 28 = 3,565,744 \text{ in}^{-1} \text{lb}\]

\[
M_c \text{ at } d \text{ from support} = \frac{V_c d}{2} - \frac{V_c d}{2} = \left(12.526 \times \frac{25}{2}\right) \times 28 - \frac{12.526(25)^2}{12 \times 2}
= 3,974.917 \text{ in}^{-1} \text{lb}\]

\[
\frac{V_c d}{M_c} = \frac{3,565,744}{3,974.917} = 0.9 < 1.0 \text{ use 0.9}
\]

\[V_c = 1.9 \sqrt{4000 \times 14 \times 28 + 2500 \times 0.0153 \times 0.9 \times 14 \times 28}
= 47105.4 \text{ lb } + 13494.6 = 60600 \text{ lb}\]

Use a two-legged No. 4 size vertical stirrup, as in Ex. 6.1.
\[
x = \frac{A_s f_{c}}{V_{c}} = \frac{0.4 \times 60000 \times 25}{189790} = 6.15 \text{ in. (156.0 mm)}
\]

For \( x = d = 7 \text{ in.} \): By similar triangles and applying the expression in Eq. 6.8 in the trias,

\[
V_{c_1} = 52530 \text{ lb.} \quad V_{c_2} = 151699 \text{ lb.}
\]

\[
x_1 = (150 - 28) \times \frac{151699}{169790} = 109.8 \text{ in.}
\]

For \( x = d/2 = 14 \text{ in.} \):

\[
V_{c_2} = 54737 \text{ lb.} \quad V_{c_3} = 98677 \text{ lb.}
\]

\[
x_2 = 122 \times \frac{98677}{169790} = 70.9 \text{ in.}
\]

At the point in the shear envelope where \( V_c = 0 \), the value of \( V_{c\lambda} \) in Eq. 6.8 is close to zero for uniformly distributed loads. Hence assume that

\[
V_c = 2.9 f_{c} \sqrt{f_{c} b_1 d} \quad \text{instead of} \quad V_c = 1.9 f_{c} \sqrt{f_{c} b_1 d} + 25000 \frac{V_{c \lambda} b_1 d}{M_s},
\]

\( \lambda = 1.0 \) for normal-weight concrete. Therefore, use \( x_1 = 35.6 \text{ in.} \) in Figure 6.11 (as in Figure 6.10 of Ex. 6.1) as being accurate enough for all practical purposes. Proportion the spacing of the stirrups as in Fig. 6.11b.

The shear diagram showing all these details is given in Figure 6.11. It can be seen that this refined solution reduced the shearing force taken by the stirrups at the critical section by the difference between the 49535 lb of Ex. 6.1 and the 60600 lb of Ex. 6.2, as shown in the darkly shaded portion. This difference is taken by the plain concrete in the web. There is a small saving in the number of stirrups that can be justified only if the beam section designed by the refined method is actually and repetitively used in a similar floor.
Figure 6.11 Stumps arrangement for Ex. 6.2: (a) shear envelope and stump design segments; (b) vertical stumps spacing.

Note in these problems that if concentrated loads act on the beam close to the midpoint or, in the case of reversible loads, Figure 6.12, almost constant stump spacing throughout the span becomes necessary. The spacing to be used would be that required at the critical section at distance d from the face of the support. Superposition of the three diagrams for a concentrated live load over that of the distributed load due to self-weight or otherwise gives the total shear force for stump spacing determination. Note that for V, values using Eq. 6.8 give a parabolic line in Figure 6.11 for the cross-hatched portion, approximated by a straight inclined line in the solution and plotted in the diagram.

6.9 DEEP BEAMS: NON-LINEAR APPROACH

Deep beams are structural elements loaded as beams but having a large depth/thickness ratio and a shear span/depth ratio not exceeding 2 for concentrated load and 4 for distributed load, where the shear span is the clear span of the beam for distributed load. Floor slabs under horizontal loads, wall slabs under vertical loads, short-span beams carrying heavy loads, and some shear walls are examples of this type of structural element.
Figure 6.12  Schematic stress distribution: (a) stress due to uniformly distributed load on beam; (b) stress due to centrally loaded beam; (c) stress due to third-point loaded beam. Self-weight is not included in the shear envelope.

Because of the geometry of deep beams, they behave in a non-linear analysis as two-dimensional rather than one-dimensional members and are subjected to a two-dimensional state of stress. As a result, plane sections before bending do not necessarily remain plane after bending. The resulting stress distribution is no longer considered linear, and shear deformations that are neglected in normal beams become significant compared to pure flexure. Consequently, the stress block becomes nonlinear even at the elastic stage. At the limit state of ultimate load, the compressive stress distribution in the concrete would no longer follow the same parabolic shape or intensity as shown in Fig. 5.23.
Figure 6.13 Elastic distribution in normal beams (L/d < 3) to 5).

Figure 6.13 illustrates the linearity of the stress distribution at midspan prior to cracking in a normal beam where the effective span/depth ratio exceeds a value of 3. Figure 6.14 shows the nonlinearity of stress at midspan corresponding to the nonlinear strain under discussion. Recognize also that the magnitude of the maximum tensile stress at the bottom fiber far exceeds the magnitude of the maximum compressive stress. The stress trajectories in Fig. 6.14 may confirm the observation. Note the steepness and concentration of the principal tensile stress trajectories at midspan and the concentration of the compressive stress trajectories at the support for both cases of loading of the beam at top or bottom.

The concrete cracks in a direction perpendicular to the tensile principal stress trajectories. As the load increases, the cracks widen and propagate, and more cracks open. Hence less and less concrete remains to resist the predetermined state of stress. Because the shear span is small, the compressive stresses in the support regions affect the magnitude and direction of the principal tensile stresses such that they become less inclined and lower in value.

In many cases, the cracks would almost be vertical or follow the direction of the compression trajectories, with the beam almost shearing off from the support in a total shear failure. Hence, in the case of deep beams, horizontal reinforcement is provided throughout the height of the beam. In addition, the vertical shear reinforcement along the span. From Fig. 6.14 and the steep gradient of the tensile stress trajectories at the lower fibers, a concentration of horizontal reinforcing bars is required to resist the high tensile stresses at the lower regions of the deep beam (Ref. 6.8, 6.9).

Additionally, the high depth/span ratio of the beam should provide an increased distance to the external shear load due to a higher compressive arch action. Consequently, it should be expected that the nominal resisting shear strength of the beam concrete in deep beams will considerably exceed the Vc value for normal beams.

In summary, shear in deep beams is a major consideration in their design. The magnitude and spacing of both the vertical and horizontal shear reinforcement differ considerably from those used in normal beams, as well as the expressions that have to be used for their design.

Another totally different approach is the strut-and-tie model approach presented in Section 6.11, including a full design example, and appears in the 318-25 Code Appendix A. The Euro Code, EC-2, (Ref. 6.28) maintains, however, that the linear elastic approach presented in Section 6.9 is one of the recommended methods, and lump the design of deep beams with the design of walls. It stipulates also that for the strut-and-tie method to be efficient and economic, the solution should be computer assisted for the analysis of such in-plane structures (Ref. 6.28).

6.9.1 Design Criteria for Shear in Deep Beams Loaded at the Top

From the discussion in Section 6.9, it can be inferred that deep beams (a/d < 2) and Vc/d < 4.0 have a higher nominal shear resistance Vc than do normal beams, where a = shear span to support face for concentrated load, and Vc = shear span for distributed load (Fig. 6.3). While the critical section for calculating the factored shear force Vc is taken as distance d from the face of the support in normal beams, the shear plane in the deep beam is considerably steeper in inclination and closer to the support. H is the distance of the failure plane from the face of the support. Vc is the clear span for uniformly distributed load, and Vc is the shear area or span for concentrated loads; the expression for distance d
uniform load: \( x = 0.15a \) \hfill (6.17a)
concentrated load: \( x = 0.50a \) \hfill (6.17b)

In either case, the distance \( x \) should not exceed the effective depth \( d \).

The factored shear force \( V' \) has to satisfy the condition:

\[
V' \leq 0.6 \left( 60 \sqrt{f'c} d \right) \hfill (6.18a)
\]

or

\[
V' = 10 \sqrt{f'c} d \hfill (6.18b)
\]

If not, the section has to be enlarged. The strength reduction factor \( \phi \) = 0.75.

The present ACI Code does not give guidelines on determining the shear value \( V' \) of the plain concrete or the maximum permissible value, although the shear capacity of the plain concrete in the deep beam has to be considerably higher than in normal beams as previously observed. A value \( V' \leq 0.6 \sqrt{f'c} d \) has been used for deep beams.
the limit value of \( V \approx 3.5 \sqrt{f_c} b_d \) in normal beams. In the strut-and-tie approach given in Section 6.11, compressive forces in the struts and tensile forces in the ties are used for determining the necessary reinforcement in lieu of the approach presented in this section.

The nominal shear resisting force \( V_s \) of the plain concrete can be taken as

\[
V_s = \left( 3.5 - 2.5 \frac{M_d}{V_s d} \right) (1.9 \sqrt{f_c} + 2500 \frac{V_{sd}}{M_d}) b_d \leq 6 \sqrt{f_c} b_d d \quad (6.19a)
\]

where \( 1.0 < 3.5 - 2.5(M_d/V_s d) \leq 2.5 \). This factor is a multiplier of the basic equation for \( V_s \) in normal beams to account for the higher resisting capacity of deep beams. If some minor uncracked concrete is not tolerated, the designer can use

\[
V_s = 2 \sqrt{f_c} b_d d \quad (6.19b)
\]

When the factored shear \( V_f \) exceeds \( 4V_s \), shear reinforcement has to be provided such that \( V_f \leq \alpha(V_f + V_s) \), where \( V_s \) is the force resisted by the shear reinforcement:

\[
V_s = \frac{A_s}{s} \left( \frac{1 + d_{f}/d}{12} \right) + \frac{A_{sh}}{s_h} \left( \frac{11 - d_{f}/d}{12} \right) f_{cd} \quad (6.20)
\]

where \( A_s = \) total area of vertical reinforcement spaced at \( s \) in the horizontal direction at both faces of the beam
\( A_{sh} = \) total area of horizontal reinforcement spaced at \( s_h \) in the vertical direction at both faces of the beam

\[
\text{maximum } s \leq \frac{d}{5} \text{ or } 12 \text{ in.} \quad \text{whichever is smaller} \quad (5.21a)
\]

\[
\text{maximum } s_h \leq \frac{d}{5} \text{ or } 12 \text{ in.} \quad (5.21b)
\]

The shear reinforcement required at the critical section must be provided throughout the deep beams.

In the case of continuous deep beams, because of the large stiffness and negligible rotation of the beam section at the supports, the continuity factor at the first interior support has a value close to 1.0. Consequently, the same reinforcement for shear can be used in all spans for all practical purposes if all the spans are equal and similarly loaded.

### 6.9.2 Design Criteria for Flexure in Deep Beams

**6.9.2.1 Simply-supported Beams.** The ACI Code does not specify a design procedure but requires a rigorous nonlinear analysis for the flexural analysis and design of deep beams. The simplified provisions presented in this section are based on the recommendations of the Euro-International Concrete Committee (CEB Ref. 6.8).

Figure 6.1 shows a schematic stress distribution in a homogenous deep beam having a span/depth ratio \( L/d \approx 1.0 \). It was experimentally observed that the moment over the span does not change significantly even after initial cracking. Since the nominal resisting moment is

\[
M_e = A_s f_c \text{ (moment arm )} \quad (6.22a)
\]
the reinforcement area $A_r$ for flexure is

$$A_r = \frac{M_{cr}}{\phi \sigma_y d} \geq \frac{3\sqrt{f_c' b d}}{f_y} \geq \frac{200 d}{f_y}$$  \hspace{1cm} (6.22b)$$

The lever arm as recommended by CEB is

$$jd = 0.2(l + 2h) \quad \text{for} \quad \frac{l}{h} < 2$$  \hspace{1cm} (6.23a)

and

$$jd = 0.6l \quad \text{for} \quad \frac{l}{h} < 1$$  \hspace{1cm} (6.23b)

where $l$ is the effective span measured center to center of supports or 1.15 clear span $l_e$, whichever is smaller. The tension reinforcement has to be placed in the lower segment of beam height such that the segment height is

$$y = 0.25h - 0.15l < 0.20h$$  \hspace{1cm} (6.24)

It should consist of closely spaced small-diameter bars well anchored into the supports.

6.9.2 Continuous Beams. Continuous deep beams can be treated in the same manner as simply supported deep beams, except that additional reinforcement has to be provided for the negative moment at the support. Figure 6.15 presents stress trajectories of the principal tensile and compressive stresses in a continuous deep beam. Comparing this diagram to Figure 6.14b for the simply supported case, one can observe the similarity of the steepness of the tensile stress trajectories at midspan. At the continuous supports, the total section is in tension. These principal stress trajectories serve as guidelines for the compression strut paths discussed in Section 6.11.

The concentration of the tensile stress trajectories at the support regions of the continuous deep beam necessitates a concentration of well-anchored horizontal shear reinforcement. The required total flexural reinforcement area

$$A_r = \frac{M_{cr}}{\phi \sigma_y d} \geq \frac{3\sqrt{f_c' b d}}{f_y} \geq \frac{200 d}{f_y}$$

as in Eq. 6.22b for the simply-supported beam. The lever arm $jd$ is, however, different and has a value

$$jd = 0.2(l + 1.5h) \quad \text{for} \quad 1 \leq \frac{l}{h} \leq 2.5$$  \hspace{1cm} (6.25a)

![Figure 6.15 Tensile and compression trajectories in a continuous deep beam. Solid line: tension trajectories; dashed line: compression trajectories.](image-url)
6.9 Deep Beams

\( \phi = 0.5 \) for \( \frac{l}{d} \leq 1.0 \)  \( (6.25b) \)

The distribution of the negative flexural reinforcement, \( A_n \), in continuous beams should be such that the steel area \( A_{n1} \) should be placed in the top 20% of the beam depth, and the balance steel area \( A_{n2} \) at the next 60% of the beam depth, as shown in Fig. 6.16. The value of \( A_{n1} \) and \( A_{n2} \) would be as follows:

\[ A_{n1} = 0.5 \left( \frac{l}{d} - 1 \right) A_n \] \( (6.26a) \)

\[ A_{n2} = A_n - A_{n1} \] \( (6.26b) \)

For cases where the ratio \( \frac{l}{d} \) has a value equal to or less than 1.0, use normal steel for \( A_{n1} \) in the top 20% of the beam depth and provide the total \( A_n \) in the next 60% of the depth. In the lower \( h_2 \) zone, the positive reinforcement coming from the beam span should pass through the support for anchorage and continuity.

6.9.3 Sequence of Deep Beam Design Steps for Shear

The following is a recommended procedure for the design of shear reinforcement in deep beams based on ACI requirements. The sequence of steps should essentially be similar to that in Section 6.7 for web reinforcement design in normal beams. Additionally, plain steel reinforcement has to be provided to resist the stresses due to bending.

1. Check whether the beam can be classified as a deep beam, that is, \( d/\phi d \leq 2.0 \) or \( l/d \leq 4.0 \) for a concentrated or uniform load, respectively.
2. Determine the critical section distances \( z \) from the face of support: \( z = 0.5d \) for concentrated load and \( z = 0.1M_0 \) for distributed load. Calculate the factored \( V_0 \) at the critical section, and check whether it is less than the maximum \( A_s \) permitted by Eqs. 6.58a or 6.18a; if not, enlarge the beam section.
3. Calculate the shear resisting capacity \( V_r \) of the plain concrete from Eq. 6.19.
4. Calculate \( V \), if \( V > 0.7V_r \), and choose \( x \) and \( y \), by assuming the size of shear reinforcement in both the horizontal and vertical directions.
5. Verify if the size and minimum spacing from step 4 satisfy Eqs. 6.21a and 6.21b; if not, revise and readjust using Eq. 6.20.
6. Select reasonable size and spacing of the shear reinforcement in both horizontal and vertical directions. Where possible, use welded wire fabric bars since they provide better anchorage of the reinforcement to the stirrups and are easier to handle and keep in position at both faces of the deep beam.

Figure 6.19: Distribution of horizontal flexural steel in continuous deep beams
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7. Design the flexural reinforcement as in Section 6.9.2 after determining the moment lever arm \(d\) for the particular case of simply supported or continuous deep beams.

8. Distribute the flexural reinforcement in accordance with Eqs. 6.26a and 6.26b and Fig. 6.16 if the beam is continuous. If the beam is simply supported, concentrate the flexural horizontal longitudinal bars in the lower (0.25d - 0.05d) ± 0.20d part of the beam depth.

9. Sketch a detailed schematic of the distribution of both the shear and the flexural reinforcement. The longitudinal flexural reinforcement must be well-anchored into the supports by embedment, hooks or welding to special devices. Bent upbars are not recommended.

6.9.4 Example 6.3: Design of Shear Reinforcement in Deep Beams

A simply supported beam having a clear span \(L_a = 10\) ft (3.05 m) is subjected to a uniformly distributed live load of \(81,000\) lb (362 kN/m) on the top. The height \(h\) of the beam is 6 ft (1.83 m) and its thickness \(t = 20\) in. (508 mm). The area of its horizontal tension steel is 8.0 in.\(^2\) (5161 mm\(^2\)), determined in Ex. 6.4. Given:

\[
\begin{align*}
\sigma' &= 4000 \text{ psi (27.7 MPa)} \\
\beta &= 60,000 \text{ psi (414 MPa)}
\end{align*}
\]

Design the shear reinforcement for the beam.

**Solution:** Use Eq. 6.21 and evaluate factored shear force \(V_u\) (Step 1).

Assume that \(d = 0.5 \times h = 0.5 \times 6\) in. (152 mm).

\[
\frac{L_a}{d} = \frac{10.0 \times 12}{6} = 20 < 4
\]

Hence treat as a deep beam.

beam self-weight = \(20 \times \frac{72}{144} \times 150 = 1500 \text{ lb/ft (21.9 kN/m)}\)

total factored load = \(1.2 \times 1500 + 1.5 \times 81,000\)

= \(131,400 \text{ lb/ft (1942 kN/m)}\)

distance of the critical section = 0.15L = 0.15 \times 10 = 1.5 \text{ ft = 18 in. (457.2 mm)}

The factored shear force \(V_u\) at the critical section is

\[
\begin{align*}
V_u &= \frac{131,400 \times 10}{2} - \frac{131,400 \times 18}{12} = 459,000 \text{ lb (2040 kN)}
\end{align*}
\]

Nominal shear strength \(V_{u,n}\) and resisting capacity \(V_r\) (Steps 2 and 3)

\[
\begin{align*}
\psi V_u &= \psi(10\sqrt{h_d} \cdot d) = 0.75(10\sqrt{508 \times 20 \times 65}) \\
&= 616,440 \text{ lb (2742 kN)} > 459,000 \text{ O.K.}
\end{align*}
\]

\[
\begin{align*}
\frac{M_u}{2} &= \frac{131,400 \times 10 \times 150}{150} - \frac{131,400 \times (1.5)^2}{2} \\
&= 837,675 \text{ ft-lb (1002,100 in.-lb)}
\end{align*}
\]

\[
\begin{align*}
\frac{M_u}{V_{eff}} &= \frac{10,052,100}{459,000 \times 0.34} = 0.34
\end{align*}
\]

\[
\begin{align*}
\frac{1.5 - 2.5}{V_{u,n}} &= 0.7 \\
&= 2.7 \text{ (2.5)} - 2.5 \text{ (2.5)} \text{ and } 2.5
\end{align*}
\]
From Eq. 6.10,

\[ V_e = 2.5 \left( 1.9 \sqrt{2400} + 2500 \frac{V_1}{M_r} \right) \sigma_d \]

\[ = 2.5 \left( 1.9 \sqrt{2400} + 2500 \times 8.862 \times 2.94 \right) \times 20 \times 65 = 538.644 \text{ lb} \]

\[ 6 \sqrt{f_{y,d}} = 493.315 \text{ lb} < 538.644 \text{ lb} \]

Hence \( V_e = 493.315 \) lb (2144 kN) controls.

**Shear reinforcement (Steps 4 and 5)**

Assume No. 3 (9.52-mm diameter) bars placed both horizontally and vertically on both faces of the beam.

\[ A_s = 2 \times 0.11 = 0.22 \text{ in.}^2 (141.9 \text{ mm}^2) = A_{sh} \]

or

\[ V_e = V_{h} = V_{v} \]

\[ V_{h} = \frac{V_e}{0.75} = \frac{493.900}{0.75} = 658.533 \text{ lb (533 kN)} \]

\[ V_{v} = \left[ \frac{A_s}{12} \left( \frac{1 + 1}{12} \right) \right] f_d \]

Assume that \( t_s = s \) (similar spacing in both the vertical and horizontal directions).

Hence

\[ 119.885 = \frac{0.22}{s} \left( \frac{1 + 120/65}{12} \right) \frac{0.22}{s} \left( \frac{11 - 120/65}{12} \right) 40,000 \times 65 \]

\[ s = 7.16 \text{ in. (179 mm)} \]

If no insignificant cracks are tolerated, \( V = 2 \sqrt{f_{y,d}} \sigma_d = 164,338 \text{ lb} \). This gives \( V = 493.900 \times 0.75 = 164,338 \text{ lb} \). For this condition, the spacing \( s \) of the shear reinforcement becomes for No. 5 bars:

\[ s = 7.16 \left( \frac{119.885 \sqrt{0.62}}{448.862 \sqrt{0.52}} \right) = 5.39 \text{ in.} \]

requiring No. 5 bars on each face both vertically and horizontally at 5.39 in. c.e. The maximum permissible spacing of vertical bars \( s_v = 10 \text{ in. or 12 in.}, \) whichever is smaller.

\[ s_v = \frac{65}{3} = 13 \text{ in.} \]

The maximum permissible spacing of horizontal bars \( s_h = 0.65 \text{ or 12 in.}, \) whichever is smaller.

\[ s_h = \frac{65}{8} = 8 \text{ in.} \]

Since similar spacing assumed in both directions, \( s = 7.16 \text{ in.} \). Use spacing \( s = s_h = 7 \text{ in. (179 mm)} \).

**Check for minimum area:**

\[ \text{minimum } A_{sh} = \frac{0.0015}{0.0015} \times 20 \times 7 = 0.21 \text{ in.}^2 < 0.22 \text{ in.}^2 \]

\[ \text{O.K.} \]

\[ \text{minimum } A_{sh} = \frac{0.0025}{0.0025} \times 20 \times 7 = 0.35 \text{ in.}^2 > 0.22 \text{ in.}^2 \]
Hence No. 3 bars are not adequate for vertical steel. No. 4 bars on both faces = 2 \times 0.20 \text{ in.}^2 = 0.40 \text{ in.}^2. Use horizontal No. 3 bars at 7 in. center to center (178 mm center to center) and vertical No. 4 bars at 7 in. center to center (12.78 mm diameter at 178 mm center to center), Fig. 6.17. Use of No. 4 bars in Eq. 6.20 for $V'$ would give a higher value of the force $V'$ that the shear reinforcement is resisting. It should be noted that the vertical shear reinforcement is more effective than the horizontal ones.

A better reinforcing system would be to use welded wire fabric in deep beams. For a comparable reinforcing area needed, use nine D20 welded wire fabric (1.5 in. \times 12.7 mm diameter) spaced at $x = 8$ in. (203 mm) center to center in the horizontal direction and $y = 9$ in. (228.6 mm) center to center in the vertical direction.

### 6.9.5 Example 6.1: Flexural Steel in Deep Beams

Design the flexural reinforcement for the beam in Ex. 6.3.

**Solution:** $l = 120$ in. (3048 mm) and $b = 72$ in. (1828 mm). Since the width of the supports is not given, assume that $I = 1.15I$, $138$ in. (3509 mm). The external factored load $V = 151,400$ lb-ft.

---

**Figure 6.17** Reinforcement for a simply supported deep beam (Ex. 6.4): (a) sectional view of beam; (b) cross-section at beam end; (c) cross-section at support.
6.9 Deep Beams

external factored moment \( M_e = \frac{w_f l^2}{8} \)

\[ = \frac{13,400 (10.0)^2}{8} = 1642.500 \text{ ft-lb} \]

\[ = 19,710,000 \text{ in.-lb} (2228 \text{ kNm}) \]

\[ b = 12 \text{ in.} \]

\[ h = 72 \text{ in.} \]

\[ \beta = 0.2138 + 2 \times 72 = 56.4 \text{ in.} \]

\[ A_c = \frac{19.7 \times 10^6}{6.9 \times 56.4 \times 60.000} = 6.47 \text{ in.}^2 (4045 \text{ mm}^2) \]

\[ > \left( \frac{200}{l} = 4.33 \text{ in.}^2 \right) > \left( \frac{3 \sqrt{6}}{l} \beta d = 4.1 \text{ in.}^2 \right) \text{ O.K.} \]

Use four No. 9 horizontal bars on each face, area = 8.00 in.². The height over which \( A_e \) is to be distributed above the lower beam face is

\[ 0.25h - 0.05f = 0.25 \times 72 - 0.05 \times 138 = 11.2 \text{ in.} \]

Spacing of flexural steel = \( \frac{11.2}{2} = 5.6 \text{ in.} \)

Space four No. 9 bars at 3.5-in. center-to-center vertical spacing on each face of the deep beam to be well anchored into the supports (20.5-mm-diameter bars at 76.2-mm spacing).

Figures 6.17a and b give, respectively, a sectional elevation and a horizontal cross section showing the details of the vertical and horizontal steel reinforcement as well as the flexural reinforcement concentrated at the lower 10.5 in. of the deep beam.

6.9.6 Example 6.5: Reinforcement Design for Continuous Deep Beams

Design the reinforcement necessary for an interior span of a continuous beam over several supports if the loading and the properties of the beam are the same as those of Ex. 6.3.

Solution: Shear reinforcement

Since the deep beam has a large stiffness, the shear continuity factor for the first interior support is assumed to equal 1.0. Hence use the same vertical and horizontal shear reinforcement as in Ex. 6.3. Use site D30 welded wire fabric (0.5 in. = 12.7-mm diameter) spaced at 6 in. (152 mm) center to center in the vertical direction and 8 in. (203 mm) center to center in the horizontal direction (Figure 6.18a).

Flexural reinforcement

Assume that \( d = 65 \text{ in.} \) from Ex. 6.3. The approximate positive factored moment at midspan is

\[ M_e = \frac{w_f l^2}{16} \]

\[ = \frac{13,400 (10.0)^2}{16} = 821,250 \text{ ft-lb} \]

\[ = 9,855,000 \text{ in.-lb} (1141 \text{ kNm}) \]

\[ + M_e = \frac{M_e}{\phi} = 10,050,000 \text{ in.-lb} (1238 \text{ kNm}) \]

For continuous deep beams,

\[ \text{lever arm } \beta d = 0.20(l + 1.5h) \]

\[ = 0.20(138 + 1.5 \times 72) = 49.2 \text{ in.} \]

\[ A_e = \frac{M_e}{\beta d} = \frac{10,050,000}{60,000 \times 49.2} = 3.71 \text{ in.}^2 \]
Figure 6.18 Continuous three-beam reinforcement: (a) sectional elevation of the beam, (b) section over support, (c) section at midpoint.

\[ \frac{A}{f_c} = \frac{2000}{6000} = \frac{200(20 \text{ in})(0.9 \times 72 \text{ in}^2)}{6000} = 4.32 \text{ in}^2 \]

\[ \frac{3\sqrt{f_c}}{f_d} = \frac{3\sqrt{4000}}{6000} = 20(0.9 \times 72) = 4.10 \text{ in}^2 \]

Hence \( A = 4.32 \text{ in}^2 \) controls.

Use three No. 8 bars on each face (three 3.8-mm diameter on each face), area = 4.74 in.\(^2\) (307 mm\(^2\)). Continue the reinforcement over all the span into the support as in Figure 6.18.

The maximum negative factored moment at an interior span is

\[ -M_n = \frac{wL^2}{12} = \frac{13.4 \times 4000(100)}{12} = 12 = 133,445 \text{ in.-lb (1600 kN-m)} \]

lever arm \( d = 0.2(l + 1.56) = 0.2(18 + 0.5 \times 72) = 49.2 \text{ in.} \)
6.10 BRACKETS OR CORBELS

Brackets or corbels are short-branched cantilevers that project from the inner face of columns to support heavy concentrated loads or beam reactions. They are very important structural elements for supporting precast beams, gantry girders, and any other forms of precast structural systems. Precast and prestressed concrete is becoming increasingly dominant, and larger spans are being built, resulting in heavier shear loads at supports.

Figure 6.19 Failure patterns: (a) diagonal shear; (b) shear friction; (c) anchorage splitting; (d) vertical splitting.
Hence the design of brackets and corbels has become increasingly important. The safety of the total structure could depend on the sound design and construction of the supporting element, in this case the corbel, necessitating a detailed discussion of this subject.

In brackets or corbels, the ratio of the shear arm or span to the corbel depth is often less than 1.0. Such a small ratio changes the state of stress of a member into a two-dimensional one, as discussed in the case of deep beams. Shear deformations would hence affect their nonlinear stress behavior in the elastic state and beyond, and the shear strength becomes a major factor. They differ from deep beams in the existence of potentially large horizontal forces transmitted from the supported beam to the corbel or bracket. These horizontal forces result from long-term shrinkage and creep deformation of the supported beam, which in many cases is anchored to the bracket.

The cracks are usually mostly vertical or steeply inclined pure shear cracks. They often start from the point of application of the concentrated load and propagate toward the bottom reentrant corner junction of the bracket to the column face as in Figure 6.19a, or start at the upper reentrant corner of the bracket or corbel and proceed almost vertically through the corbel toward its lower fibers, as shown in Figure 6.19b. Other failure patterns in such elements are shown in Figure 6.19c and d. They can also develop through a combination of the ones illustrated. Bearing failure can also occur by crushing of the concrete under the concentrated load-bearing plate, if the bearing area is not adequately proportioned.

As will be noticed in the subsequent discussion, detailing of the corbel or bracket reinforcement is of major importance. Failure of the element can be attributed in many cases to incorrect detailing that does not realize full anchorage development of the reinforcing bars.

As with deep beams, a totally different approach from the Shear Friction design approach of Section 6.10.1 presented herein, a Strut-and-Tie design approach as in Section 6.11 can also be used for the design of corbels. The correct force path in the strut-and-tie model has to be developed by the designer to render a safe design.

### 6.10.1 Shear Friction Hypothesis for Shear Transfer in Corbels

Corbels cast at different times than the main supporting columns can have a potential shear crack at the interface between the two concretes through which shear transfer has to develop. As discussed in the case of deep beams, the smaller the ratio a/h is, the larger the tendency for pure shear to occur through essentially vertical planes. This behavior is more accentuated in the case of corbels with a potential interface crack between two dissimilar concretes.

The shear friction approach in this case is recommended by the ACI as shown in Figure 6.19b. An assumption is made of an already cracked vertical plane (as in Figure 6.19b) along which the corbel is considered to slide as it reaches its limit state of failure. A coefficient of friction μ is used to transform the horizontal resisting forces of the well-anchored closed ties into a vertical nominal resisting force larger than the external factored shear load. Hence the nominal vertical resisting shear force...
to give:

\[ V_s = A_{sf} f_{ck} \]

\[ A_{sf} = \frac{V_s}{f_{ck}} \]

(6.27a)

(6.27b)

where \( A_{sf} \) is the total area of the horizontal, anchored closed shear ties.

The external factored vertical shear has to be \( V_c \leq \phi V_s \), where for normal concrete,

\[ V_c = 0.2 f_{ck} b_c d \]

(6.28a)

or

\[ V_c = 800 b_c d \]

(6.28b)

whichever is smaller. The required effective depth \( d \) of the corbel can be determined from Eq. 6.28a or 6.28b, whichever gives a larger value.

For all-lightweight or sand-lightweight concretes, the shear strength \( V_c \) should not be taken greater than \( (0.2 - 0.07 \frac{a}{d}) f_{ck} b_c d \) or \((800 - 200 \frac{a}{d}) b_c d \) in pounds.

If the shear friction reinforcement is inclined to the shear plane such that the shear force produces some tension in the shear friction steel,

\[ V_s = A_{fr} (a \sin \alpha + \cos \alpha) \]

(6.28c)

where \( \alpha \) is the angle between the shear friction reinforcement and the shear plane. The reinforcement area becomes

\[ A_{fr} = \frac{V_s}{f_{ck} (a \sin \alpha + \cos \alpha)} \]

(6.28d)

The assumption is made that all the shear resistance is due to the resistance at the crack interface between the corbel and the column. The ACI coefficient of friction \( \mu \) has the following values:

- Concrete cast monolithically: 1.4\( \mu \)
- Concrete placed against hardened roughened concrete: 1.0\( \mu \)
- Concrete placed against unroughened hardened concrete: 0.8\( \mu \)
- Concrete anchored to structural steel: 0.7\( \mu \)

![Figure 6.21](image)

**Figure 6.21** Reinforcement schematic for corbel design by shear friction hypothesis.
where $\lambda = 1.0$ for normal-weight concrete, 0.85 for sand-lightweight concrete, and 0.75 for all-lightweight concrete.

High values of the friction coefficient $\mu$ are used so that the calculated shear strength values are in agreement with experiments. If considerably higher strength concretes are used in the corbels, such as polymer-modified concretes, to interface with the normal concrete of the supporting columns, higher $\mu$ values could logically be used for such cases than those listed above. Work by the author in Ref. 6.13 substantiates the use of higher values.

Part of the horizontal steel $A_{sh}$ is incorporated in the top tension tie, and the remainder of $A_{sh}$ is distributed along the depth of the corbel as in Figure 6.22. Evaluation of the top horizontal primary reinforcement layer $A_{t}$ will be discussed in the next section.

6.10.2 Horizontal External Force Effect

When the corbel or bracket is cast monolithically with the supporting column or wall and is subjected to a large horizontal tensile force $N_a$, produced by the beam supported by the corbel, a modified approach is used, often termed the strut theory approach. In all cases, the horizontal factored force $N_a$ cannot exceed the vertical factored shear $V_c$. As seen in Figure 6.22, reinforcing steel $A_{t}$ has to be provided to resist the force $N_a$.

$$A_t = \frac{N_a}{f_y} \quad (6.29)$$

and

$$A_{sh} = \frac{V_a h + N_a (h - d)}{f_y d} \quad (6.30)$$

where $A_{sh}$ is the reinforcement area resisting the moment.

Reinforcement $A_{t}$ also has to be provided to resist the bending moments caused by $V_a$ and $N_a$.

Reinforcement $A_{sh}$ to resist tensile force $N_a$ should be determined from $N_a = \phi A_{sh} f_y$.

Tensile force $N_a$ should not be taken less than 0.2 $V_a$ unless special provisions are

![Figure 6.22: Composite action in corbel.](image-url)
made to avoid tensile forces. Tensile force $N_{tr}$ should be regarded as a live load even when tension results from creep, shrinkage or temperature changes.

The value of $N_{tr}$ considered in the design should not be less than 0.20$V_c$. The flexural steel area $A_f$ can be approximately obtained by the usual expression for the limit state at failure of beams, that is,

$$A_f = \frac{M_r}{6f_{yd}} \quad (6.31)$$

where $M_r = V_{tc} + N_{tr}$ ($h - d$). The axis of such an assumed section lies along a compression strut inclined at an angle $\beta$ to the tension tie $A_r$, as shown in Figure 6.22. The volume $C_c$ of the compressive block is

$$C_c = 0.85f_{yd}ch = \frac{T_r}{\cos \beta} \frac{A_f}{\cos \beta} = \frac{V_{tc}}{\sin \beta} \quad (6.32a)$$

for which the depth $b_c$ of the block is obtained perpendicular to the direction of the compressive strut,

$$b_c = \frac{A_f f_y}{0.85f_{yd}b \cos \beta} \quad (6.32b)$$

The effective depth $d$ minus the $b_c/2 \cos \beta$ in the vertical direction gives the lever arm $jd$ between the force $T_r$ and the horizontal component of $C_c$ in Fig. 6.22. Therefore,

$$jd = d - \frac{b_c}{2 \cos \beta} \quad (6.32c)$$

If $jd$ is substituted in Eq. 6.31,

$$A_f = \frac{M_r}{6f_{yd}jd - b_c/2 \cos \beta} \quad (6.33)$$

To eliminate several trials and adjustments, the lever arm $jd$ from Eq. 6.32c can be approximated for all practical purposes in most cases as

$$jd = 0.85d \quad (6.34a)$$

so that

$$A_f = \frac{M_r}{0.85f_{yd}d} \quad (6.34b)$$

The area $A_f$ of the primary tension reinforcement (tension tie) can now be calculated and placed as shown in Figure 6.23.

---

*Figure 6.23 Reinforcement schematic for corbel design by strut theory.*
\[ A_s = \frac{2}{3} A_{tv} + A_e \]  
(6.35)

or

\[ A_s \geq A_t + A_e \]  
(6.36)

whichever is larger:

\[ p = \frac{A_{fx}}{bd} \geq 0.04 \frac{f_f}{f_y} \]  
(6.37)

where \( A_e \) = area of reinforcement resisting tensile force \( N_{te} \).

If \( A_e \) is assumed to be the total area of the closed stirrups or ties parallel to \( A_s \),

\[ A_e \geq 0.5 (A_{tv} - A_e) \]  
(6.38)

The bearing area under the external load \( V_e \) on the bracket should not project beyond the straight portion of the primary tension bars \( A_{tv} \), nor should it project beyond the interior face of the transverse welded anchor bar shown in Figure 6.23.

### 6.10.3 Sequence of Corbel Design Steps

As discussed in the preceding section, a horizontal factored force \( N_{te} \), a vertical factored force \( V_{pe} \), and a bending moment \([V_{pe} = N_{te} (h - d)]\) basically set on the corbel. To prevent failure, the corbel has to be designed to resist these three parameters simultaneously by one of the following two methods, depending on the type of corbel construction sequence, that is, whether the corbel is cast monolithically with the column or not:

1. For monolithically cast corbel with the supporting column, by evaluating the steel area \( A_s \) of the closed stirrups that are placed below the primary steel tensile area \( A_{tv} \). Part of \( A_s \) is due to the steel area \( A_t \) from Eq. 6.29 resisting the horizontal force \( N_{te} \).

2. Calculating the steel area \( A_s \) by the shear friction hypothesis if the corbel and the column are not cast simultaneously, using part of \( A_t \) along the depth of the corbel stem and incorporating the balance in the area \( A_s \) of the primary top steel reinforcing layer.

The primary tension steel area \( A_{tv} \) is the major component of both methods 1 and 2. Calculations of \( A_{tv} \) depend on whether Eq. 6.35 or 6.36 governs. If Eq. 6.35 controls, \( A_e = A_{tv} - A_s \) is used and the remaining \( 1/3 \) is distributed over a depth \( l \) adjacent to \( A_s \).

If Eq. 6.36 controls, \( A_e = A_t \times \frac{l}{l} \) is distributed over a depth \( l \) adjacent to \( A_s \).

In both cases, the primary tension reinforcement and the closed stirrups automatically yield the total amount of reinforcement needed for either type of corbel. Since the mechanism of failure is highly inelastic, and randomness can be expected in the propagation of the shear crack, it is sometimes advisable to choose a larger calculated value of the primary top steel area \( A_{tv} \) in the corbel regardless of whether the corbel element is cast simultaneously with the supporting column.

The horizontal closed stirrups are also a major element in reinforcing the corbel, as seen from the foregoing discussions. Occasionally, additional inclined closed stirrups are also used.

The following sequence of steps is proposed for the design of the corbel:

1. Calculate the factored vertical force \( V_e \), and the nominal resisting force \( V_{pe} \), of the section such that \( V_{pe} \geq V_e / \delta \), where \( \delta = 0.75 \) for all calculations. \( V_{pe} / \delta \) should be

\[ 0.29f_y h_d \]  

or \( 0.39f_y d \) if \( h \leq d \). If \( h > d \), the concrete section at the support should be equal.
2. Calculate \( A_t = V/\mu \) for resisting the shear friction force and use in the subsequent calculation of the primary tension top steel \( A_{tu} \).

3. Calculate the flexural steel area \( A_f \) and the direct tension steel area \( A_{tu} \) where
   \[
   A_f = \frac{V_u + N_u(h - d)}{f_y \text{bd}} \quad \text{and} \quad A_{tu} = \frac{N_u}{f_y}.
   \]

4. Calculate the primary steel area:
   (a) \( A_s = \frac{1}{4} A_{tu} + A_{tu} \)
   (b) \( A_s = A_{tu} + A_{tu} \)

and select whichever is larger. If case (a) controls, the remaining \( A_{tu} \) needs to be provided as closed stirrups parallel to \( A_{tu} \) and distributed within a \( 1d \) distance adjacent to \( A_{tu} \), as in Figure 6.2. If case (b) controls, use in addition \( 1A_s \) as closed stirrups distributed within a distance \( 1d \) adjacent to \( A_{tu} \), as in Figure 6.23.

5. Select the size and spacing of the corbel reinforcement with special attention to the detailing arrangements, as many corbel failures are due to incorrect detailing.

A flowchart for proportioning corbels is given in Figure 6.24.

6.10.4 Example 6.6: Design of a Bracket or Corbel

Design a corbel to support a factored vertical load \( V_u = 60,000 \text{ lb (160 kN)} \) acting at a distance \( d = 5 \text{ in. (127 mm)} \) from the face of the column. It has a width \( b = 10 \text{ in. (254 mm)} \), a total thickness \( h = 18 \text{ in. (457 mm)} \), and an effective depth \( d = 14 \text{ in. (356 mm)} \). Given:

- \( f' = 5000 \text{ psi (34.5 MPa)} \): normal-weight concrete
- \( f_y = 60,000 \text{ psi (414 MPa)} \)

 supporting column size: \( 12 \times 18 \text{ in.} \), and corbel width \( = 18 \text{ in.} \).

Assume the corbel to be either cast after the supporting column was constructed or both cast simultaneously. Neglect the weight of the corbel.

Solution:

**Step 1**

\[
V_* \geq \frac{V_u}{0.75} = \frac{60,000}{0.75} = 80,000 \text{ lb}
\]

\[
0.2/bd = 0.2 \times 5000 \times 10 \times 14 = 400,000 \text{ lb} > V_*
\]

800bd = 800 \times 10 \times 14 = 10,000 \text{ lb} > V_* \quad \text{O.K.}

**Step 2**

(a) Monolithic construction; normal-weight concrete \( \mu = 1.4 \):

\[
A_{tu} = \frac{V_u}{f_y \mu} = \frac{106,667}{60,000 \times 1.4} = 1.27 \text{ in.}^2 (80 \text{ mm}^2)
\]

(b) Nonmonolithic construction; \( \mu = 2.0 \).
Choose the larger $A_e = 1.78$ in as controlling.

\[ A_e = 1.78 \text{ in} \]

Figure 6.24: Flowchart for design of columns.

Chapter 6: Biaxial and Diagonal Tension in Beams
minimum \( N_c = 0.2 C = 0.2 \times 80,000 = 16,000 \) lb

\[ A_t = \frac{M_o + V_d \pm V_s (d - d)}{6f_{yd} d} \quad \text{where} \quad d = 0.85d \]

\[ = \frac{80,000 \times 5 + 16,000(1 - 14)}{0.75 \times 60,000(0.85 \times 14)} = 0.87 \text{ in.}^2 (531 \text{ mm}^2) \]

\[ A_n = \frac{N_c}{4f_{yd}} = \frac{16,000}{0.75 \times 60,000} = 0.36 \text{ in.}^2 (278 \text{ mm}^2) \]

**Step 4**

Check the controlling area of primary steel \( A_n \)

(a) \( A_n = \left( \frac{A_n}{A_n} + A_s \right) = 0.78 + 0.36 = 1.55 \text{ in.}^2 \)

(b) \( A_n = A_s = 0.87 + 0.36 = 1.23 \text{ in.}^2 \)

minimum \( A_n = 0.84 \times \frac{f_{yd}}{f_{y}} = 0.44 \times 60,000 \times 0.14 = 4.67 \text{ in.}^2 \)

\( < 1.55 \text{ O.K.} \)

Provide \( A_n = 1.55 \text{ in.}^2 (960 \text{ mm}^2) \). Horizontal closed stirrups: Sine case (a) control.

\( A_s = 0.5(A_n - A_s) = 0.5(1.55 - 0.36) = 0.60 \text{ in.}^2 \)

**Step 5**

Select bar sizes:

(a) Required \( A_n = 1.55 \text{ in.}^2 \); use three No. 7 bars = 1.80 \text{ in.}^2 (three bars of diameter 22.2 mm = 1.81 \text{ mm}^2)

(b) Required \( A_s = 0.60 \text{ in.}^2 \); use three No. 3 closed stirrups = 2 \times 3 \times 0.1 = 0.66 \text{ in.}^2 \) spread over \( 3d = 9.33 \text{ in. vertical distance. Hence use three No. 3 closed stirrups at } 3 \text{ in. center} \); also use three framing size No. 3 bars and one welded No. 3 anchor bar.

Details of the bracket reinforcement are shown in Figure 6.25. The bearing area under the load has to be checked and the bearing pad designed such that the bearing stress at the factored load \( V_s \) should not exceed 70% of \( 6(0.85f_{yd}) \), where \( f_{yd} \) is the yield strength.

**Photo 6.6** High-strength concrete corbel at failure. (Nawy et al.)
Design \( V_c = 80,000 \) \( \frac{d}{b} = \phi \cdot (0.5 f'/A_i) \), where \( \phi = 0.70 \),
\[
A_i = 0.70 \times 0.85 \times 5000
\]
Use a plate 5\( \frac{1}{8} \) in. \( \times 5\) in. Its thickness has to be designed based on the manner in which \( V_c \) is applied.

For the alternative strut-and-tie solution, see Example 6.8.

6.11 STRUT AND TIE MODEL ANALYSIS AND DESIGN OF CONCRETE ELEMENTS

6.11.1 Introduction

As an alternative to the usual approach for plane sections before bending remaining plane after bending, the strut-and-tie model is applied effectively in regions of discontinuity. These regions could be the support sections in a beam, the zones of load application, the discontinuity caused by abrupt changes in a section, such as brackets, beam daps, pile caps, or column sections, portal frames, and others. Consequently, structural elements can be divided into segments called \( B \)-regions, where the standard beam theory applies, with the assumption of linear strains, and the others as \( D \)-regions, where the plane sections hypothesis is no longer applicable. Figure 6.26 (adapted from Ref. 6.1) demonstrate the locations of \( B \) and \( D \) regions.

The analysis essentially follows the truss analogy approach, where parallel inclined cracks are assumed and expected to form in the regions of high shear. The concrete between the inclined cracks carries inclined compressive forces such as in Fig. 6.7A(a) and
(b), acting as diagonal struts. Thus, provision of transverse stirrups along the beam span, as in Fig. 6.7B(c), results in truss-like action in which the longitudinal steel provides the tension chord of the truss as it is, hence the "strut-and-tie" expression.

Strut-and-tie modeling has been introduced in some codes including ACI 318-02. Appendix A. Simplifying assumptions in design have to be made when applying this approach to different structural systems. These simplifications are necessitated because of the wide range of alternatives in the selection of the paths of forces, which represent the compressive struts and the tension ties intersecting at the "nodal" points, and the choice of locations where the corresponding reinforcement is to be placed.

Therefore, depending on the interpretation of the designer, simplifications of the paths of forces that are chosen to represent the real structure can considerably differ. Since this node-path modeling method is a plastic method with stress concentration conditions and load concentration, it does not provide a check for the serviceability levels inherent in the semi-elastic methods, but represents strength-limit states at the critical sections. Thus, after excessive deformation and cracking, the idealizations made in the choice of the force paths render this method less accurate for design purposes, particularly because no unique design solutions are possible. This approach is more an art than an engineering science in the selection of the models. Significant overdesign is, therefore, required and extensive full-scale tests are needed for different structural systems. Such extensive tests were conducted in the case of anchor blocks in post-tensioned beams, as discussed by the author in Reference 6.26. It should be thus emphasized that this approach is a design method that enables analyzing semi-elastic regions, particularly at

Figure 6.26 Segrions and D-regions in beams: (a) continuous beam; (b) beam with concentrated load; (c) deeper-ended beam on column support
affected by shear and torsion, with infinite possible configurations for identifying load paths in structural systems (Ref. 6.21 and Ref. 6.27) and that it does not provide a check on serviceability, as it principally deals with high-overload conditions and with load-carrying capacity.

6.11.2 Strut-and-Tie Mechanism

For equilibrium, at least three forces have to act at a joint, termed as the node, as in Figure 6.27, where \( C \) = compression vector and \( T \) = tension vector. The nodes are classified in accordance with the sense of the forces intersecting at the nodal point. As an example, a \( C-C-T \) node resists one tensile force and two compressive forces. A typical representation of the strut-and-tie model of a simply supported deep beam is shown in Figure 6.39, and for a continuous beam in Fig. 6.29. A \( C-C-T \) nodal zone can be represented as a hydrostatic nodal zone if the tie is assumed to extend through the node and anchored by a plate on the far end of the node (Ref. 6.1, Appendix A). Typical nodal zones are shown in Figs. 6.30 and 6.31 (Ref. 6.1), including the possible distribution of the steel reinforcement through the nodal zones. Fig. 6.32 demonstrates the simplified truss model for simply supported deep beams loaded on the top fibers. Note that nodes \( A \) and \( B \) at the beam support are compressive nodes as seen by the crushing of the concrete in Fig. 6.32(d).

In order to design the critical \( D \)-region, the following steps need to be taken:

1. Isolate each \( D \)-region
2. Compute the stresses, which act on the boundaries of the \( D \)-region, replacing them with one or more resultant forces on each boundary.
3. Select a truss model to transfer the resultant forces across the \( D \)-region.

![Diagram of strut-and-tie mechanism nodes](image)
Figure 6.28: Slab-anode model of a simply supported deep beam subjected to concentrated load on top.

Figure 6.29: Typical strut-and-tie model for continuous beams subjected to concentrated loads on top: (a) positive moment strut, (b) negative moment strut.
Figure 6.30  Typical nodal zones: (a) three struts going at node; (b) two struts AE and CE acting at node; (c) support nodal zone; (d) subdivided nodal zone.

If more than three forces act at a nodal point as shown in Fig. 6.30 (b), it becomes necessary to exercise engineering judgment in resolving the system of forces such that only three forces act at the nodal point. That is why no unique solution is possible, as assumptions based on significant idealizations can widely differ (Ref. 6.27). The angle between the axes of struts and ties that intersect through the node should not be too small, namely, not less than 25°, in order to avoid any incompatibilities that can result because of the lengthening of the ties and the shortening of the struts occurring in the same direction. Figure 6.32(c) represents a simplified idealized truss model for the principal compressive and tensile stress trajectories resulting from the applied distributed load at the top deep beam fibers. The assumed truss model is one alternative. Other possible alternative models can also be used, provided that they satisfy equilibrium and compatibility. Figure 6.33 (Refs. 6.22, 6.23), to follow, is a modified more rigorous model for Example 6.7, than the simplified Fig. 6.34 of the solution.

It is important to recognize that the decisions made in steps 2 and 3 are very critical in arriving at a sufficiently representative model and a safe structure. The axes of the struts and ties are selected to coincide with the axes of the compression and tension fields and the forces in the struts and ties computed. Serviceability limit checks have thereafter to be applied. The vertical and horizontal components equilibrate the forces in the inclined strut, as is usually done in a truss analysis (see Fig. 6.34) of the forces in the nodal zones.
6.11.3 ACI Design Requirements

(1) Nodal Forces

\[ \phi F_0 \geq F_n \]  \hspace{1cm} (6.38)

where, \( F_0 \) = nominal strength of a strut, tie, or nodal zone, lb.
\( F_n \) = factored force acting on a strut, tie, bearing area, or nodal zone, lb.

where \( \phi = 0.75 \) for both struts and ties (similar to the strength reduction for shear)

(2) Strength of Struts

\[ F_m = f_n A_n \]  \hspace{1cm} (6.39)

where \( F_m \) = nominal strength of strut, lb
\( A_n \) = effective cross-sectional area at one end of a strut, taken perpendicular to the axis of the strut, in.²
\( f_n \) = effective compressive strength of the concrete in a strut or nodal zone, psi

\[ f_n = 0.85 f_c^0 \]  \hspace{1cm} (6.40)

(in the ACI Code, \( f_n \) is designated as \( f_{nc} \))
Figure 6.32: Truss model and stress distribution in simply supported deep beams: (a) lines of principal stress trajectories for beams loaded on top; (b) elastic stress distribution across beam depth; (c) idealized truss model (adapted from Refs. 6.22, 6.23); (d) cracking pattern.

where

- $\beta_1 = 1.0$ for struts which have the same cross-sectional area of the midstrut cross-section in case of bubble struts.
- $= 0.75$ for struts with reinforcement resisting transverse tensile forces
- $= 0.40$ for struts in tension members or tension flanges
- $= 0.50$ all other cases

For $f'_c$ not greater than 6000 psi, the strut configuration and the compressive forces in the strut can be satisfied if

$$\sum \beta_{1j} \sin \gamma_j \geq 0.003,$$

where $A_s$ is the total area of reinforcement at spacing $s$, in a layer of reinforcement with bars at an angle $\alpha$ to the axis of the strut.
3. **Longitudinal Reinforcement**

\[ F_{cs} = f_{cu} A_{cs} + A_{r} f'_{c} \]

where \( A_{cs} \) = area of compression reinforcement in a strut, in.\(^2\)

\( f'_{c} \) = stress in compression reinforcement, psi.

4. **Strength of Ties**

\[ F_{t} = A_{ts} f_{t} + A_{p} (f_{pt} + \Delta f_{p}) \]

where \( F_{t} \) = nominal strength of tie, lb.

\( A_{ts} \) = area of non-prestressed reinforcement in a tie, in.\(^2\)

\( A_{p} \) = area of prestressing reinforcement, in.\(^2\)

\( f_{t} \) = effective stress after losses in prestressing reinforcement

\( \Delta f_{p} \) = increase in prestressing stress beyond the service load level

\( f_{pt} \), \( f_{pt} + \Delta f_{p} \) should not exceed \( f_{cu} \). When no prestressing reinforcement is used, \( A_{p} = 0 \) in Equation 6.42.

\[ h_{cs} = F_{nl}/F_{cs} \]

where \( h_{cs} \) = maximum effective height of concrete concentric with the tie, used to dimension nodal zone, in.

\( \Delta f_{p} \) can be taken as 60,000 psi for bonded prestressed reinforcement, or 10,000 psi for non-bonded reinforcement.

If the bars in the tie are in one layer, the effective height of the tie can be taken as the diameter of the bars in the tie plus twice the cover to the surface of the bars. The reinforcement in the ties have to be anchored by hooks, mechanical anchorages, post-tensioning anchors, or straight bars, all with full development length.

5. **Strength of Nodal Zones**

\[ F_{n} = f_{cu} A_{nc} \]

(6.54)
where $F_c = \text{nominal strength of a face of a nodal zone, lb.}$

$A_n = \text{area of face of a nodal zone or a section through a nodal zone, in.}^2$

It can be assumed that the principal plane directions in the struts and ties act parallel to the planes of the struts and ties. Under such a condition, the stresses on faces perpendicular to these planes are principal stresses.

(6) Confinement in the Nodal Zone

The ACI 318-05, Appendix A, stipulates that unless confining reinforcement is provided within the nodal zone and its effect is supported by analysis and experimentation, the computed compressive stress on a face of a nodal zone due to the strut and tie forces should not exceed the values given by Eq. 6.45 (Ref. 6.1):

$$f_c = 0.85 f_{c}$$

where $f_{c} = 1.0$ in nodal zones bounded by struts or bearing stresses

$= 0.6$ in nodal zones anchoring one tie

$= 0.4$ in nodal zones anchoring two or more ties

In the case of corbels, the area $A_s$ to control shear cracks has to satisfy the expression

$$A_s \geq 0.5 (A_n - A_o)$$

where $A_o = \text{area of reinforcement resisting tensile force } N_o$.

6.11.4 Example 6.7: Design of Deep Beam by Strut-and-Tie Method

Solve Example 6.3 using the strut-and-tie method in designing the flexural and shear reinforcement for the indicated deep beam.

Solution:

(1) Truss model selection

The uniformly distributed load on the beam top is idealized by two concentrated loads as shown in Fig. 6.32 and detailed in Fig. 6.33. The crossing truss model can be considered to simulate the stress trajectories of the principal stresses. $f_{th} = 100$ ksi = 1.07 - 2.17, hence a deep beam.

Strut inclination angle $\theta$ in Figure 6.32 (c) is interpolated between $\theta = 68^\circ$ for $f_{th} = 1.0$ and $\theta = 54^\circ$ for $f_{th} = 1.2$ (Ref. 6.25)

Hence, $\theta = 68^\circ - (68^\circ - 54^\circ)(1.07 - 1.0)$

$= 68^\circ - 0.67 \times 14 = 58.62^\circ$

Assume supports center line span = 11 ft 5 in.

Assume $c = \text{cover to centroid of tensile reinforcement} = 7$ in.

Vertical distance of node C in Figure 6.33(a) from the centroid of the tensile reinforcement $= \frac{20}{3} = 6.67$ in.

Length CD = 11 ft 5 in. - 2(6.67) = 11.424 ft (use 11.42 ft)

Intensity of total factored distributed load from Ex. 6.3 $w_o = 131.400$ lb/ft

Idealized equivalent concentrated load is

$$P_o = \frac{w_o \times 10}{2} = \frac{131.400 \times 10}{2} = 657.000 \text{ lb/strut (Fig. 6.33)}$$
Figure 6.34. Two-end tie model in example 6.7. (a) Idealized truss model; (b) ties forces (C = compression, T = tension); (c) forces on joint C (C-C-C node); (d) forces on joint A (C-C-T node).
Chapter 6  Shear and Diagonal Tension in Beams

Compressive forces in each of the compressive struts EC and FD = 876,000 lb.

(2) Stirrup selection

Trying No. 5 vertical stirrup bars, \( A_s = 2 \times 0.31 = 0.62 \) in. \[ \text{stirrup} \]

Force per vertical stirrup bar on each of the two faces of the deep beam (stirrup):

\[ = 0.62 \times 60,000 = 37,200 \text{ lb.} \]

Number of vertical stirrups needed is \[ = 876,000 / 37,200 = 23.5 \]

Spacing = \[ = 11.0 \times 12 = 5.52 \text{ in. on center.} \]

Use No. 5 vertical bars at 5 1/8 in. c/e at each of the two faces of the beam (0.68 in.).

From Eq. 6.21b Minimum \[ A_s = 0.0025 \times 20 \times 5.5 = 0.275 \text{ in.}^2 < 0.68 \text{ in.}^2 \]

O.K.

(3) Vertical and horizontal components of forces in the truss model

The compressive forces in the truss shown in Figure 6.33(b) are computed in the usual manner, as the vector sine and cosine components of the inclined struts at the assumed \( \theta = 58.62^\circ \).

Figure 6.33(c) gives the intersecting nodal forces in the D-region of node C, giving a strut force in CA = 1,078,082 lb. (C-C node).

Figure 6.33(d) gives the intersecting forces in the D-region of node A, giving a tie force in truss member AB = 334,393 lb. (C-C node).

Evidently, idealizing the distributed load into four or six equivalent concentrated loads would have reduced the compressive forces in the struts and the tensile forces in the ties, leading to reduced reinforcement areas that ideally became relatively closer to the reinforcement area obtained in Ex. 6.3 (see truss model in Fig. 6.34).

(4) Horizontal and vertical reinforcement across depth of beam web to control cracking

Horizontal web reinforcement is not required as part of the truss. However, in order to control cracking, ACI-318 Code requires an area \[ A_s = 0.0015 b_s t_s \] as in Example 6.3.

Assuming a spacing of 12 in. on centers:

\[ \text{Min } A_s = 0.0015 b_s t_s = 0.0015 \times 20 \times 12 = 0.35 \text{ in.}^2 \]

Use No. 4 bars at 12 in. on centers as horizontal reinforcement in each of the two faces of the deep beam (0.40 in. each).

Vertical web reinforcement to control cracking: assume a spacing of 8 in. on centers.

\[ \text{Min } A_s = 0.0025 b_s t_s = 0.0025 \times 20 \times 8 = 0.40 \text{ in.}^2 \]

Use No. 4 bars at 8 in. on centers as vertical reinforcement in each of the two faces of the deep beam.

(5) Strength of struts

From Equation 6.40, compressive strength of concrete in a strut or nodal zone is

\[ f_c = 0.85 f_c', \text{ where } f_c' = 0.75 \times 43,000 = 25500 \text{ psi.} \]

Required strength of struts \[ C, D, A \text{ is } \]

\[ f_{cu} = 1,026,082 \text{ lb.} \]

From Equation 6.39,

\[ f_{cu} = f_{cu} A_s \leq 1,026,082 \times 2.55 / A_s \text{, hence } A_s \leq 1,026,082 \times 2.55 / 402 \text{ in.}^2 \]
Width of CM, DB = 402/20 = 201 in., which is within the available area of the deep beam, OK.

(6) Strength of ties

Required strength \( F_e \) = 534,293 lb.

From Equation 6.42,

\[
F_e = A_e f_e = A_e (f_e + 4f_t)
\]

or 534,293 = \( A_e \times 60,000 \)

\[
A_e = \frac{534,293}{60,000} = 8.9 \text{ in.}^2
\]

Trying No. 10 bars, \( n = \frac{8.9}{1.25} = 7.0 \)

Use 8 No. 10 bars in 4 layers of two bars at 3 in. on centers.

\( d_e = 2.15 \text{ in.} + 3 \times 1.5 = 7 \text{ in.}, \) assumed in constructing the true model dimensions.

From Equation 6.43, the maximum height of concrete concrete with the tie for dimensioning the nodal zone is

\[
f_{con} = F_{e} / f_{ce} = \frac{534,293}{2.550} = 210 \text{ in.}
\]

Actual tie height = 2.15 + 3 + 3 + 2.15 = 13.3, say 14 in., accept.

Anchor the No. 10 bars using hooks at bar ends with full development length. Check the development length.

(7) Strength of nodal zones

From Equation 6.45.

Maximum allowable concrete strength in the nodal zone anchoring non-confined two or more ties is

\[ f_{c} = 0.85 f_{ce}, \] where \( f_{ce} = 0.6 \)

or \( f_{c} = 0.85 \times 0.6 \times 4,000 = 2,040 \text{ psi.} \)

Surface area of node perpendicular to CM:

\[
A_c = 20 \left( \frac{14}{\cos \theta} \right) = 21 \times \frac{14}{0.521} = 537 \text{ in.}^2
\]

From Equation 6.44, the nominal strength of the nodal force is

\[
F_n = f_c A_c = 2,040 \times 537 = 1,095,480 \text{ lb.} > 1,026,382 \text{ lb., OK.}
\]

Confined of the nodal zone is not required, since the stress in the concrete in the nodal zone did not exceed the calculated permissible \( f_{c} = 2,040 \text{ psi.} \)

Hence, adopt the design.

Another true model simulating the uniformly distributed load on the top of the beam by low concentrated loads instead of two, could have reduced the amount of the horizontal reinforcement. Such a true model could have the form shown in Fig. 6.33 (Ref. 6.32) in idealizing the stress trajectories of the principal stresses in the deep beam. Careful engineering judgment has to be exercised in the selection of the path of forces on the basis of the principles outlined in Section 6.1.1, to determine whether the resulting reinforcement is excessive or relatively efficient. Principles of equilibrium and compatibility have to be maintained in any chosen model (Ref. 6.37)

Comparison of the solution in Example 6.7 to that in Example 6.3 demonstrates the conservative values obtained in the strut-and-tie solution. This can possibly be justified because of the inherent large variability and wide range of assumptions that can be made by the designer in the selection of the path of forces that produce the truss model.
6.11.5 Example 6.8: Design of Brackets and Corbels by the Strut-and-Tie Method

Design the corbel in Example 6.6 by the strut-and-tie method.

Solution:
Column size: 12 x 18 in.
Corbel width = 18 in.

(1) Ties Model Selection
Assume the corbel is monolithically cast with the column. The total depth b = 18 in. and effective depth d = 14 in. are based on the requirement that the vertical dimension of the corbel outside the bearing area is at least one half the column face width of 14 in. (column size: 12 x 14 in.)

Select a simple strut-and-tie model as shown in Fig. 6.34, assuming that the center of tie AB is located at a distance of 4 in. below the top extreme corbel fiber, using one layer of reinforcing bars. Also assume that horizontal tie DO lies on a horizontal line passing at the re-entrant corner C of the corbel. The solid lines in Fig. 6.34 denote tension tie action (T), and the dashed lines denote compression strut action (C). The nodal points A, B, C, D result from the selected strut-and-tie model. Note that the entire corbel is a D-region structure because of the resulting statical discontinuities in the geometry of the corbel and the vertical and horizontal loads.

(2) Strut-and-tie stress forces

From Example 6.6: \( V_u = 80,000 \) lb,

\[ N_{ax} = 0.20 \cdot V_u = 16,000 \text{ lb} \]

The following are the strut member forces calculated from states in Fig. 6.34:

a) Compression strut BC:

Length BC = \( \sqrt{(7)^2 + (14)^2} = 15.652 \) in.

\[ F_{BC} = 80,000 \cdot \frac{15.652}{14} = 89,443 \text{ lb} \]

b) Tension tie BA:

\[ F_{BA} = 80,000 \cdot \frac{7}{14} = 40,000 = 56,000 \text{ lb} \]

c) Compression strut AC:

\[ F_{AC} = \frac{26,000 \sqrt{(9)^2 + (14)^2}}{8} = 112,072 \text{ lb} \]

d) Tension tie AD:

\[ F_{AD} = \frac{112,072 \cdot 14}{\sqrt{(8)^2 + (14)^2}} = 88,000 \text{ lb} \]

e) Compression strut CC:

\[ F_{CC} = 80,000 \cdot 9,000 = 178,000 \text{ lb} \]

f) Tension tie CD:

\[ F_{CD} = 56,000 \cdot \frac{7}{14} = 16,000 \text{ lb} \]

(3) Steel Bearing Plate Design:

\[ f_s = 0.85 f'c' \text{ where } f'c' = 3,000 \text{ psi} \]

Area of plate is

\[ A_1 = \frac{80,000}{0.75(0.85(f'c')^2)} = \frac{80,000}{0.75 \cdot 0.85 \cdot 3000} = 25.10 \text{ in}^2 \]

The 5 x 5 in plate and 0.10 in. thick steel is used in this section.
Figure 6.34 Strut-and-Tie Model in Example 6.8.

(4) Tie Reinforcement Design

\[ A_{t,1} = \frac{56,000}{0.75 \times 60,000} = 1.25 \text{ in.}^2 \]

Use 3 # 6 bars = 1.32 in.\(^2\), or, conservatively, 3 # 7 bars = 1.80 in.\(^2\) as in Example 6.6.

These top bars in one layer have to be fully developed along the longitudinal column reinforcement.

\[ A_{t,2} = \frac{16,000}{0.75 \times 60,000} = 0.26 \text{ in.}^2 \]

Use 2 # 6 tie bars = 0.88 in.\(^2\) to form part of the cage shown in Fig. 6.35.
(5) **Horizontal Reinforcement** $A_h$ for Crack Control of Shear Cracks

$$A_h = 0.50(A_{xx} - A_{xx}^*)$$

where $A_{xx}$ is reinforcement resisting the internal force $N_{xx}$,

$$N_{xx} = \frac{V_{uu}}{\gamma_{xx}} = \frac{16000}{0.75 \times 60000} = 0.02 \text{ in.}^2$$

Hence, $A_h = 0.50(0.25 - 0.36) = 0.45 \text{ in.}^2$.

Try 3 #3 closed ties evenly spaced vertically as shown in Fig. 6.35, giving $A_h = 3(0.11) = 0.33 > 0.45 \text{ in.}^2$, O.K.

Because $\theta = 0.75$ is used for calculating the effective concrete compressive strength in the struts in the following section, where $f_{cu} = 0.85 f_{cu}^*$, the minimum reinforcement provided has also to satisfy:

$$\sum \frac{A_{xx}}{bh} \sin \theta = 0.013$$

$$= \frac{20 \times 11}{14 \times 3.0} \sin 60^\circ / 5^\circ = 0.0045 \geq 0.003 \text{ O.K.}$$

Hence, adopt 3 #3 closed ties at 3.0 in. c. to c. spacing.
6.12 S/Design Expressions and Example for Shear Design

(6) Stress Capacity Evaluation

(i) Strike CC

The width, \( w \), of nodal zone A has to satisfy the allowable stress limit on the nodal zones, namely, node B below the bearing plate and node C in the re-entrant corner to the column. Both nodes are CCT nodes and considered unloaded.

Because the non-confinement of the nodes \( f_{cc} \), \( f_{cb} < 0.85 \) \( f_{cc} \), where \( f_{cb} < 0.80 \) for a nodal zone anchoring one. Hence, \( f_{cc} = 0.85 \times 0.80 \times 5000 = 3400 \) psi.

\[ F_{cc} = f_{cc} b_{cc} w_{cc} = 0.75 \times 3400 \times 18 \times w_{cc} = 45.9 \text{ kips} \]

where \( w_{cc} \) is min. width of the strut. Taking moment about node D, \( F_{cc} = (10 - w_{cc}) / 20 = 0.5 (5 + 10) = 10 \times 18 \) to give min. \( w_{cc} \), 2.4 in. But allowable cover \( w_{cc} \), 2.4 in; hence \( F_{cc} = 45.9 \times 4.0 = 83.6 \text{ kips} > \text{actual 17 kips} \). O.K.

(ii) Strike BC

Nominal strength is limited to \( F_{bc} = f_{bc} A_{bc} \), where \( f_{bc} = 0.85 \), \( f_{bc} \)

\[ F_{bc} = 0.85 \times 0.75 \times 5000 = 3188 \text{ kips} \]

\( A_{bc} \), the smaller strut cross-sectional area at the two ends of the strut, namely, at node C, while at node B, the node width may be assumed equal to the steel plate width of 5.50 in.

\[ A_{bc} = 0.75 \times 3188 \times 40.32 = 96.4 \text{ kips} > \text{required } F_{bc} = 89.4 \text{ kips} \). O.K.

(iii) Strike AC

Required min. width, \( w \), of strut AC

\[ w_{min} = \frac{F_{ac}}{\sigma_{ac}} = \frac{112.87 \text{ kips}}{0.75 \times 3188 \times 18} = 2.62 \text{ in.} \]

Examination of the corbel and column depth of 12 in., shows there is a minimum clear cover of 2.6 in. from the exterior concrete surface. Hence the width, \( w_{min} \), of all struts fit within the corbel geometry.

Adopt the design as shown in Fig. 6.35.

6.12 S/Design Expressions and Example for Shear Design

Equation 6.6:

\[ V_c = \left( \lambda \sqrt{f_c} + 12f_p \right) \frac{V_c}{M_{ds}} \]

Equation 6.9:

\[ V_c = \lambda \frac{\sqrt{f_c}}{6} b_c d \]

where \( \lambda = 1.0 \) for normal weight concrete; 0.75 for all lightweight concrete.

Equation 6.10:

\[ V_c = \lambda \left( 1 + \frac{N_{ec}}{14A_c} \right) \frac{\sqrt{f_c}}{6} b_c d \]

where \( N_{ec} \) is expressed in MPa.

Equation 6.15a:

\[ V_c = \frac{A_c f_c d}{s} \]

Equation 6.15b:

\[ V_c = \frac{A_c f_c d}{(V_c / h) / V_c} \]

Min \( A_c = \frac{b_c d}{3f_c} \) where \( b_c \) and \( d \) are expressed in mm and \( f_c \) in MPa.

Limitations on Spacing Shortcuts

1. \( V_c - V_c > \frac{\sqrt{f_c}}{3} b_c d \) \: \( b_{max} = \frac{d}{4} \leq 601 \text{ in.} \)

- \( V_c \) and \( V_c \) are the shear forces in the column and strut, respectively.
- \( f_c \) is the concrete strength.
- \( b_c \) is the corbel width.
- \( d \) is the strut depth.
- \( s \) is the spacing of the struts.
6.12.1 Example 6.8: SI Shear Design

Solve Ex. 6.1: using SI units

\[ \begin{align*}
V_s &= 27.6 \, \text{MPa} \\
\lambda &= 1.0 \text{ for normal-weight concrete} \\
f_c &= 444 \, \text{MPa} \\
\ell &= 7.82 \, \text{m} \\
b_s &= 356 \, \text{mm} \\
d_s &= 710 \, \text{mm} \\
l &= 765 \, \text{mm} \\
A_s &= 6 \text{ No. 9} \text{ bars (diameter 28.6 mm)} = 3860 \, \text{mm}^2.
\end{align*} \]

Closest area using metric bars from Fig. B.3b in the appendix:

\[ 2 \text{ No. 25 M} + 4 \text{ No. 30 M} = 2 \times 500 + 4 \times 700 = 3800 \, \text{mm}^2. \]

\[ \text{Solution: beam self-weight} = 366 \times 765 (23.8 \times 10^3) \, \text{kN/m} = 6420 \, \text{kN/m} = 6.4 \, \text{kN/m}. \]

Total factored load \( w_s = 1.2 \times 6 + 1.6 \times 104 = 174 \, \text{kN/m}. \)

Factored shear force at face of support, \( V_s = \frac{274 \times 7.82}{2} = 663 \, \text{kN} \)

Half-span, \( V_s \text{ at } \ell \text{ from support} = \frac{3810 - 710}{3810} \times 663 = 539 \, \text{kN} \)

Required, \( V_s = \frac{V_s}{\phi} = 0.75 = 717 \, \text{kN} \)

\[ V_s = \lambda \frac{\sqrt{f_c}}{8} b_s d = 1.0 \frac{\sqrt{276}}{8} 356 \times 710 \, \text{N} = 221 \, \text{kN}. \]

Check for adequacy of section:

\[ V_s = \left( \frac{2}{3} \sqrt{f_c} \right) b_s d + \left( \frac{1}{3} \sqrt{276} \right) 356 \times 710 \times \frac{1}{1000} \]

\[ = 410 \, \text{kN} > 717 \, \text{kN}. \]

Hence, the section is adequate. Since \( V_s > V \), stirrups are needed.

Web Steel Reinforcement:

Try No. 3 stirrup, or, from Fig. B.2b, try No. 10 M metric bar.

\[ A_s = 2 \times 100 = 200 \, \text{mm}^2. \]

\[ s = \frac{A_s f_e d}{V_s \phi - V_s} = \frac{200 \times 410 \times 710 \, \text{N}}{221 \times 119 \, \text{N}} = 149 \, \text{mm}. \]
Plane \( x = s/4 \) maximum spacing:

\[
V_u = V_c = 221 = 496 \text{ kN}
\]

\[
\frac{1}{3} f_y b_d = \frac{\sqrt{f_y}}{\sqrt{3}} \times 356 \times 710 \times \frac{1}{1000} = 445 \text{ kN}
\]

\( V_c < V_u \) = 496 \text{ kN}

Find plane for \( x = s/4 \) at a distance \( x_r \) from midspan:

\[
V_{cr} = V_c + 445 = 221 + 445 = 666 \text{ kN}
\]

\[
x_r \text{ from midspan} = \frac{(3810 - 710) \times 666}{717} = 2879 \text{ mm}
\]

Plane \( x = s/2 \) maximum spacing:

\[
x = \frac{d}{2} = \frac{A_f f_d}{V_{cr} - V_c}
\]

or

\[
V_{cr} = V_c + \frac{A_f f_d}{s/2}
\]

\[
= 221 + \frac{300 \times 414 \times 710}{710/2} \times \frac{1}{1000} = 366 \text{ kN}
\]

\[
x \text{ from midspan} = \frac{(3810 - 710) \times 366}{717} = 1670 \text{ mm}
\]

---

**Figure 6.36** Shear envelope and stirups arrangement (SI units) for Ex. 6.7.
Photo 6.7 Construction Photograph of the Borgata Hotel and Casino Complex, Atlantic City, New Jersey, during the casting of the slabs and columns in 2003; Courtesy the Borgata Management and Carey, Harman and Associates, Design Engineers, Philadelphia, PA. Details are given in Photo 1.9.

Plane $x_1$ at shear force $V$;

$$V = 221 \text{ kN}$$

$$x_{1, \text{ from midspan}} = \frac{3810 - 710251}{712} = 956$$

Discontinue stirrups at plane where \( \frac{V}{V_0} \leq 1 \).

Minimum $A_s$:

$$\frac{b_0}{S_0} = \frac{556 \times 710251}{3 \times 414} = 102 \text{ mm}^2$$

< actual $A_s = 200 \text{ mm}^2$. O.K.

$x_s$ for $\frac{V}{V_0} = 9562 = 478 \text{ mm}$ from midspan.

SELECTED REFERENCES

6.1 ACI Committee 318, Building Code Requirements for Structural Concrete (ACI 318-05) and Commentaries (ACI 318R-05), American Concrete Institute, Farmington Hills, MI, 2005, pp. 445.


6.3 Taylor, H. P. J., The Fundamental Behavior of Reinforced Concrete Beams in Bending and Shear, Special Publication SP-42, Vol. 1, American Concrete Institute, Farmington Hills, MI, 1974, pp. 43-77.


PROBLEMS FOR SOLUTION

6.1. A simply supported beam has a clear span \( l = 22 \) ft \((6.70 \text{ m})\) and is subjected to an external uniform service dead load \( w_d = 90 \) lb per ft \((13.1 \text{kN/m})\) and live load \( w_l = 1200 \) lb per ft \((17.5 \text{kN/m})\). Determine the maximum factored vertical shear \( V_c\) at the critical section. Determine the nominal shear resistance \( V_r\) by both the short method and by the more refined method of taking the contribution of the flexural steel into account. Design the size and spacing of the diagonal tension reinforcement. Given:

\[
\begin{align*}
b_c &= 12 \text{ in. (305 mm)} \\
d &= 17 \text{ in. (432 mm)} \\
h &= 20 \text{ in. (508 mm)} \\
A_s &= 6.0 \text{ in.}^2 \((3700 \text{ mm}^2)\) \\
f' &= 4000 \text{ psi (27.6 MPa), normal-weight concrete} \\
f'_c &= 60,000 \text{ psi (413.7 MPa)}
\end{align*}
\]

Assume that no torsion exists.

6.2. Solve Problem 6.1 assuming that the beam is made of sand-lightweight concrete and that it is subjected to an axial service compressive load of 2500 lb acting at its plastic centroid.

6.3. A cantilever beam is subjected to a concentrated service live load of 25,000 lbs \((111 \text{kN})\) acting at a distance of 3 ft \((1.00 \text{ m})\) from the wall support. Its cross section is 10 in. \times 20\text{ in.} with an effective depth \(d = 17\text{ in. (432 mm)}\). Design the stirrups needed. Given:

\[
\begin{align*}
f' &= 3000 \text{ psi (20.7 MPa), normal-weight concrete} \\
f'_c &= 40,000 \text{ psi (275.9 MPa)}
\end{align*}
\]

6.4. The fixed reaction span of a continuous beam has a clear span of 18 ft \((5.49 \text{ m})\) and is subjected to an intensity of external uniform service live load \( w_l = 1000 \) lb per linear foot \((20 \text{kN/m})\) and a service dead load \( w_d = 2200 \) lb per linear foot \((32 \text{kN/m})\) not including its self-weight. Design the section for flexure and diagonal tension, including the size and spacing of the stirrups, assuming that the beam width \(b = 15\text{ in. (381 mm)}\). Assume that the beam is not subjected to torsion and that all spars are equal. Given:

\[
\begin{align*}
f' &= 3000 \text{ psi (20.7 MPa), normal-weight concrete} \\
f'_c &= 60,000 \text{ psi (413.7 MPa)}
\end{align*}
\]

6.5. A continuous beam has two equal spans \( l = 18 \text{ ft (4.9 m)}\) and is subjected to an external service dead load \( w_d = 350 \) lb per ft \((5.1 \text{kN/m})\) and a service live load \( w_l = 500 \) lb per ft \((13.2 \text{kN/m})\). In addition, an external service concentrated dead load \( P_c = 20,000 \) lb and an external service concentrated live load \( P_l = 28,500 \) lb \((127 \text{kN})\) are applied to one midspan only. Design the diagonal tension reinforcement necessary. Given:

\[
\begin{align*}
f' &= 3000 \text{ psi (20.7 MPa), normal-weight concrete} \\
f'_c &= 60,000 \text{ psi (413.7 MPa)}
\end{align*}
\]

6.6. Design the vertical stirrups for a beam having the shear diagram shown in Figure 6.37 assuming that \( V_c = 2\sqrt{f'd} \). Given:

\[
\begin{align*}
l &= 14 \text{ in. (356 mm)} \\
d &= 20 \text{ in. (508 mm)}
\end{align*}
\]
6.3. Calculate the nominal shear strength \( V_n \) of the plain concrete in the web of the continuous normal weight concrete beam shown in Figure 6.36 using the more refined expression for evaluating the shear. Given:

\[
\begin{align*}
\gamma_c &= 0.025 \\
\gamma_d &= 1.0 \\
\beta &= 0.40 \\
M_e &= \frac{w_{avg} L}{8} = 20,000 \text{ ft-lb} (122.7 \text{ kN}m)
\end{align*}
\]

Figure 6.36
Chapter 6 Shear and Diagonal Tension in Beams

\[ M_c = 55,000 \text{ ft-lb (74.8 kN-m)} \]
\[ f_c = 5000 \text{ psi (34.5 MPa)}, \text{ lightweight concrete} \]
\[ f_p = 60,000 \text{ psi (414 MPa)} \]

Also compute the intensity of factored load \( w \), per foot to which this span is subjected.

6.8. A simply supported beam with a clear span of 10 ft (3.0 m) and an effective center-to-center span of 11 ft 6 in. (3.5 m). The total depth of the beam is \( h = 8.6 \text{ in.} (2.2 \text{ m}) \). It is subjected to a uniform factored load on the top fibers of intensity \( w = 120,000 \text{ lb/ft} (1601.8 \text{ kN/m}) \), including its self-weight. Design the beam for flexure and shear by (a) non-linear approach for flexure, and (b) strut-and-tie approach.

Given: \( f_p = 4200 \text{ psi (29.0 MPa)}, \text{ normal-weight concrete} \]
\[ f_p = 60,000 \text{ psi (414 MPa)} \]

Assume the beam to be loaded only in its plane; and that wind and earthquake are not a consideration.

6.9. Solve Problem 6.8 if the same beam was continuous over three spans and was subjected to the same intensity of load.

6.10. Design a bracket to support a concentrated factored load \( V_c = 125,000 \text{ lb (556.0 kN)} \) acting at a lever arm \( u = 4 \text{ in.} (101.6 \text{ mm}) \) from the column face; horizontal factored force \( N_c = 40,000 \text{ lb (176 kN)} \). Given:

\[ b = 28 \text{ in. (711 mm)}, \text{ normal-weight concrete} \]
\[ f_c = 5000 \text{ psi (34.5 MPa)}, \text{ normal-weight concrete} \]
\[ f_p = 60,000 \text{ psi (414 MPa)} \]

Column size = \( 12 \times 26 \text{ in.} (305 \times 711 \text{ mm}) \). Corbel width = 26 in.

Used both the shear-friction approach and the strut-and-tie method in your solution.

Assume that the bracket was cast after the supporting columns were cast and that the column surface at the bracket location was not roughened before casting the bracket. Detail the reinforcing arrangements for the bracket.

6.11. Solve Problem 6.10 if the structural system was made from monolithic, lightweight concrete in which the corbel or bracket is cast simultaneously with the supporting columns. Use both the strut- and-tie method and the shear friction approach in your solution.
7.1 INTRODUCTION

Torsion occurs in monolithic concrete construction primarily where the load acts at a distance from the longitudinal axis of the structural member. An end beam in a floor panel, a spandrel beam receiving load from one side, a canopy or a bus-stand roof projecting from a monolithic beam on columns, peripheral beams surrounding a floor opening, or a helical staircase are all examples of structural elements subjected to twisting moments. These moments occasionally cause excessive shearing stresses. As a result, severe cracking can develop well beyond the allowable serviceability limits unless special torsional reinforcement is provided. Photos 7.2 and 7.3 illustrate the extent of cracking at failure of a beam in torsion. They show the out-of-plane plane of twist caused by the imposed torsional moment. In actual spandrel beams of a structural system, the extent of damage due to torsion is usually not as severe, as seen in Photos 7.4 and 7.5. This is due to the redistribution of stresses in the structure. However, loss of integrity due to torsional distress should always be avoided by proper design of the necessary torsional reinforcement.

Photo 7.1. Newark International Airport, terminal, New Jersey. (Courtesy of Port of New York- New Jersey Authority.)
Chapter 7 Torsion

Photo 7.2 Reinforced plaster beam at failure in pure torsion. (Rutgers tests: Law, Nassy, et al.)

(a)

(b)

Photo 7.3 Plain mortar beam in pure torsion: (a) top view; (b) bottom view. (Rutgers tests: Law, Nassy, et al.)

Photo 7.4 Reinforced concrete beams in torsion testing setup. (Courtesy of Thomas T. C. Hsu.)
An introduction to the subject of torsional stress distribution has to start with the basic elastic behavior of simple sections, such as circular or rectangular sections. Most concrete beams subjected to twist are components of rectangles. They are usually flanged sections such as T beams and L beams. Although circular sections are rarely a consideration in normal concrete construction, a brief discussion of torsion in circular sections serves as a good introduction to the torsional behavior of other types of sections.

Shear stress is equal to shear strain times the shear modulus at the elastic level in circular sections. As is the case of flexure, the strain is proportional to its distance from the neutral axis (i.e., the center of the circular section) and is maximum at the extreme fibers. If \( r \) is the radius of the element, \( J = \pi r^4/2 \), its polar moment of inertia, \( \nu \), the elastic shearing stress due to an elastic twisting moment, \( T \), has the following value:

\[
\nu = \frac{T r}{J}
\]

When deformation takes place in the circular shaft, the axis of the circular cylinder is assumed to remain straight. All radii in a cross-section and remain straight (i.e., without warping) and rotate through the same angle about the axis. As the circular element starts to behave plastically, the stress in the plastic outer ring becomes constant while the stress in the inner core remains elastic, as shown in Figure 7.1. As the whole cross-section
becomes plastic, \( h = 0 \) and the shear stress where \( \tau_p \) is the nonlinear shear stress due to an ultimate twisting moment \( T_u \), where the subscript \( f \) denotes failure.

\[
\tau_p = \frac{3 T_u}{4 J}
\]

In rectangular sections, the torsional problem is considerably more complicated. The originally plane cross sections undergo warping due to the applied torsional moment. This moment produces axial as well as circumferential shear stresses with zero values at the corners of the section and the centroid of the rectangle and maximum values on the periphery at the middle of the sides, as seen in Figure 7.2. The maximum torsional shearing stress would occur at midpoints \( A \) and \( B \) of the larger dimension of the cross-section. These complications plus the fact that reinforced concrete sections are neither homogeneous nor isotropic make it difficult to develop exact mathematical formulations based on physical models such as Eqs. (a) and (b) for circular sections.

For over sixty years, the torsional analysis of concrete members has been based on either (1) the classical theory of elasticity developed through mathematical formulations coupled with membrane analogy verifications (St. Venant's) or (2) the theory of plasticity represented by the sand-heaps analogy (Nadai's). Both theories were applied essentially to the state of pure torsion. But experiments revealed that the elastic theory is not entirely satisfactory for the accurate prediction of the state of stress in concrete in pure torsion. The behavior of concrete was found to be better represented by the plastic approach. Consequently, almost all developments in torsion as applied to concrete and reinforced concrete have been in the latter direction.

### 7.2 PURE TORSION IN PLAIN CONCRETE ELEMENTS

#### 7.2.1 Torsion in Elastic Materials

St. Venant presented in 1853 his solution to the elastic torsional problem with warping due to pure torsion that develops in noncircular sections. Prandtl in 1903 demonstrated the physical significance of the mathematical formulations by his membrane analogy model. The model establishes particular relationships between the deflected surface of the loaded membrane and the distribution of torsional stresses in a bar subjected to twisting moments. Figure 7.3 shows the membrane analogy behavior for rectangular as well as L-shaped sections.
For small deformations, it can be proved that the differential equation of the deformed membrane surface has the same form as the equation that determines the stress distribution over the cross-section of the bar subjected to twisting moments. Similarly, it can be demonstrated that (1) the tangents to a contour line at any point of a deflected membrane give the direction of the shearing stress at the corresponding cross-section of the actual member subjected to twists; (2) the minimum slope of the membrane at any point is proportional to the magnitude of shear stress at the corresponding point in the actual member; (3) the twisting moment to which the actual member is subjected is proportional to twice the volume under the deflected membrane.

It can be seen from Figures 7.2 and 7.3 that the torsional shearing stress is inversely proportional to the distance between the contour lines. The closer the lines are, the higher the stresses, leading to the previously stated conclusion that the maximum torsional shearing stress occurs at the middle of the longer side of the rectangle. From the membrane analogy, the maximum stress has to be proportional to the steepest slope of the tangents at points A and B.

If $\Delta$ is the maximum displacement of the membrane from the tangent at point A, then from basic principles of mechanics and St. Venant's theory,

$$\Delta = \frac{h^2G\theta}{(1 - \nu^2)EI}$$

(7.16)
where \( G \) is the shear modulus and \( \theta \) is the angle of twist. But \( \nu_{\text{max}} \) is proportional to the slope of tangent; hence
\[
\nu_{\text{max}} = k_i b G \theta
\]
(7.1b)
where the \( k_i \)'s are constants. The corresponding torsional moment \( T_i \) is proportional to twice the volume under the membrane, or
\[
T_i = \frac{2}{3} b bh \left( \frac{2}{3} b bh \right) = k_i b bh
\]
or
\[
T_i = k_i b bh \theta G b
\]
(7.1c)
From Eqs. 7.1b and 7.1c,
\[
\nu_{\text{max}} = \frac{T_i}{kbh} = \frac{T_i}{J_i}
\]
(7.1d)
The denominator \( kbh \theta \) in Eq. 7.1d represents the polar moment of inertia \( J_i \) of the section. Comparing Eq. 7.1d to Eq. (a) for the circular section shows the similarity of the two expressions except that the factor \( k_i \) in the equation for the rectangular section takes into account the shear strains due to warping. Equation 7.1d can be further simplified to give
\[
\nu_{\text{max}} = \frac{T_i}{kbh}
\]
(7.2)
It can also be written to give the stress at planes inside the section, such as an inner concentric rectangle of dimensions \( x \) and \( y \), where \( x \) is the shorter side, so that
\[
\nu_{\text{max}} = \frac{T_i}{kxy}
\]
(7.3)
It is important to note in using the membrane analogy approach that the torsional shear stress changes from one point to another along the same axis as \( AB \) in Fig. 7.3, because of the changing slope of the analogous membrane, rendering the torsional shear stress calculations lengthy.

7.2.2 Torsion in Plastic Materials
As indicated earlier, the plastic sand-heep analogy provides a better representation of the behavior of brittle elements such as concrete beams subjected to pure torsion. The torsional moment is also proportional to twice the volume under the heap, and the maximum torsional shearing stress is proportional to the slope of the sand heap. Figure 7.4 is a two- and three-dimensional illustration of the sand heap. The torsional moment \( T_i \) in Fig. 7.4 is proportional to twice the volume of the rectangular heap shown in parts (b) and (c). It can also be recognized that the slope of the sand heap sides as a measure of the torsional shearing stress is constant in the sand-heep analogy approach, whereas it is continuously variable in the membrane analogy approach. This characteristic of the sand heap considerably simplifies the solutions.

7.2.3 Sand-heep Analogy Applied to L Beams
Most concrete elements subjected to torsion are flanged sections, most commonly L beams comprising the external wall beams of a structural floor. The L beam in Figure 7.5 is chosen in applying the plastic sand-heep approach to evaluate its torsional moment capacity and shear stresses to:...
7.2 Pure Torsion in Plain Concrete Elements

Figure 7.4 Sand-heap analogy in plastic pure torsion: (a) sand-heap L-section; (b) sand-heap rectangular section; (c) plan of rectangular section; (d) torsional shear stress.

The sand heap is broken into three volumes:

\[ V_1 = \text{pyramid representing a square cross-sectional shape} = \frac{y_1 b_2^2}{6} \]

\[ V_2 = \text{tangential portion of the web representing a rectangular cross-sectional shape} = y_2 b_2 (b - b_1) \]

\[ V_3 = \text{top representing the flange of the beam, transferring part PDM to NQM} = y_3 b_1 (b - b_2)^2 \]

Torsional moment is proportional to twice the volume of the sand heaps; hence

\[ T_\tau = 2 \left[ \frac{y_1 b_2^2}{3} + \frac{y_2 b_2 (b - b_1)}{2} + \frac{y_3 b_1 (b - b_2)^2}{2} \right] \quad (7.4) \]

Also, torsional shear stress is proportional to the slope of the sand heaps; hence

\[ y_1 = \frac{y_2 b_2}{2} \quad (7.5a) \]

\[ y_2 = \frac{y_3 b_1}{2} \quad (7.5b) \]

Substituting \( y_1 \) and \( y \) from Eqs. 7.5a and 7.5b into Eq. 7.4 gives us
$$v_{\text{final}} = \frac{T_c}{(b_2/h)h - b_2} + \frac{(b_2/h)h - b_2}{2(b_1/h)h - b_2}$$  \hspace{1cm} (7.6)$$

If both the numerator and denominator of Eq. 7.6 are divided by \((b_1,h)^2\) and the terms rearranged, we have

$$v_{\text{final}} = \frac{T_c h/(b_2,h)^2}{\left[\frac{1}{6}(3 - b_2/h)^2 + \frac{1}{2}(b_2/h)(2b_2/h - b_2/h)\right]}$$  \hspace{1cm} (7.7a)$$

If one assumes that \(C_1\) is the denominator in Eq. 7.7a and \(J_e = C_1(b_1,h)^2\), Eq. 7.7a becomes

$$v_{\text{final}} = \frac{T_c h}{J_e}$$  \hspace{1cm} (7.7b)$$

where \(J_e\) is the equivalent polar moment of inertia and a function of the shape of the beam cross section. Note that Eq. 7.7b is similar in format to Eq. 7.1d from the membrane analogy except for the different values of the denominators \(J\) and \(J_e\). Equation 7.7a can readily be applied to rectangular webs with \(b_2 = 0\).
It must also be recognized that concrete is not a perfectly plastic material; hence the actual residual strength of the plain concrete section has a value lying between the membrane analogy and the stress-bow analogy values.

Equation 7.7b can be rewritten designating \( T_e = T \), as the nominal torsional resistance of the plain concrete and \( T_{min} = T_e \) using ACI terminology, so that

\[
T_e = k_{1} \sqrt{T_e v_c} \quad (7.8a)
\]

\[
T_e = k_{2} \sqrt{T_e v_c} \quad (7.8b)
\]

where \( T_e \) is the smaller torsion of the rectangular section.

Extensive work by Hau, confirmed by others, has established that \( k_2 \) can be taken as 1. This value originated from research in the skew-bending theory of plain concrete. It was also established that \( 0.8 \sqrt{T_e} \) can be considered as a limiting value of the pure torsional strength of a member without torsional reinforcement. Using a reduction factor of 2.5 for the first cracking torsional load \( T_{cr} = 2.4 \sqrt{T_e} \) and using \( k_2 = 4 \) in Eq. 7.8 results in

\[
T_e = 0.8 \sqrt{T_e} 2.5 \quad (7.9a)
\]

where \( T_e \) is the shorter side of the rectangular section. The high reduction factor of 2.5 is used to effect any effect of shear and bending moments that might be present.

If the cross-section is a T or \( L \) section, the area can be broken into component rectangles as in Figure 7.6, such that

\[
T_e = 0.8 \sqrt{T_e} 2.5 \quad (7.9b)
\]

7.2.4 Skew-Bending Theory

This theory considers in detail the internal deformational behavior of the series of transverse warped surfaces along the beam. Initially proposed by Lessig, it had subsequent contributions from Collins, Hsu, Zia, Gerov, and Elgot among the several researchers in this field. T. C. Hau made a major contribution experimentally to the development of the skew-bending theory as it presently stands. In his book (Ref. 7.13), Hau details the development of the theory of torsion as applied to concrete structures and how the skew-bending theory formed the basis of the 1989 ACI Code provisions on torsion. The complexity of the torsional problem can only permit in this textbook only the brief discussion that follows.

![Figure 7.6 Component rectangles for \( T_e \) calculation.](image-url)
The failure surface of the normal beam cross section subjected to bending moment \( M \), remains plane after bending, as in Fig. 7.7a. If a twisting moment \( T \) is also applied exceeding the capacity of the section, cracks develop on three sides of the beam cross-section and compressive stresses on portions of the fourth side along the beam. As torsional loading proceeds to the limit state at failure, a skewed failure surface results due to the combined torsional moment \( T \), and bending moment \( M \). The neutral axis of the skewed surface and the shaded area in Figure 7.7b denote the compression zone would no longer be straight but subtend a varying angle \( \theta \) with the original plane cross-sections.

Prior to cracking, neither the longitudinal bars nor the closed stirrups make any appreciable contribution to the torsional stiffness of the section. At the post-cracking stage of loading, the stiffness of the section is reduced but its torsional resistance is considerably increased, depending on the amount and distribution of both the longitudinal bars and the transverse closed ties. It has to be emphasized that little additional torsional strength can be achieved beyond the capacity of the plain concrete in the beam unless both longitudinal tension bars and transverse ties are used.

The skew-bending theory idealizes the compression zone by considering it to be of uniform depth. It assumes the cracks on the remaining three faces of the cross section to be uncracked and with the steel ties (stirrups) at those faces carrying the tensile forces at the cracks and the longitudinal bars resisting shear through dowel action with the concrete. Figure 7.8a shows the forces acting on the skewed bent plane. The polygon in Figure 7.8b gives the shear resistance \( F_x \) of the concrete, the force \( T \), of the active longitudinal steel bars in the compression zone, and the normal compressive block force \( C \).

The torsional moment \( T \) of the resisting shearing force \( F_x \), generated by the shaded compressive block area in Figure 7.8a is thus

\[
T_x = \frac{F_x}{\cos 45°} \times \text{its arm about forces } F_x \text{ in Fig. 7.8a}
\]

Figure 7.7 Skew bending due to tension: (a) bending before twist; (b) bending and twist.
Figure 7.8 Forces on the skewly bent planes: (a) all forces acting on skew plane at failure; (b) vector forces on compression zone.

\[
T_x = \sqrt{2} F_y (0.8x) \quad (7.10a)
\]

where \( x \) is the shorter side of the beam. Extensive tests (Re's 7.9 and 7.13) to evaluate \( F_y \) in terms of internal stress in concrete, \( k_y \sqrt{f_c} \), and the geometrical torsional constants of the section, \( k_{xy} \), led to the expression

\[
T_x = \frac{2.4}{\sqrt{x}} x^2 y \sqrt{f_c} \quad (7.10b)
\]

### 7.3 TORSION IN REINFORCED CONCRETE ELEMENTS

Torsion rarely occurs in concrete structures without being accompanied by bending and shear. The foregoing should give a sufficient background on the contribution of the plain concrete in the section toward resisting part of the combined stresses resulting from torsional, axial, shear, or flexural forces. The capacity of the plain concrete to resist torsion when in combination with other loads could, in many cases, be lower than when it resists the same factored external twisting moments alone. Consequently, torsional reinforcement has to be provided to resist the excess torque.
Inclusion of longitudinal and transverse reinforcement to resist part of the torsional moments introduces a new element in the set of forces and moments in the section. If

\[ T_r = \text{required total nominal torsional resistance of the section including the reinforcement} \]
\[ T_p = \text{nominal torsional resistance of the plain concrete} \]
\[ T_r = \text{torsional resistance of the reinforcement} \]

then

\[ T_r = T_p + T_r \]  \hspace{1cm} (7.11)

\( T_r \) is assumed equal to zero for design simplification, and all the torsion is assumed to be borne by the longitudinal steel bars and the closed transverse stirrups. To study the contribution of the longitudinal steel bars and the closed stirrups, one has to analyze the system of forces acting on the warped cross-sections of the structural element at the limit state of failure.

A modified space truss analogy is presented comparable to the plane truss analogy used for the design of shear stirrups. In this theory, both the longitudinal reinforcement and the transverse stirrups (ties) are utilized as components of the space truss (see Sec. 7.3.2).

### 7.3.1 Space Truss Analogy Theory

This theory was originally developed by Rausch and extended later by Lamper and Collins, with additional work by Hsu, Thurlimann, Eilgren, and others. Further refinement was introduced by Collins and Mitchell (Ref. 7.12) as a compression field theory. Hsu (Refs. 7.14, 7.15) proposed combining the equilibrium, compatibility, and the softened constitutive laws of concrete in a unified theory that can predict with reasonable accuracy the shear and torsional behavior of beams (the softened truss model). The shear flow concept is utilized in deriving the relevant expressions for shear equilibrium.

The space truss analogy is an extension of the model used in the design of the shear-resisting stirrups, in which the diagonal tension cracks, once they start to develop, are resisted by the stirrups. Because of the nonplanar shape of the cross-sections due to the twisting moment, a space truss composed of the stirrups is used as the diagonal tension members, and the idealized concrete strips at a variable angle between the cracks are used as the compression members, as shown in Figure 7.9.

It is assumed in this theory that the concrete beam behaves in torsion similar to a thin-walled box with a constant shear flow in the wall cross-section, producing a constant torsional moment. The use of hollow-walled sections rather than solid sections proved to give essentially the same ultimate torsional moment, provided that the walls are not too thin. Such a conclusion is borne out by tests, which have shown that the torsional strength of the solid sections is composed of the resistance of the closed stirrup cage, consisting of the longitudinal bars and transverse stirrups, and the idealized concrete inclined compression struts in the plane of the cage wall. The compression struts are the inclined concrete strips between the cracks in Figure 7.9.

The CBT-FIP code is based on the space truss model. In this code, the effective wall thickness of the hollow beam is taken as \( D_w \), where \( D_w \) is the diameter of the circle inscribed in the rectangle connecting the corners of the longitudinal bars, that is, \( D_w = \sqrt{2} \), in Figure 7.9. A rational method to derive the effective wall thickness was given by Hsu (Ref. 7.15). This nonlinear analysis takes into account the warping compatibility condition of the wall. In summary, the absence of the core does not affect the strength of such members in torsion; hence the acceptability of the space truss analogy approach based on hollow sections.
7.3 Torsion in Reinforced Concrete Elements

7.3.2 Equilibrium in Element Shear

A unit square membrane element of thickness $h$ is subjected to shear flow $q$ due to pure shear as in Figure 7.10 (Ref. 7.15). Reinforcement in both the longitudinal (E–W) direction $i$ and transverse (N–S) direction $j$ is subjected to a unit stress $f_{ij}$ and $f_{ji}$, respectively, such that the shear flow $q$ can be defined by the equilibrium equations

$$ q = (F_i) \tan \theta $$  \hspace{1cm} (7.12a)

where unit $F_i = A_i f_{ij}$, and

$$ q = (F_j) \cot \theta $$  \hspace{1cm} (7.12b)

where unit $F_j = A_j f_{ji}$. $A_i$ is the cross-sectional area of the reinforcement, and $\theta_i$ and $\theta_j$ are the spacings in the $i$ and $j$ directions, respectively.

From the geometry of the triangles in Figure 7.10, the shear flow can also be defined as

$$ q = (f_{ij} \sin \theta \cos \theta) $$  \hspace{1cm} (7.12c)

If the reinforcement in both directions is assumed to have yielded, Eqs. 7.12a and b give

$$ \tan \theta = \frac{F_j}{F_i} $$  \hspace{1cm} (7.13a)

and

$$ q_r = \sqrt{F_i F_j} $$  \hspace{1cm} (7.13b)

where the subscript $r$ denotes the yielding of reinforcement.
7.3.3 Equilibrium in Element Torsion

The case of a hollow tube of any shape and variable thickness is considered (Figure 7.11). It is subjected to pure torsion. St. Venant’s theory stipulates that the cross-sectional shape remains unchanged in elastic small deformations, and the warping deformation perpendicular to the cross-section would be the same along the member’s axis. Hence it can be assumed that only shear stresses develop in the tube wall in the form of shear flow \( q \) in Figure 7.11a and that the in-plane normal stresses in the wall vanish. If an infinitesimal wall element \( ABCD \) is isolated as in Fig. 7.11b, the shear flow in the \( c \) direction has to be equal to the shear flow in the \( r \) direction or

\[
\tau_{r1} = \tau_{c1} \quad (7.14)
\]

On this basis, the shear flow \( q \) is considered constant throughout the cross-section (Ref. 7.15). The torsional force over an infinitesimal distance \( dl \) along the shear flow path is \( qdl \).

Figure 7.11 Hollow tube equilibrium torsion forces: (a) section of tube subjected to torsion \( T \); (b) unit shear element from tube wall of varying thickness \( h \). Note: \( l \) and \( l' \) denote the longitudinal and transverse \( x \)-directions, respectively.
so that the torsional resistance to the external torsional moment $T$ in Figure 7.11a becomes

$$T = q \int r \, dt$$

(7.15)

It can be seen from Figure 7.11a that $r \, dt$ in the integral is equal to twice the area of the shaded triangle formed by $r$ and $dt$. A summation of the total area around the cross-section gives

$$\int r \, dt = 2A_0$$

(7.16)

where $A_0$ is cross-sectional area bounded by the shear flow center line. Substituting $2A_0$ into Eq. 7.15 gives

$$q = \frac{T}{2A_0}$$

(7.17)

By neglecting warping, the shear element subjected to pure torsion in the tube wall of Figure 7.11a becomes identical to the membrane shear element in Figure 7.16a. Hence, substituting for the shear flow $q$ from Eq. 7.17 into Eqs. 7.12a, b, and c, three equations of equilibrium for torsional result,

$$T = \frac{F_t}{p_0} (2A_0) \tan \theta$$

(7.18a)

where $F_t = F_{t0} + p_0 \pi$ is the perimeter of the shear flow path. $F_t$ is the total longitudinal force due to torsion.

$$T = F_{t0}(2A_0) \cot \theta$$

(7.18b)

$$T = (p_0 \pi)(2A_0) \sin \theta \cot \theta$$

(7.18c)

Equation 7.18b can be written at yield as

$$T = \frac{2A_0 f_{f_0}}{x} \cot \theta$$

(7.19)

where $T_y$ is the maximum torsional moment strength.

The required torsional reinforcements in the transverse and longitudinal directions become

$$A_{t} = \frac{T_y}{2A_0 f_{f_0} \cot \theta}$$

(7.20)

$$A_{l} = \frac{f_{f_0}}{f_{r_0}} (A_{t} \cot \theta)$$

(7.21a)

where $A_{l}$ is the area of one longitudinal bar.

If $x$, the longitudinal reinforcement spacing represents the perimeter $p_0$ of the center line of the outermost closed transverse torsional reinforcement, then

$$A_{l} = \frac{A_{t}}{x} \frac{f_{f_0}}{f_{r_0}} p_0 \cot \theta$$

(7.21b)

where $A_{t}$ is total area of all longitudinal torsional steel in the section.

The factored torsional moment strength, $T_{f_{t}}$, must equal or exceed the external torsion, $T$, due to the factored loads. In the calculation of $T_{f_{t}}$ (ACI 318-02, Ref. 7.18), all the torque is assumed to be resisted by the closed stirrups and longitudinal steel, with the torsional moment $T$, resisted by the concrete compression struts assumed as zero. At the
7.4 SHEAR–TORSION–BENDING INTERACTION

Consider the rectangular boxes in Figures 7.9 and 7.12. The shear flow \( q \) will not be the same on the four walls of the box when subjected to combined shear and torsion, as shown in Figure 7.12c. Failure can precipitate in two distinct modes:

(a) \textit{Bottom tension steel yielding.} If the failure mode is caused by yielding of the longitudinal bottom stringer (tensile steel) and the transverse stirrups due to combined shear and torsion, the following expression can be derived from equilibrium (Ref. 7.15):

\[
\frac{M}{M_{0}} + \left(\frac{V}{2.0F_{b}}\right)\frac{T_{b}}{2A_{b}F_{b}} + \left(\frac{T}{2.5A_{b}F_{b}}\right)\frac{T_{b}}{2A_{b}F_{b}} = 1
\]  

(7.22)

If \( M_{0}, V_{0}, \) and \( T_{0} \) are the moments and forces acting \textit{alone}, they can be defined as follows:

\[
M_{0} = F_{0}\alpha_{0}
\]

(7.23a)

\[
V_{0} = 2.5\alpha_{0} \sqrt{\left(\frac{F_{0}}{2F_{b}}\right)\frac{A_{f}}{s}} \quad \text{for a two-web box}
\]

(7.23b)

\[
T_{0} = 2.5A_{b} \sqrt{\left(\frac{2F_{b}}{2F_{b}}\right)\frac{A_{f}}{s}}
\]

(7.23c)

where \( \alpha_{0} = 2(t_{b} + x_{t}) \).

\[
R = \frac{F_{b}}{F_{0}}
\]

(7.23d)

Figure 7.11: The section shear flow \( q, \) \( q_{d}, \) and \( q_{t} \) combined shear and torsion.
A nondimensional interaction surface relationship can be obtained by introducing Eq. 7.23 into Eq. 7.22 such that:

\[
\frac{M}{M_c} = \left( \frac{V}{V_{c}} \right)^2 + \left( \frac{T}{T_{c}} \right)^2 R = 1 \tag{7.24a}
\]

(b) Top compression yielding. If the failure mode is caused by yielding of the longitudinal top chord (compression steel) and the transverse stirrups, Eq. 7.24a becomes

\[
\left( \frac{M}{M_c} \right)^2 + \left( \frac{V}{V_{c}} \right)^2 + \left( \frac{T}{T_{c}} \right)^2 = 1 \tag{7.24b}
\]

From both Eqs. 7.24 a and b the interaction of \( V \) and \( T \) is circular for a constant bending moment \( M \) for both failure surfaces. The intersection of the two failure surfaces for these two failure modes forms a peak interaction curve between \( V \) and \( T \) such that Eqs. 7.24a and b give

\[
\left( \frac{V}{V_{c}} \right)^2 + \left( \frac{T}{T_{c}} \right)^2 = 1 + \frac{R}{2R} \tag{7.25a}
\]

Equation 7.25a for \( R = 0.25, 0.5, \) and 1.0 on the peak planes gives the circular plots shown in Figure 7.13.

A third mode of failure is caused by yielding to the top bar, in the bottom bar, and in the transverse reinforcement, all on the side where shear flows due to shear and torsion are additive, that is, the left wall (Ref. 7.15). A modified form of Eq. 7.25a results as follows:

\[
\left( \frac{V}{V_{c}} \right)^2 + \left( \frac{T}{T_{c}} \right)^2 + \frac{1}{2} \left( \frac{VT}{VT_{c}} \right)^2 = 1 + \frac{R}{2R} \tag{7.25b}
\]

7.5 ACI DESIGN OF REINFORCED CONCRETE BEAMS SUBJECTED TO COMBINED TORSION, BENDING, AND SHEAR

7.5.1 Torsional Behavior of Structures

The torsional moment acting on a particular structural component such as a spanned beam can be calculated using normal structural analysis procedures. Design of the particular component needs to be based on the limit state at failure. Therefore, the nonlinear
behavior of a structural system after torsional cracking must be identified in one of the
following two conditions: (1) no redistribution of torsional stresses to other members
after cracking and (2) redistribution of torsional stresses and moments after cracking to
effect deformation compatibility between intersecting members.

Stress resultants due to torsion in statically determinate beams can be evaluated
from equilibrium conditions alone. Such conditions require a design for the full-factored
external torsional moment, because no redistribution of torsional stresses is possible.
This state is often termed equilibrium torsion. An edge beam supporting a cantilever
canopy as in Figure 7.14 is such an example.

The edge beam has to be designed to resist the real external factored twisting mo-
ment due to the cantilever slab; otherwise, the structure will collapse. Failure would be
cased by the beam not satisfying conditions of equilibrium of forces and moments re-
sulting from the large external torque.

In statically indeterminate systems, stiffness assumptions, compatibility of strains at
the joints, and redistribution of stresses may affect the stress resultants, leading to a re-
duction in the resulting torsional shearing stresses. A reduction in permitted in the value
of the factored moment used in the design of the member if part of this moment can be
redistributed to the intersecting members. The ACI Code permits a maximum factored
torsional moment at the critical section of from the face of the supports for reinforced
concrete members as follows:

\[ T_e = \phi_4 \sqrt{f'_c} \frac{A_s}{P_o} \]  

(7.26)

where

- \( A_s \) = area enclosed by outside perimeter of concrete cross section
- \( P_o \) = outside perimeter of concrete cross section \( A_s \) in.
- \( 2(\tan + y) \)

For pre-stressed concrete members at \( \frac{h}{b} \) from the face of the support

\[ T_e = \phi_4 \sqrt{f'_c} A_s \left( 1 + \frac{2}{4 \sqrt{f'_c}} \right) \]  

(7.27)

where \( \bar{f}_c \) = average compressive stress in the concrete at the centroidal axis due to effec-
tive prestress only after allowing for all losses (\( \bar{f}_c \) in the ACI Code is denoted as \( f_{pc} \)).

Figure 7.14 No redistribution torsion (equilibrium torsion).
Neglect of the full effect of the total external torsional moment in this case does not, in effect, lead to failure of the structure but may result in excessive cracking if \( \sqrt{f} [A_{e}^{*} / A_{t}] \) is considerably smaller in value than the actual factored torque. An example of compatibility torsion can be seen in Figure 7.15.

Beams \( B \), apply twisting moments \( T_{A} \) at sections 1 and 2 of spandrel beam \( A'B \) in Figure 7.15(b). The magnitudes of relative stiffnesses of beam \( A'B \) and transverse beams \( B \) determine the magnitudes of rotation at intersecting joints 1 and 2. Because of the development of torsional plastic hinges near joints \( A \) and \( B \), the end moments of beams \( B \) at their intersections with spandrel beam \( A'B \) will not be fully transferred as twisting moments to the column supports at \( A \) and \( B \). They would be greatly reduced, because moment redistribution results in transfer for most of the end bending moments from ends 1 and 2 to ends 3 and 4, as well as the midspans of beams \( B_{1}, B_{2}, \) at each spandrel beam supports \( A \) and \( B \) and at the critical section at distance \( d \) from these supports is determined from Eq. 7.26 for reinforced concrete and Eq. 7.27 for prestressed concrete.

If the actual factored torque due to beams \( B \) is less than that given by Eqs. 7.26 or 7.27, the beam could be designed for the lesser torsional value. Torsional moments are neglected, however, if for reinforced concrete.

Figure 7.15  Torsion redistribution (compatibility): (a) isometric view of one panel and (b) plan of a typical one-way floor system.
and for prestressed concrete

$$T_n < \phi \sqrt{f_c} \frac{A_s^2}{p_s} \left(1 + \frac{f_y}{4\sqrt{f_c}} \right)$$  \hspace{1cm} (7.29)

7.5.2 Torsional Moment Strength

The size of a cross-section is chosen on the basis of reducing uncracking and preventing the creation of the surface concrete caused by the inclined compressive stresses due to shear and torsion defined by the left-hand side of the expressions in Eqs. 7.30a and b. The geometrical dimensions for torsional moment strength in both reinforced and prestressed members are limited by the following expressions

(a) Solid sections

$$\sqrt{\frac{V_n}{(b,d)}} \left(1 + \frac{T_n b_s}{(1.7A_s) f_y} \right) \leq \phi \left(\frac{V_n}{(b,d)} + 8 \sqrt{f_c} \right)$$  \hspace{1cm} (7.30a)

(b) Hollow sections

$$\sqrt{\frac{V_n}{(b,d)}} \left(1 + \frac{T_n b_s}{(1.7A_s) f_y} \right) \leq \phi \left(\frac{V_n}{(b,d)} + 8 \sqrt{f_c} \right)$$  \hspace{1cm} (7.30b)
7.5 ACI Design of Reinforced Concrete Beams Subjected to Combined Torsion, Bending, and Shear

Reinforced concrete:

\[ V_r = 2k \sqrt{f_t} b_d \]  \hspace{1cm} (7.30c)

Prestressed concrete for \( f_{pc} > 0.4f_{cc} \):

\[ V_r = \left( \frac{0.6k}{\sqrt{f_t}} + \frac{700}{M_s} \right) b_d \]  \hspace{1cm} (7.30d)

\[ \geq \frac{V_d}{M_s} \approx 1.0 \]

\[ \approx 1.7b_d \leq 5.0k \sqrt{f_t} b_d \]

where \( A_{sk} \) = area enclosed by the center line of the outermost closed transverse torsional reinforcement, in.²

\( p_c \) = perimeter of center line of outermost closed transverse torsional reinforcement, in.

The areas \( A_{sk} \) for different sections are given in Figure 7.16.

The sum of the stresses at the left-hand side of Eqs. 7.30a and b should not exceed the stresses causing shear cracking plus 8 \( \sqrt{f_t} \). This is similar to the limiting strength \( V_r \leq 8 \sqrt{f_t} \) for shear without torsion. The upper limit of stress in terms of the nominal shear strength \( V_c \) of the plain concrete in the web permits applying the two expressions in both reinforced and prestressed concrete elements.

7.5.2.1 Hollow Sections Wall Thickness. The shear stresses due to shear and to torsion both develop in the walls of the hollow section, as shown in Figure 7.17a. Note that in a solid section the shear stresses due to torsion still concentrate in the outer zones of the section as in Figure 7.17b and as discussed in Section 7.3.3.

If the wall thickness in the hollow section varies around its perimeter, the section geometry has to be evaluated at such a location where the left-hand side of Eq. 7.30c has a maximum value. Also, if the wall thickness \( t < A_{sk}/p_c \), the left-hand side of Eq. 7.30b should be taken as

\[ \frac{V_r}{b_d} = \frac{T_r}{1.7A_{sk}} \]

The wall thickness \( t \) is the thickness where stresses are being checked.

Figure 7.16 Torsional geometric parameters.
7.5.3 Torsional Web Reinforcement

As indicated in Section 7.3.1, meaningful additional torsional strength due to the addition of torsional reinforcement can be achieved only by using both stirrups and longitudinal bars. Ideally, equal volumes of steel in both the closed stirrups and the longitudinal bars should be used so that both participate equally in resisting the twisting moments. This principle is the basis of the ACI expressions for proportioning the torsional web steel. If \( s \) is the spacing of the stirrups, \( A_t \) is the total cross-sectional area of the longitudinal bars, and \( A_s \) is the cross-section of one stirrup leg, the transverse reinforcement for torsion has to be based on the full external torsional moment strength value \( T_e \), namely, \( T_e/\phi \), where

\[
T_e = \frac{2A_t A_s f_y}{s} \cot \theta \tag{7.31a}
\]

(see the derivation of Eq. 7.19).

where

- \( A_t \) = gross area enclosed by the shear flow path, in.\(^2\)
- \( A_s \) = cross-sectional area of one leg of the transverse closed stirrups, in.\(^2\)
- \( f_y \) = yield strength of closed transverse torsional reinforcement not to exceed 60,000 psi
- \( \theta \) = angle of the compression diagonals (struts) in the space truss analogy for torsion (see Figure 7.9)

Transposing terms in Eq. 7.31a, the transverse reinforcement area becomes

\[
A_s = \frac{T_e}{2A_t f_y} (\cot \theta)^{-1} \tag{7.31b}
\]

The area \( A_t \) has to be determined by analysis (Refs. 7.14 and 7.15), except that the ACI 318 Code permits taking \( A_t = 0.85 A_s \) in lieu of the analysis.

As discussed in Sec. 7.3, the factored torsional resistance \( T_e \) must equal or exceed the factored external torsional moment \( T_e \). All the torsional moment is assumed in the ACI 318-85 code to be resisted by the closed stirrups and the longitudinal steel with the torsional resistance, \( T_e \), of the concrete disregarded; that is, \( T_e = 0 \) on the assumption that the concrete compression struts between the inclined cracks have negligible resistance to torsion. The shear \( T_e \), resisted by the concrete is assumed to be maintained by the pressure of steel. This shear is

\[
\frac{T_e}{A_s} = \frac{T_e}{2A_t f_y} (\cot \theta)^{-1}
\]
The angle $\theta$ subtended by the concrete compression diagonals (stirrups) should not be taken smaller than 30° nor larger than 60°. It can be obtained by analysis as detailed in Refs. 7.12 and 7.15. According to Eq. 7.21b, the additional longitudinal reinforcement for torsion should not be less than

$$A_L = \frac{1}{2} \frac{f_y}{f_y} \frac{f_y}{f_y} \cot^2 \theta \tag{7.32}$$

where $f_y = \text{yield strength of the longitudinal torsional reinforcement, not to exceed 60,000 psi}$, and $A_L = \text{total area of longitudinal torsional steel in the cross section}$.

The same angle $\theta$ should be used in both Eqs. 7.31 and 7.32. It should be noted that as $\theta$ gets smaller, the amount of stirrups required by Eq. 7.31 decreases. At the same time the amount of longitudinal steel required by Eq. 7.32 increases.

In lieu of determining the angle $\theta$ by analysis (Ref. 7.15), the ACI Code allows a value of $\theta$ equal to:

(i) 45° for nonprestressed members or members with less prestress than in (ii)
(ii) 37.5° for prestressed members with an effective prestressing force larger than 40% of the tensile strength of the longitudinal reinforcement.

7.5.3 Minimum Torsional Reinforcement. It is necessary to provide a minimum area of torsional reinforcement in all regions where the factored torsional moment $T_o$ exceeds the value given by Eqs. 7.28 and 7.29. In such a case, the minimum area of the transverse closed stirrups required should be

$$A_s + 2A_T = \frac{500b_T}{f_{yt}} \text{ or } A_s = 0.75 \frac{b_T}{f_y}$$

whichever is larger.

The minimum spacing should not exceed the smaller of $p_2/8$ or 12 in.

The minimum total area of the additional longitudinal torsional reinforcement should be determined by

$$A_{L_{min}} = \frac{5 \sqrt{f_y}}{f_y} A_s - \frac{A_s}{S} p_2$$

where $A_s$ should not be taken less than $2b_T/f_y$.

The additional longitudinal reinforcement required for torsion should be distributed around the periphery of the closed stirrups with a maximum spacing of 12 in. The longitudinal bar or tendons should be placed inside the closed stirrups, with at least one longitudinal bar or tendon in each corner of the stirrup. The bar diameter should be at least one-sixteenth of the stirrup spacing, but not less than a No. 3 bar. Also, the torsional reinforcement should extend for a minimum distance of $b_T + d$ beyond the point theoretically required for torsion, because torsional diagonal cracks develop in a helical form extending beyond the cracks caused by shear and flexure. $b_T$ is the width of that part of the cross-section containing the stirrups resisting torsion. The critical section in beams is at a distance $d$ from the face of the support for reinforced concrete elements and at $b_T/2$ for prestressed concrete elements, $d$ being the effective depth and $b_T$ the total depth of the section.

7.5.4 Design Procedure for Combined Torsion and Bending

The following is a summary of the recommended sequence of design steps. A flowchart describing the sequence of operations in graphical form is shown in Figure 7.18.
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Figure 7.18 Flowchart for the design reinforcement for combined shear and torsion in solid sections: (a) Torsional web steel; (b) shear web steel.

1. Classify whether the applied torsion is equilibrium or compatibility torsion. Determine the critical section and compute the factored torsional moment $T_r$. The critical section is taken as $d$ from the face of the support in reinforced concrete beams and $h/2$ in prestressed concrete beams. If $T_r$ is less than $\phi \sqrt{f_c' A_{cp}} \bar{p}_{np}$ for nonprestressed members or less than $\phi \sqrt{f_c' A_{cp}} \bar{p}_{ps} \sqrt{1 + \frac{T_r}{4f_c'}}$, for prestressed members, torsional effects are neglected (the ACI code is directed at $f_{cT}$).

2. Check whether the factored torsional moment $T_r$ creates equilibrium or compatibility torsion. For compatibility torsion, limit the design torsional moment to the lesser of the actual moment $T_{aT}$ or $T_{dT} = \frac{\phi}{2} \sqrt{f_c' A_{cp}} \bar{p}_{ps} \sqrt{1 + \frac{T_r}{4f_c'}}$ for prestressed concrete members. The value of the design nominal strength $T_{dT}$ is to be at least equivalent to the factored $T_r/\phi$, proportioning the section such that:
Figure 7.18 Continued

(a) for solid sections:
\[ \sqrt{\frac{V_c}{b_d h}} + \frac{T_{cp}}{1.7A_{st}f_y} \leq \phi \left( \frac{V_c}{b_d h} + 8 \sqrt{f_y} \right) \]

(b) For hollow sections:
\[ \frac{V_c}{b_d h} + \frac{T_{cp}}{1.7A_{st}f_y} \leq \phi \left( \frac{V_c}{b_d h} + 8 \sqrt{f_y} \right) \]

If the wall thickness is less than \( A_{o/w} / \rho_o \), the second term should be taken as \( T_{c/1.7A_{st}f_y} \).

3. Select the required \textit{terminal} closed stirrups to be used as transverse reinforcement, using a maximum yield strength of 60,000 psi, such that
\[ \frac{A_t}{s} = \frac{T_{c}}{2A_{st}f_y} (\cot \theta)^{-1} \]

Unless using \( A_{o/w} \) and \( \theta \) values obtained from analysis (Ref. 7.14), use \( A_{o/w} = 0.85A_{o/w} \) and \( \theta = 45^\circ \) for nonprestressed members with an effective posttension not less than the tensile strength of the longitudinal reinforcement. The additional longitudinal reinforcement should be
Figure 7.18 Continued

\[ A_s = \frac{A_{s \text{min}}}{f_{c}} \times f_{y} \times \cos^{2} \theta \]

but not less than

\[ A_{s \text{min}} = \frac{SF_{c}A_{c}}{f_{y}} = A_{s} \times f_{c} \]

where \( A_{s} \) shall not be less than \( \frac{1}{3}b_{w}/s \). Maximum allowable spacing of transverse stirrups is the smaller of \( b_{w}/12 \) or 12 in., and bars should have a diameter of at least one-sixteenth of the stirrup spacing, but not less than a No. 3 bar size.

4. Calculate the required shear reinforcement \( A_s \) for unit spacing in a transverse section. \( V_{s} \) is the factored external shear force at the critical section; \( V_{c} \) is the nominal shear resistance of the concrete in the web, and \( V_{s} \) is the shearing force to be resisted by the stirrups:

\[ A_{s} = \frac{V_{s}}{s \times f_{y \text{all}}} \]

where \( V_{s} = V_{c} - V_{s} \) and...
for reinforced concrete,

\[ V_r = 2A_v \sqrt{f_p b_d d} \]

for prestressed concrete if \( a_p > 0 \) \( h \) for prestressed beams are

\[ V_r = 1.5A_v \sqrt{f_p b_d d} \]

where \( h = 1.0 \) for normal-weight concrete

- 0.85 for medium-lightweight concrete
- 0.75 for all-lightweight concrete

The value of \( V_r \) has to be at least equal to the factored \( V_{rf} / b \).

5. Obtain the total \( A_{stir} \), the area of the closed stirrups for torsion and shear, and design the stirrups such that

\[ A_{stir} = 2A_t + A_s \approx \frac{50b_d d}{f_y} \]

Extend the stirrups a distance \( h + d \) beyond the point theoretically no longer required, where \( h = \) width of the cross section containing the closed stirrup resisting torsion.

7.5.5 Example 7.1: Design of Web Reinforcement for Combined Torsion and Shear in a T-beam Section

A T-beam cross section has the geometrical dimensions shown in Figure 7.19. A factored external shear force acts at the critical section, having a value \( V_r = 40,000 \text{ lb (180 kN)} \). It is subjected to the following torques: (a) equilibrium factored external torsional moment \( T_r = 450,000 \text{ in-lb (514 kN-m)} \); (b) compatibility factored \( T_c = 65,000 \text{ in-lb (7.3 kN-m)} \); (c) compatibility factored \( T_c = 250,000 \text{ in-lb (29.9 kN-m)} \). Given:

- Bending reinforcement: \( A_t = 3.4 \text{ in}^2 (290 \text{ mm}^2) \)
- \( f_y = 4000 \text{ (27.6 MPa)}, \) normal-weight concrete
- \( f_y = f_c = 60,000 \text{ (414 MPa)} \)

Design the web reinforcement needed for this section.

![Figure 7.19 Component rectangles of the T-beam.](image-url)
Solution: (a) Equilibrium torsion:

Factored torsional moment (Step 1)
Assume that the flanges are not confined with ties.

\[ T_e = \frac{T}{\phi} = \frac{450,000}{0.75} = 600,000 \text{ in.-lb (678 kN-m)} \]

The total torsional moment must be provided for in the design.

\[ A_{G} = 14 \times 25 = 350 \text{ in.}^2 \]

\[ \rho_{G} = 2 \sqrt{(s_{x} + s_{y})} = 2 \sqrt{4 + 25} = 78 \text{ in.} \]

If the flanges were confined with closed ties,

\[ A_{G} = 14 \times 25 + 2(4 \times 12) = 446 \text{ in.}^2 \]

\[ \rho_{G} = 2 \sqrt{(4 + 25) + 2(4 + 3 \times 6)} = 142 \text{ in.} \]

From Eq. 7.28, torsional moment for which torsion can be neglected is

\[ T_{e} = \Phi \sqrt{\frac{A_{G}}{\rho_{G}}} = 0.75 \sqrt{\frac{350}{78}} \]

\[ = 74.49 \text{ in.-lb} < 450,000 \]

Hence design for full torsion.

Sectional properties (Step 2)

\[ A_{m} = 0.85 A_{G} \text{ where } A_{m} \text{ is the area enclosed by the center line of the outermost closed stirrups. Assuming 1.5-in. close cover and No. 4 stirrups, from Fig. 7.11,} \]

\[ s_{x} = 4 \times (1.5 + 0.25) = 10.5 \text{ in.} \]

\[ s_{y} = 25 \times (1.5 + 0.25) = 31.5 \text{ in.} \]

\[ A_{G} = 10.5 \times 21.5 = 226 \text{ in.}^2 \]

\[ A_{m} = 0.85(10.5 \times 21.5) = 192 \text{ in.}^2 \]

\[ d = 25 - (2.5 + 0.4 + 0.25) = 22.75 \text{ in.} \]

\[ p_{m} = 2(10.5 + 21.5) = 64 \text{ in.} \]

Use \( \theta = 45^\circ \), cos \( \theta = 0.707 \).

Check adequacy of section (Step 3)

For the section to be adequate, it should satisfy Eq. 7.30:

\[ \sqrt{\frac{V_{y}}{b_{x}d}} + \frac{\tau_{G}b_{y}}{1.7A_{G}} \leq \phi \left( \sqrt{\frac{V_{y}}{b_{x}d}} + 8 \sqrt{\tau_{G}} \right) \]

\[ V_{y} = 2 \sqrt{b_{x}d} = 2 \sqrt{400 \times 14 \times 22.5} = 39,845 \text{ lb.} \]

\[ \sqrt{\frac{V_{y}}{b_{x}d}} + \frac{\tau_{G}b_{y}}{1.7A_{G}} \leq \sqrt{\left( \frac{40,000}{14 \times 22.5} \right) + \left( \frac{450,000 \times 64}{1.7 \times 226} \right)} \]

\[ = \sqrt{16,124 + 110.015} = 385 \text{ psi (2.44 MPa)} \]

\[ \phi \left( \sqrt{\frac{V_{y}}{b_{x}d}} + 8 \sqrt{\tau_{G}} \right) = 0.75 \left( \frac{39,845}{14 \times 22.5} + 8 \sqrt{200} \right) \]

\[ = 0.75(265.5 + 506.6) = 474 \text{ psi (3.1 MPa) > 355 psi} \]

Hence the section is adequate.
7.6 ACI Design of Reinforced Concrete Beams Subjected to Combined Torsion, Bending, and Shear

**Torsional reinforcement (Step 4)**

From Eq. 7.31,

\[
A_t = \frac{V_T}{f_t \cdot \tan \theta} = \frac{\text{0.06 lb}}{5,000 \cdot 60 \text{ ksi} \cdot 12 \text{ in} \cdot \text{ft/rev \ leg}} = 0.026 \text{ in}^2/\text{ft/rev \ leg}
\]

**Shear reinforcement**

\[
V_T = 2 \sqrt{f_y \cdot d} = 30.645 \text{ lb}
\]

\[
V_T = \frac{20,000}{0.75} = 53,333 \text{ lb} > V_T; \text{ also } \frac{1}{2} V_T
\]

For minimum shear reinforcement, hence, provide shear stirrups,

\[
V_T = V_y = 53,333 - 30.645 = 15,488 \text{ lb}
\]

\[
A_s = \frac{V_y}{f_y} = \frac{15,488 \text{ lb}}{60 \text{ ksi} \cdot 12 \text{ in/rev \ leg}} = 0.01 \text{ in}^2/\text{ft/rev \ leg}
\]

\[
A_s = \frac{V_y}{f_y} \cdot \frac{1}{2} = 2 \times \frac{0.026}{2} + 0.01 = 0.062 \text{ in}^2/\text{ft/rev \ leg}
\]

Try No. 3 (9.5-mm diameter) closed stirrups; area of two legs = 0.22 in.².

\[
S = \text{area of stirrup from section} = 0.22 \text{ in.}^2
\]

\[
r_{y2} = \text{required } A_s \text{ in.}
\]

\[
0.062 \text{ in.}^2
\]

Maximum allowable spacing, \( r_{y2} \), is smaller of \( \frac{r}{y} \), or 12 in., where \( r_{y2} \), 2\( (x + y) = 64 \text{ in.} \)

From before \( r_{y2} = 8 \text{ in.} > 3.35 \text{ in.} \)

\[
0.75 \sqrt{r} = 0.75 \sqrt{4900} = 47 < 50 \text{ in Eq. 7.33}
\]

hence, from Eq. 7.33,

\[
A_y = \frac{S_3 \cdot f_y}{A_y} = \frac{50 \times 14 \times \frac{3.5}{2}}{60.000} = 0.04 \text{ in}^2
\]

Less than 0.22 in.²; does not control. Hence use No. 3 closed stirrups at 3.5 in. center to center.

If No. 4 closed stirrups are used, spacing can be increased to 65 in. c. to c.

\[
A_t = \frac{\lambda}{2} \cdot \frac{f_y}{f_t} \cdot \frac{r_{y2}}{r_{y2}} \cdot \text{cos} \theta
\]

\[
= 0.026 \times 60 \text{ ksi} \times 1.0 = 1.66 \text{ in}^2
\]

minimum \( A_t = \frac{5 \sqrt{A_{y2}}}{f_y} - \frac{2}{r_{y2}} = \frac{5 \sqrt{4000 \times 350}}{60 \times 500} - 0.025 \times 64 \times 60.000
\]

\[
= 1.84 - 1.66 = 0.18 \text{ in.}^2 < 1.66 \text{ in.}^2
\]

Hence, \( A_t = 1.66 \text{ in.}^2 \) controls.

**Distribution of tension longitudinal steel**

Torsional \( A_t = 1.66 \text{ in.}^2 \) Assume that \( 1/4 A_t \) goes to the top corners and \( 1/4 A_t \) goes to the bottom corners of the stirrups, to be added to the flexural bars. The balance, \( 1/2 A_t \), would thus be distributed equally to the vertical faces of the beam web cross section at a spacing not to exceed 12 in. center to center.

midspan \( \Sigma A_t = \frac{A_t}{4} + \frac{A_t}{4} = \frac{1.66}{4} + 3.34 = 3.82 \text{ in.}^2 \)
Provide five No. 8 (25.4-mm-diameter) bars at the bottom. Provide two No. 4 (12.7-mm-diameter) bars with an area of 0.40 in.² at the top. Provide two No. 4 (12.7-mm-diameter) bars on each vertical face. Figure 7.20 shows the geometry of the cross section.

(b) Compatibility Torsion

Factored torsional moment (Step 1)

Given $T_{ef} = 65,000$ in.-lb (7.3 kN-m) $< T_s = 74,496$ in.-lb from part (a). Hence disregard torsion and provide stirrups for shear only.

From part (a),

$$\frac{A_s}{A} = 0.010; \text{ Min. } A_s = 0.04 \text{ in.}^2 < 0.22 \text{ in.}^2 \text{ for No. 3 stirrups, hence does not control.}$$

For No. 3 stirrups, $\bar{z} = 0.22/0.010 = 22$ in. closer to center.

maximum $\bar{z} = \frac{d}{2} = \frac{22.5}{2} = 11.25$ in.

Use No. 3 closed stirrups at 10 in. c-c at the critical section.

(c) Compatibility Torsion

Factored torsional moment (Step 1)

Since $T_s = 269,000$ in.-lb (31.0 kN-m) is greater than $74,496$ in.-lb from case (b), hence stirrups have to be provided. Because this is a compatibility torsion, the section can be designed by Eq. 7.26 for

$$T_s = 4A \sqrt{f_y} \frac{A_s}{A_p}$$

$$= 4 \times 74,496 \text{ from case (a) } = 297,984 \text{ in.-lb}$$

This is $< 269,000$, hence use $T_s = 269,000$ in.-lb for the torsional design of the section.

required $T_s = \frac{T_s}{4} = \frac{269,000}{0.75} = 355,333$ in.-lb

Torsional reinforcement (Step 2)

From case (a), $A_s = 192$ in.², $p_s = 64$ in.

$$A_s = \frac{T_s}{2A_s f_y \cot \theta} = \frac{355,333}{2 \times 192 \times 0.6000 \times 1.0}$$

Figure 7.20 Web reinforcement details, Ex. 7.1(a).
Figure 7.21 Web reinforcement details, Ex. 7.1(c).

For case (a),

\[
A_y = 0.010 \text{ in.}^2/\text{in.}/2 \text{ legs}
\]

\[
A_y = 2 \frac{A_y}{s} = 2 \times 0.015 = 0.040 \text{ in.}^2/\text{in.}/2 \text{ legs}
\]

Using No. 3 stirrups, \( s = 0.220 \times 0.040 = 0.550 \text{ in.} \). This is less than \( f_y/8 = 8 \text{ in.} \) or 12 in. Hence, use No. 2 closed stirrups at 5 in. (i.e., 9.5-mm diameter at 138 mm c-c) at the critical section.

\[
A_y = \frac{A_y}{s} f_y \text{ c-c} = 0.015 \times 64 = 1.0 \times 0.96 \text{ in.}^2
\]

\[
\min A_y = \frac{5 \sqrt{f_y/2}}{f_y} \left( \frac{A_y}{s} f_y \right)
\]

\[
= \frac{5 \sqrt{60.000 \times 350}}{60.000} = 0.015 \times 64 = 0.96 \text{ in.}^2
\]

\[
= 1.0 - 0.96 = 0.04 \text{ in.}^2 < 0.96 \text{ in.}^2
\]

\( A_y = 0.96 \text{ in.}^2 \) controls.

Distribution of torsion longitudinal bars

Torsional \( A_y = 0.96 \text{ in.}^2 \); to \( A_y = 0.24 \text{ in.}^2 \). Using the same logic as that followed in case (a), provide five No. 8 (25.4-mm-diameter) bars at the bottom face. The area required, \( A_y = 0.54 \text{ in.}^2 \); the area provided is \( 5.84 \text{ in.}^2 \). The required area at the top corner and at each vertical face = 0.84 in.². Provide two No. 4 bars (12.7-mm diameter) at the top two corners and at each of the vertical sides, giving 0.40 in.² in each area. Figures 7.20 and 7.21 show the geometry of the section reinforcement.

7.5.6 Example 7.2: Equilibrium Torsion Web Steel Design

A normal-weight 7.0 concrete, concrete canopy slab on continuous spans spans 24 ft (7.32 m) on several supports, as shown in Figure 7.22. It carries a uniform service live load of 30 psf (1.44 kPa) on the canopy. Design the interior span spanned beam A1 - A2 for diagonal tension and torsion. Assume no wind or earthquake and neglect creep and shrinkage effects. Given:

\[
\begin{align*}
&f_y = 4000 \text{ psi} (27.6 \text{ MPa}) \\
&f_y = f_y = 60,000 \text{ psi (415 MPa)}
\end{align*}
\]
Figure 7.22 Plan and sectional elevation, Ex. 7.2: (a) plan; (b) section A-A.

- Extension column = 12 in. × 20 in. (300 mm × 500 mm)
- Midspan A₁ = 1.50 in.² (996 mm²)
- Support Aₗ = 2.4 in.² (1545 mm²)
- Support Aₚ = 0.8 in.² (516 mm²)

Solution:

- Factored torsional moment (Step 1)
  - Beam A₁-A₃ is a case of eccentric distribution because the torsional resistance of the beam is required to maintain equilibrium. Hence, the section has to be designed to resist the total external factored torsional moment.
  - Service dead load of the cantilever slab = 8.0 psf (0.38 kPa)
  - Service live load = 30 psf (1.44 kPa)
  - Torsional load U = 1.2 × 1000 = 1.2 × 30 = 36 psf (168 kPa)
  - Total load on the cantilever slab = 148 × 24 × 7 = 20,224 lb (93.0 kN)

- This load acts at the center of gravity of the loading, as shown in Figure 7.22a. The maximum factored moment at the center line of the support = \( \frac{1}{2} \times (20.224 \times 4) = 16,444 \text{ ft lb} \).

- Note that the reaction at the supports is half of the total torque acting on the slab, as shown in Figure 7.23, because the center of gravity of the twisting moment is midway be-
7.5 ACI Design of Reinforced Concrete Beams Subjected to Combined Torsion, Bending, and Shear

Figure 7.23 Distribution of torsional moment.

The torsional moment at the critical section (97.5 m) from the face of the support is

\[ T_\tau = 56,448 \left( \frac{22 - 10 + 27.5}{12} \right) = 41,748 \text{ ft-lb} \]
\[ = 502,976 \text{ in.-lb (70.3 kN-m)} \]

The required torsional moment is
\[ T_\tau = \frac{55,664}{0.75} = 74,219 \text{ ft-lb (94 kN-m)} \]
\[ = 667,491 \text{ in.-lb} \]

Shear force distribution (Step 2)

Since the beam is to be designed for combined shear and torsion, the distribution of the shear force along the span needs to be determined.

Shear load = \[ \frac{30 \times 12}{144} \times 150 \times 1.2 = 450 \text{ lb/ft} \]

Figure 7.24 Torsion envelope for beam A1 - A0, Ex. 7.2.
Total factored shear at face of support is

\[ V_L = \frac{2}{3} (45 \times 24 + 28224) = 19,512 \text{ lb} \]

\[ V_r = \frac{19,512}{12 + 22.5} = 1443 \text{ lb} \]

\[ V_r' = \frac{1443}{3} = 481 \text{ lb} \]

Section properties (Step 3):
From Figure 7.23, assuming 1.5-in. clear cover and No. 4 stirrups and that the flange is not confined with closed ties.

\[ A_w = 12 \times 12 = 36 \text{ in}^2 \]
\[ x_n = 2(r + x) = 2(12 + 30) = 84 \text{ in} \]

If the flange is confined, a flange width of 3 x slab thickness would have been taken. In such a case,

\[ A_w = 12 \times 24 = 288 \text{ in}^2 \]
\[ p_n = \beta (1/2 + 0.25) = 8.5 \text{ in} \]
\[ y_n = \frac{20}{2(1.5 + 0.25)} = 26.5 \text{ in} \]
\[ d_n = \frac{20}{2(1.5 + 0.25)} = 20 \text{ in} \]
\[ f_n = 0.35 \times 26.5 = 27.35 \text{ in} \]
\[ A_{w_n} = 8.5 \times 26.5 = 225 \text{ in}^2 \]
\[ A_b = 0.65A_{w_n} = 191 \text{ in}^2 \]
\[ a = \frac{45^\circ}{4} = 10^\circ \]

Check if torsion has to be considered.

From Eq. 7.28,

\[ T_e = \frac{2}{3} \sqrt{\frac{A_{w_n}^2}{p_n}} \cdot 0.75 \sqrt{\frac{300^2}{84}} \]
\[ T_e = 3715 \text{ in.} \cdot \text{lb} < 500 \text{ in.} \cdot \text{lb} \]

Hence, torsional moment has to be considered.

Figure 7.29 Component rectangles.
Section adequacy check (Step 3)

\[ V_t = 2 \sqrt[3]{bd} = 2 \sqrt[3]{400 \times 12 \times 27.5} = 41,740 \text{ lbs} \]

\[ \sqrt[3]{\frac{V_t}{bd}} = \frac{T_{pl}}{1.7\lambda_w} = \sqrt[3]{\frac{41,740}{12 \times 27.5}} = \frac{500,076 \times 70}{1.7 \times (225)^2} \]

\[ = \sqrt[3]{1912 + 166,036} = 410 \text{ psi} \]

\[ \theta \left( \frac{V_t}{bd} + 8 \sqrt[3]{T_{pl}} \right) = 0.75 \left( \frac{41,740}{12 \times 27.5} + 8 \times 410 \right) = 0.75(126 + 505) = 473 \text{ psi} > 410 \text{ psi}; \text{ hence section is adequate} \]

Since this is an equilibrium torsion, there is no need to evaluate the value of \( T_{pl} \) that the section can sustain using Eq. 7.26.

Torsional reinforcement (Step 3)

From Eq. 7.31b,

\[ T_{pl} = 500,076 \times 0.75 = 667,068 \]

\[ \frac{A_v}{x} = \frac{24 \lambda_v \cos \theta}{2 \times 191 \times 60,000 \times 1.0} = 0.03 \text{ in.}^2/\text{in.}/\text{one leg} \]

Shear reinforcement (Step 4)

\( V_t = 41,740 \) lb from before. Required \( V_u = 19,441 \) lb, from before < 41,740 lb. Also < \( \frac{1}{3} V_t \), hence no minimum shear reinforcement needed.

\[ \frac{\Delta_u}{\Delta_{pl}} = 2 \frac{x}{x} = 2 \times 0.03 + 0 = 0.06 \text{ in.}/\text{two legs} \]

\[ \frac{0.75 \sqrt{T_{pl}}}{75} = 0.75 \sqrt{400} = 47 < 50; \text{ hence from Eq. 7.33} \]

\[ \frac{\Delta_{pl}}{\Delta_{pl}} = \frac{500}{66000} = 0.01 < 0.06, \text{ O.K.} \]

Try No. 3 closed stirrups = \( 2 \times 0.11 = 0.22 \) in. (bar size has the larger of at least No. 3 bar or \( 0.36 \)).

\[ s = \text{area of the cross section required } \frac{A_v}{x} = 0.22 \times \frac{0.06}{3.67} = 0.03 \text{ in.}^2/\text{in.} \]

Maximum allowable \( s = \text{least of } \frac{p}{(p)_{pl} \text{ or } 12 \text{ in.}} \)

\[ \frac{p}{8} = \frac{70}{8} = 8.75 \text{ in.} \]

Therefore, provide No. 3 (0.5-mm diameter) closed stirrups at 3.5 in. center to center (89 mm c-c) at the critical section up to the face of the support. Since the maximum spacing is 8.75 in. and \( V_t \) is larger than the factored \( V_{pl} \), the increase in spacing along the span toward the support is determined only with respect to the decrease in \( T_{pl} \) along the span. Assume that the stirrups start being spaced at \( s = 8.5 \) in. at a plane \( x \), distance from face of the support, having a torsional moment \( T_{pl} \).

For \( s = 8.5 \)

\[ \frac{\Delta_{pl}}{\Delta_{pl}} = 0.22 \times \frac{0.026}{8.5} = 0.026 \]

\[ T_{pl} = 0.026 \times 55,664 = 24,121 \text{ lb} \]
From similar triangles in Fig. 7.24,

$$x = 27.5 \mp \left( \frac{36.5}{5.0} \times \frac{24.121}{2.5} \times 106.5 \right) = 88 \text{ in.}$$

Torsion is disregarded at $T_{ad}$ if

$$T_{ad} = \sqrt{\frac{36}{84}} \cdot 97,579 \text{ in.-lb} = 1332 \text{ ft-lb}$$

$$x_2 = 27.5 + \left( \frac{106.5}{55.664} \times 18.5 \right) = 118.5 \text{ in.}$$

Extend closed stirrups a distance $a + d$ beyond $x_2$, that is, $118.5 + 12 + 27.5 = 158 \text{ in.}$; thus use closed stirrups throughout the span. Figure 7.26 shows schematically the spacing of the closed No. 3 stirrups.

**Longitudinal torsional reinforcement**

From Eq. 7.32,

$$A_i = \frac{A_{ci}}{f_{ct}} \times \frac{f_{ct}}{f_{ct}} \cdot \cos \theta = 0.03 \times 70 \times \frac{60,000}{60,000} \times 1.0 = 2.1 \text{ in.}$$

From Eq. 7.34,

$$A_{i,pad} = \frac{5 \sqrt{\frac{f_{ct}}{f_{ct}}} \cdot A_{ci} \cdot A_{ci} \cdot f_{ct} \cdot f_{ct}}{f_{ct}} = \frac{5 \sqrt{90} (360)}{60,000} \cdot 0.03 \times 70 \times \frac{60,000}{60,000} = 0$$

Support (column)

\[25 \text{ spans at 3.5 in. } + 87.5 \text{ in.} \]

Midspan

\[5 \text{ at 8.5 in.} \]

Longitudinal bar

87.5 (2227 mm)

42.5 in. (1080 mm)

10 in.

12 ft (3.66 m)

Use 12 No. 3 stirrups for the whole span.

**Figure 7.26** Closed stirrup arrangement for Ex. 7.2.
Figure 7.27  Web reinforcement details: (a) support section; (b) midspan section.

Use $A_t = 2.1$ in.$^2$ (1355 mm$^2$). To distribute $A_t$ evenly on all four faces of the beam, use $A_{t1}$ at each vertical face and $A_{t2}$ at the top two corners and $A_{t3}$ at the bottom two corners or tension side to be added to the flexural reinforcement, $A_{t4} = 2.14 = 0.53$. Use two No. 5 bars = 0.62 in.$^2$ (12.7 mm diameter) on each vertical side for both the support and midspan sections.

**Support section:**

$$
\Sigma A_t = \frac{A_t}{4} + A_t = 0.53 + 2.4 = 2.93 \text{ in.}^2
$$

Use four No. 8 bars = 3.16 in.$^2$ (25.4 mm diameter).

$$
\Sigma A_t' = \frac{A_t}{4} + A_t' = 0.53 + 0.8 = 1.33 \text{ in.}^2
$$

Use two No. 8 bars = 1.58 in.$^2$.

**Midspan section:**

$$
\Sigma A_t = \frac{A_t}{4} + A_t = 0.53 + 1.50 = 2.03 \text{ in.}^2
$$

Use three No. 8 bars = 2.37 in.$^2$ at bottom.

Since the torque decreases as the midspan is approached, two of the top No. 8 longitudinal bars can be cut off prior to reaching the midspan section. Figure 7.27a and b give the reinforcing details of the beam at the support and midspan sections, respectively.

### 7.5.7 Example 7.3: Compatibility Torsion Web Steel Design

A parking-garage floor system of one-way slabs on beams is shown in Figure 7.28. Typical panel dimensions are 12 ft. 6 in. x 20 ft (3.81 m x 6.10 m) on centers. Design the exterior spandrel beam $A_t = R$, for combined torsion and shear, assuming that the sections are adequately designed for bending. Given:

- Service live load = 50 psf (2.4 kPa)
- Slab thickness = 3 in. (76 mm)
- $f'_c = 4000$ psi (27.6 MPa), normal-weight concrete
- $f'_c = f'_c = 6000$ psi (413.7 MPa)
- Height floor to floor = 10 ft
- Exterior columns = 14 in. x 24 in. (356 mm x 610 mm)
- Interior columns = 24 in. x 24 in. (610 mm x 610 mm)
all beams = 14 in. × 30 in. (356 mm × 762 mm)
required flexural reinforcement for beam $A_1 = B_1$
midspan $A_1 = 1.69$ in.$^2$
support $A_1 = 2.16$ in.$^2$
support $A_1 = 0.90$ in.$^2$

Solution: Beam $A_1 - B_1$ is a case of compatibility torsion because it is part of a continuous floor system where redistribution of moments takes place. The torsional moment due to $C_2 - C_1$ at intersection $C_3$ is redistributed in directions $C_1 - C_2$ due to the flexibility and rotation of the beam section at $C_3$ compared to its rigidity at $A_1$ and $B_1$. Hence the maximum factored torsional value to be applied to the section at each of the two ends (Figure 7.29a) is to be the lesser value of the actual $T$, or that obtained from Eq. 7.26.

Section properties (Step 1)
From Figure 7.29, assuming 1.5-in. cover and No. 4 closed stirrups, $x_t$ and $y_t$ are the smaller and larger dimensions, respectively, of the section, and $x_1$ and $y_1$ are the inner dimensions to the center of the stirrups. The flange is not confined with torsional closed ties.

$$A_{w} = 14 \times 30 = 420 \text{ in.}^2$$
$$x_1 = 14 - 2(1.5 + 0.25) = 10.5$$
$$y_1 = 30 - 2(1.5 + 0.25) = 26.5 \text{ in.}$$
$$x = 2(x_1 + y_1) = 2(10.5 + 26.5) = 74 \text{ in.}$$
$$y = 30 - (1.5 + 0.5 + 0.25) = 27.5 \text{ in.}$$
$$A_{x} = 10.5 \times 26.5 = 278 \text{ in.}^2$$
$$A_{y} = 0.8A_{w} = 336 \text{ in.}^2$$
$$\theta = 45^\circ, \quad \cot \theta = 1.0$$
7.5 ACI Design of Reinforced Concrete Beams Subjected to Combined Torsion, Bending, and Shear

1. Factored torsional moments (Step 2.3)

The maximum factored torsional value to be applied to the section at each of the two ends (Figure 7.29a), a compatibility torsional moment, is from Eq. 7.26 for compatibility torsion

\[
T_c = 4\pi \sqrt{\frac{A_t}{R_s}}
\]

\[
= 0.75 \times 4 \sqrt{\frac{4000 \times 200}{88}} = 380.336 \text{ in}-\text{lb}
\]

\[
= 31.658 \text{ ft}-\text{lb} \times (43.8 \text{ kN-m})
\]

\[
T_c = 31.658 \times \frac{12}{0.75} = 507.120 \text{ in}-\text{lb} \times (57.3 \text{ kN-m})
\]

2. Fixed end moments in beam C1-C2

\[
\text{service dead load} = \frac{12}{50} \times 12.5 = 130 = 1146 \text{ lb/ft} (16.7 \text{ kN/m})
\]

\[
\text{service live load} = 12 \times 1146 = 1375 \text{ lb/ft} (34.5 \text{ kN/m})
\]

\[
\text{factored load} U = 1.2 \times 1146 = 1375 \text{ lb/ft} (34.5 \text{ kN/m})
\]

\[
\text{fixed end moment} = w_u \frac{50}{12} = 2375 \left( \frac{50}{12} \right) = 494.792 \text{ ft}-\text{lb}
\]

The factored torque in compatibility torsion that beam C1-C2 applies at connection C1 is

\[
T_c = 2 \times 31.658 = 63.316
\]

This value is less than the factored end moment \( w \mu \gamma / L \) at end C1. Hence the torsional moment to be used at midpoint of \( A_1 - B_1 \) is \( T_c = 63.316 \text{ ft}-\text{lb} \). Perform the moment distribution shown in Figure 7.29b to determine the reactions \( R_{1i} \) and \( R_{2i} \).

3. Beam reaction at C1, and the resulting shear in beam A1-B1

\[
\Sigma M_2 = 0 \quad \text{or} \quad 50 R_{1i} - 710,493 - 63,390 = 0
\]

\[
R_{1i} = \frac{-710,493 + 63,390 + 2,568,750}{50} = 48,433 \text{ lb}
\]

Factored self-weight of \( A_1 - B_1 \) is 12 \( \frac{14 \times 30}{144} \) = 525 lb/ft

Distance of critical section in \( A_1 - B_1 \) from column center line \( d = 142 = (30 + 2.5) + 142 = 34.5 \text{ in.} \)

\[
V_i = \frac{48,433}{2} + 525 \left( \frac{12.5 - 34.5}{12} \right) = 28,270 \text{ lb}
\]

Beams \( A_1 - B_1 \) would be subjected to the torsion and shear envelopes shown in Figure 7.30.

Section adequacy check (Step 4)

\[
V_i = 2 \sqrt{\frac{h_1 d}{1.74 A_1}} = 2 \sqrt{\frac{4000 \times 14}{14 \times 27.5}} = 48,700 \text{ lb}
\]

From Eq. 7.30a,

\[
\sqrt{V_i^2 + T_c^2} = \sqrt{28,270^2 + \left( \frac{380,336 \times 74}{1.7 \times 27.5^2} \right)^2}
\]
Figure 7.29 (a) Component rectangles; (b) bending moments for beam C₁ - C₂.

\[
V = \sqrt{5381 + 45,890} = 226 \text{ psi}
\]

\[
\phi \left( \frac{V}{f_y} + 8 \sqrt{f_y} \right) = 0.75 \left( \frac{48,700}{14 \times 27.5} + 8 \sqrt{4000} \right) = 476 \text{ psi} > 226 \text{ psi}; \text{ hence section is adequate.}
\]

_Torsional reinforcement (Step 5)_

From Eq. 7.31b,

\[
\frac{A_t}{A} = \frac{T_c}{A_f c} = \frac{507.1}{3} = 169.03
\]

\[
= 0.317 \text{ in}./\text{in.}./\text{sec}
\]
Figure 7.30 (a) Torsion and (b) shear factored force envelopes for beam A = 6, Ex. 7.3.

Shear reinforcement (Step 6)

\[ V_s = 48,700 \text{ lb and } V_f = 28,270 \text{ from before.} \]

\[ \text{required } V_s = \frac{28,270}{0.75} = 37,693 \text{ lb } < V_s \]

but larger than \( IV_s = 24,350 \text{ lb}; \) hence minimum shear reinforcement needed.

\[ 0.75 \sqrt{f_y} = 0.75 \sqrt{4000} = 47 < 50, \]

hence, from Eq. 7.33, minimum web steel is

\[ A_s = \frac{506}{f_y} = \frac{59 \times 14}{60,000} = 0.012 \text{ in.}^2 \text{/in. - two legs} \]

\[ A_s = \frac{2A_t}{A_t} = 2 \times 0.017 + 0.012 = 0.046 \text{ in.}^2 \text{/two legs} \]

Try No. 3 closed stirrups \( 2 \times 0.11 = 0.22 \text{ in.}^2 \) (but size has to be the larger of at least No. 3 or \( 0.16 \) for longitudinal bars).

\[ s = \frac{\text{area of cross section}}{\text{required } A_{fs}} = \frac{0.22}{0.046} = 4.78 \text{ in. - c-c} \]

Maximum allowable \( s = \text{less of } p_s/6 \text{ or } 12 \text{ in.; } p_s/6 = 748 = 9.25 \text{ in.. Due to constant torsion imposed by beam } C_2 - C_3 \text{ at midspan (Figure 7.28), use same spacing of the closed No. 3 stirrups throughout the span. Space the stirrups at 4.75 in. center to center.} \]
Longitudinal torsional reinforcement

From Eq. 7.32,

\[ A_t = \frac{A}{t} \left( \frac{f_t}{f_y} \right)^2 \times \frac{8}{0.017 \times 74 \times 60,000 \times 1.0} = 1.26 \text{ in.}^2 \]

From Eq. 7.34,

\[ A_{tmax} = \frac{2 \sqrt{f_t f_{fmax}}}{f_{fmax}} \times \frac{A_t f_{fmax}}{f_{fmax}} \times \frac{\sqrt{V} - V_{max}}{60,000} = 0.017 \times 74 \times 50,000 \times 0.95 \text{ in.}^2 \]

Use \( A_t = 1.26 \) (81 mm²).

To distribute \( A_t \) evenly on all four faces of the beam, use \( \frac{A_t}{4} \) at each vertical face with \( \frac{A_t}{4} \) at the top two corners and \( \frac{A_t}{4} \) at the bottom two corners at tension side to be added to the flexural reinforcement, \( A_t/4 = 1.26/4 = 0.32 \text{ in.}^2 \). Use three No. 6 bars = 0.50 in.² (12.7 mm diameter) on each vertical face for both the support and midspan sections. (Three No. 3 bars could be used, but are less rigid in handling.)

Support section:

\[ \Sigma A_t = \frac{A_t}{4} + A_t = 0.32 + 1.26 = 2.38 \text{ in.}^2 \]

Use six No. 6 bars = 2.04 in.² (six bars, 19.1 mm diameter) at top.

\[ \Sigma A_t = 0.32 + 0.00 = 2.32 \text{ in.}^2 \]

Use three No. 6 bars = 1.32 in.² at bottom.

Midspan section:

\[ \Sigma A_t = 0.32 + 1.60 = 2.01 \text{ in.} \]

Use five No. 6 bars = 2.36 in.² at bottom, with three of these bars to continue up to the support (five bars, 19.1 mm diameter).

Figures 7.31a and b give details of the combined torsional-flexural reinforcement in the spanned beam.

Figure 7.31 Web reinforcement details. (a) Support section, (b) midspan section.
7.6 SI METRIC TORSION EXPRESSIONS AND EXAMPLE FOR TORSION DESIGN

In order to design for combined torsion and shear using the SI (System International) method, the following equations replace the corresponding expressions in the PI (pound-inch) method:

Equation 7.26: \[ T_e = \frac{\phi \sqrt{f_{c}^*}}{3} A_{c}^* \]

Equation 7.27: \[ T_{c} = \frac{\phi \sqrt{f_{c}^*}}{3} A_{c}^* \sqrt{1 + \frac{8 f_{y}^*}{f_{c}^*}} \]

Equation 7.28: \[ T_{c} = \frac{\phi \sqrt{f_{c}^*}}{12} A_{c}^* \]

Equation 7.29: \[ T_{c} = \frac{\phi \sqrt{f_{c}^*}}{12} A_{c}^* \sqrt{1 + \frac{8 f_{y}^*}{f_{c}^*}} \]

Equation 7.30a: \[ \frac{V_{d}}{b_{d} d} + \frac{T_{d}}{1.7 A_{d}^*} = \phi \left( \frac{V_{d}}{b_{d} d} + \frac{8 \sqrt{f_{c}^*}}{12} \right) \]

Equation 7.30b: \[ \frac{V_{d}}{b_{d} d} + \frac{T_{d}}{1.7 A_{d}^*} = \phi \left( \frac{V_{d}}{b_{d} d} + \frac{8 \sqrt{f_{c}^*}}{12} \right) \]

Equation 7.30c (reinforced): \[ V_{d} = \lambda \sqrt{f_{c}^*} b_{d} d \]

Equation 7.30d (prestressed): \[ V_{d} = \left( \frac{\lambda \sqrt{f_{c}^*}}{20} + \frac{5V_{d} d}{M_{d}} \right) b_{d} d \]

\[ \geq 0.5 \sqrt{f_{c}^*} b_{d} d \]

\[ \leq (0.6 \lambda \sqrt{f_{c}^*}) b_{d} d \quad \text{and} \quad \frac{V_{d} d}{M_{d}} \leq 1.0 \]

Equation 7.31a: \[ T_{c} = \frac{2 A_{c}}{3} f_{c}^* \cot \theta \]

where \( f_{c}^* \) is in MPa, \( s \) in mm, \( A_{c} \), and \( A_{s} \) in mm², and \( T_{c} \) in kN-m.

Equation 7.31b: \[ A_{c} = \frac{T_{c}}{2A_{s} f_{c}^*} \cot \theta \]

Equation 7.32: \[ A_{s} = \frac{A_{c}}{2} \frac{f_{c}^*}{f_{y}^*} \cot \theta \]

where \( f_{c}^* \) and \( f_{y}^* \) are in MPa, \( p_{s} \) and \( s \) in mm, and \( A_{c} \) and \( A_{s} \) in mm².

Equation 7.33: \[ A_{s} = \frac{2A_{s} \geq 0.55b_{d} d}{f_{y}^*} \quad \text{or} \quad (A_{s} + 2A_{y}) = \frac{s}{10} \sqrt{f_{c}^*} \]

whichever is larger

Equation 7.34: \[ A_{s} = \frac{5\sqrt{f_{y}^*} A_{c}}{12 f_{y}^*} - \frac{A_{c}}{2} \frac{f_{y}^*}{f_{c}^*} \]

where \( A_{s} \) should not be taken less than \( 0.175 b_{d} d / f_{y}^* \).

Maximum allowable spacing of transverse stirrups is the smaller of 1 mm or 300 mm, and bars should have a diameter of at least \( \phi \) of the stirrups spacing, but not less than No. 10 M bar size. Maximum \( f_{c}^* \) or \( f_{y}^* \) should not exceed 400 MPa.
7.6.1 Example 7.4: SI Torsion Design

Solve Ex. 7.1 using SI units.

Data
- \( f_t = 276 \) Mpa (MPa = N/mm²)
- \( f_y = f_y' = 414 \) MPa
- \( V_c = 180 \) kN

(a) equilibrium \( T_e = 51.4 \) kN-m
(b) compatibility \( T_e = 7.3 \) kN-m
(c) compatibility \( T_e = 30.3 \) kN-m

\[ b_s = \text{350 mm} \quad A_{s} = \text{2190 mm}^2 \]
\[ d = \text{570 mm} \]
\[ h = \text{615 mm} \]
\[ h_y = \text{101 mm} \]

Solution: (a) Equilibrium torsion, \( T_e = 51.4 \) kN-m
(No conﬁning ties in the ﬂanges; hence disregarded when computing \( A_{pr} \). Same applies to \( p_{pr} \).

\[ A_{pr} = 356 \times 635 = 226,000 \text{ mm}^2 \]
\[ p_{pr} = 2(x + y) = 2(356 + 635) = 1982 \text{ mm} \]

From Eq. 7.28, torsional moment for which torsion can be neglected is

\[ T_e = \frac{12}{12} \frac{\sqrt[3]{A_{pr}}}{p_{pr}} \]
\[ = \frac{12}{12} \frac{\sqrt[3]{226,000}}{1982} \]
\[ = 8.5 \times 10^4 \text{ N-mm} = 8.5 \text{ kN-m} < 51.4 \text{ kN-m} \]

in case (a); hence design for torsion.

\[ T_e = \frac{V_c}{g} = 51.4 \quad g = 0.75 \quad 68.5 \text{ kN-m} \]

Sectional properties

\[ A_{g} = 0.85A_{as} \]

where \( A_{as} \) is the area enclosed by the center line of the outermost closed stirrups. Assume 40-mm clear cover and No. 10 M bar (diameter = 11.3 mm, \( A_s = 100 \text{ mm}^2 \)).

\[ x_s = 356 - 2 \left( \frac{40 + 11.3}{2} \right) = 264 \text{ mm} \]
\[ y_s = 635 - 2 \left( \frac{40 + 11.3}{2} \right) = 543 \text{ mm} \]
\[ A_{as} = x_s y_s = 264 \times 543 = 143,400 \text{ mm}^2 \]
\[ A_g = 0.85A_{as} = 122,060 \text{ mm}^2 \]
\[ d = 635 - \left( \frac{40 + 11.3}{2} \right) = 578 \quad \text{use } d = 570 \text{ mm} \]
\[ p_{pr} = 2(x_s + y_s) = 2(264 + 544) = 1616 \text{ mm} \]

Use \( b = 45^\circ \); cot \( b = 1.0 \).
Check adequacy of section

For the section to be adequate, it should satisfy Eq. 7.30a:

\[
\left( \frac{V_c}{A_{s}} \right)^2 + \left( T_{p} \right) = \phi \left( \frac{V_c}{A_{s}} \right) \frac{8 \sqrt{f_{c}}} {12}
\]

\[
V_c = \lambda \sqrt{f_{c}} \sum_{i=6}^{} = \frac{1.0 \times 278.2}{6} \times 356 \times 570
\]

\[= 177,800 \text{ N} = 177.8 \text{ kN}\]

\[
\left( \frac{V_c}{A_{s}} \right)^2 + \left( T_{p} \right) = \sqrt{\left( 180 \times 10^{3} \right)^2 + \left( 285.4 \times 10^{3} \times 1616 \right)^2} \]

\[= \sqrt{(0.79)^2 + (2.38)^2}
\]

\[= 1.78 \text{ N/mm}^2\]

\[
\phi \left( \frac{V_c}{A_{s}} \right) \frac{8 \sqrt{f_{c}}} {12} = 0.75 \left( \frac{177.8 \times 10^{3}}{356 \times 570} \times \frac{8 \sqrt{278.2}} {12} \right)
\]

\[= 0.75(0.88 + 3.50)
\]

\[= 3.37 \text{ MPa} > 1.78 \text{ MPa}\]

Hence, the section is adequate.

**Torsional reinforcement (Step 3)**

From Eq. 7.31 b,

\[T_s = 68.5 \text{ kN} \cdot m = 68.5 \times 10^{3} \text{ N} \cdot \text{mm}\]

From Eq. 7.31 h,

\[A_s = \frac{T_s}{2 \tau_s \cot \theta} = \frac{68.5 \times 10^{3}}{2 \times 122.050 \times 414 \times 1.0}
\]

\[= 0.666 \text{ mm}^2/\text{mm/one leg}\]

**Shear reinforcement**

\[V_c = \lambda \sqrt{f_{c}} \sum_{i=6}^{} = 178 \text{ kN}\]

From before, required \(V_c = 180/0.75 = 240 \text{ kN} > V_c \geq 1V_c\) for minimum shear web reinforcement. Provide closed stirrups.

\[V_c = V_s - V_c = 240 - 178 = 62 \text{ kN}\]

\[A_s = \frac{V_s}{f_{c} \sum_{i=6}^{}} = \frac{62 \times 10^{3}}{414 \times 570} = 0.25 \text{ mm}^2/\text{mm/two legs}\]

\[A_s = \frac{2A_s}{x} + A_s = 2 \times 0.666 + 0.25 = 1.6 \text{ mm}^2/\text{mm/two legs}\]

Try No. 10 M closed stirrups (11.3-mm diameter, \(A_s = 100 \text{ mm}^2\)). Area of two legs = \(2 \times 100 = 200 \text{ mm}^2\).

\[x = \frac{\text{area stirrups cross section}}{\text{required } A_s} = \frac{200}{1.6} = 125 \text{ mm}\]

Maximum allowable spacing, \(x_{max}\) is smaller of 125 mm or \(40 \times (x_1 + y_1) = 1616 \text{ mm} from before\), \(x_1 = 1616 = 202 \text{ mm} > 125 \text{ mm}\). From Eq. 7.33,

\[\frac{1}{16} \sqrt{f_{c}} = \frac{1}{16} \sqrt{278.2} = 0.33 < 0.35, \text{ hence}\]
\[ A_{cr} = \frac{0.35b_d f_y}{f_{cr}} = \frac{0.35 \times 356 \times 125}{414} = 37 \text{ mm}^2 \]

Hence, use No. 10 M closed stirrups at 125 mm center to center.

From Eq. 7.32,
\[ A_i = \frac{A_{cr}}{r} \frac{f_y}{f_{cr}} \cos^2 \theta = 0.666 \times 1616 = 1076 \text{ mm}^2 \]

From Eq. 7.34,
\[ A_{sw} = \frac{5 \sqrt{f_{cr} A_{cr}}}{12 f_y} = \frac{A_i}{r} \frac{f_y}{f_{cr}} \]
\[ = \frac{5 \sqrt{37.6 \times 210.560}}{12 \times 414} = 0.666 \times 1616 \]
\[ = 1195 - 1076 = 119 \text{ mm}^2 \]

where
\[ A_i = \frac{0.175 b_d f_y}{f_{cr}} = \frac{0.175 \times 356}{414} = 0.15 < 0.666 \text{ O.K.} \]

Hence, \( A_i = 1076 \text{ mm}^2 \) controls.

Assume that \( I \cdot A_i \) goes to the top corners and \( I \cdot A_i \) goes to the bottom of the stirrups to be added to the flexural bars. The balance, \( I \cdot A_i \) would thus be distributed equally on the vertical faces of the beam web cross section at a spacing not to exceed 300 mm e-e.

Midspan \( \sum A_i = \frac{A_i}{4} + A_i = \frac{1076}{4} = 2190 = 2400 \text{ mm}^2 \)

From Fig. B.2b, provide five No. 28 M longitudinal bars, \( A_i = 2500 \text{ mm}^2 \) at the bottom. Provide two No. 15 M bars at the top corners of the stirrups (400 mm\(^2\)) and two No. 15 M bars at each vertical face of the web.

(b) Compatibility tension, \( T_e = 7.3 \text{ kN-m} \) (Step 4)

From part (a), \( T_e \) value for tension to be neglected = 7.3 kN-m < 8.5 kN-m.

Hence disregard tension and provide stirrups for shear only.

From part (a), \( A_i = 0.26 \text{ mm}^2 / \text{mm}^2 / \text{two legs} \).

For No. 10 mm stirrups, \( r = 2000.26 = 770 \text{ mm} \).

Maximum \( r = 152 = 579.7 = 285 \text{ mm} \).

Use No. 10 M closed stirrups at 220 mm center to center at critical section.

(c) Compatiblity Tension, \( T_e = 30.5 \text{ kN-m} \)

\( T_e = 30.5 \times 5.5 \text{ kN-m} \) from part (a); hence, closed stirrups have to be provided. Since this is a compatibility tension, the section can be designed from Eq. 7.2 for

\[ T_e = \frac{\phi \sqrt{f_{cr} A_{cr}}}{3} \frac{f_y}{f_{cr}} = 4 \times 8.5 \text{ from part (a)} \]
\[ = 34 \text{ kN-m} > 30.5 \text{ kN-m} \]
Hence, use $T_s = 30.3 \text{ kN-m}$ for the torsional design of the section.

required $T_s = T_{cr} \cdot \frac{20.3}{0.75} = 40.4 \text{ kN-m}$

**Torsional reinforcement (Step 5)**

From part (a), $A_s = 122,000 \text{ mm}^2$, $P_s = 1616 \text{ mm}$. Using No. 10 M closed stirrups,

$$s = \frac{5 \times 100}{1.06} = 189 \text{ mm}, \text{ say 180 mm}$$

This is less than $l_p$, or 390 mm. Hence, use No. 10 M closed stirrups at 180 mm c-c (diameter of 11.3 mm) at the critical section.

$$A_s = \frac{A_t}{s} \frac{P_t}{P_s} f_s^2 f_y^2 = 40.4 \times 1616 = 646 \text{ mm}^2$$

$$A_{req} = \frac{5 \sqrt{f_y^2 A_s}}{f_s} \frac{A_t}{s} \frac{P_t}{P_s} f_y^2 = 1195 \text{ (from before)} - (0.14 \times 1616) = 549 \text{ mm}^2, \text{ controls}$$

Use $A_t = 646 \text{ mm}^2$.

**Distribution of tension longitudinal bars**

- torsional $A_t = 646 \text{ mm}^2$, $A_t' = 162 \text{ mm}^2$.

Using the same logic as that followed in part (a), provide at bottom an area $A_t = 2190 \times 162 = 2350 \text{ mm}^2$, that is, five No. 25 M bars ($A_t = 2500 \text{ mm}^2$) and two No. 15 M (400 mm$^2$) bars at top corners and each of the two vertical faces of the web.

**SELECTED REFERENCES**


7.17. American Concrete Institute, *Building Code Requirements for Concrete* (ACI 318-05) and Commentary (ACI 318R-05). American Concrete Institute, Farmington Hills, MI, 2005, 482 pp.

PROBLEMS FOR SOLUTION

7.1. Calculate the maximum allowable torsional capacity \( T_s \) for the sections shown in Figure 7.32 for compatibility torsion.

4\( \frac{3}{4} \) closed ties in all webs

\[ f_c' = 4000 \text{ psi} (27.6 \text{ MPa}), \text{ normal-weight concrete} \]

\[ f_y = f_p = 60,000 \text{ psi} (413.7 \text{ MPa}) \]

clear cover = 1.5 in.

\[ \text{Figure 7.32 Cross sections for Problem 7.1.} \]

7.2. A cantilever beam is subjected to a concentrated service live load of 26,000 lb (98 kN) acting at a distance of 3 ft 6 in. (1.07 m) from the wall support. In addition, the beam has to resist an equilibrium factored torsion \( T_t = 300,000 \text{ in.-lb} (33.9 \text{ kN-m}) \). The beam cross section is 12 in. x 25 in. (305 mm x 635 mm) with an effective depth of 22.5 in. (571.5 mm). Design the stirrups and the additional longitudinal steel needed.

Given:

\[ f_t = 3500 \text{ psi} \]

\[ f_y = f_p = 60,000 \text{ psi} \]

\[ A_t = 4.0 \text{ in.}^2 (2580.64 \text{ mm}^2) \]

7.3. The first interior span of a four-span continuous beam has a clear span \( L = 18 \text{ ft} (5.49 \text{ m}) \). The beam is subjected to a uniform external service dead load \( w_u = 1700 \text{ plf} (24 \text{ kN/m}) \) and a service live load \( w_l = 2200 \text{ plf} (32.1 \text{ kN/m}) \). Design the section for flexure, diagonal tension, and...
tension. Select the size and spacing of the closed stirrups and extra longitudinal steel that might be needed for tension. Assume that the beam width $b_w = 15$ in. (381.0 mm) and that redistribution of torsional stresses is possible such that the external (torque $T_e$) can be assumed as $664 V/\sqrt{A_t}$ [in. of lb]. Given:

\[ f_t = 5000 \text{ psi (34.5 MPa), normal-weight concrete} \]
\[ f_e = f_c = 6000 \text{ psi (415.7 MPa)} \]

7.4. A continuous beam has the shear and torsion envelopes shown in Figure 7.33. The beam dimensions are $b_w = 14$ in. (355.6 mm) and $d = 35$ in. (885 mm). It is subjected to factored shear forces $V_{e1} = 75,000$ lb (333.6 kN), $V_{e2} = 60,000$ lb, and $V_{e3} = 45,000$ lb. Design the beam for torsion and shear and detail the web reinforcement.

Given:

\[ f'_t = 4000 \text{ psi (27.6 MPa), lightweight concrete} \]
\[ f_e = f_c = 6000 \text{ psi (415.7 MPa)} \]

The required reinforcement is as follows:

- midspan $A_r = 3.0 \text{ in.}^2$
- support $A_r = 3.6 \text{ in.}^2$, $A_r' = 0.7 \text{ in.}^2$

![Figure 7.33](image)

**Figure 7.33** (a) Shear and (b) torsion envelopes.

7.5. Design the rectangular beam shown in Figure 7.34 for bending, shear, and torsion. Assume that the beam width $b = 12$ in. (305 mm). Given:

\[ f'_t = 4000 \text{ psi (27.6 MPa)} \]
\[ f_e = f_c = 6000 \text{ psi (415.8 MPa)} \]
7.6. An exterior spandrel beam $A_1-B_1$, part of the monolithic floor system shown in Figure 7.35, has a center-to-center span of 36 ft and a slab thickness $h = 6$ in. (153 mm) on beams 15 in. x 36 in. in cross section. It is subject to a service live load = 50 psf (2.4 kPa). Design the shear and torsion reinforcement necessary to resist the external factored loads. Given:

- $f'_c = 4000$ psi (27.6 MPa), normal-weight concrete
- $f_y = f'_c = 60,000$ psi (413 MPa)

Assume that the required flexural reinforcement for beam $A_1-B_1$ is

- midspan $A_s = 2.0$ in.$^2$
- support $A_s = 3.0$ in.$^2$, $A_t = 1.6$ in.$^2$
8

SERVICEABILITY OF BEAMS AND ONE-WAY SLABS

8.1 INTRODUCTION

Serviceability of a structure is determined by its deflection, cracking, extent of corrosion of its reinforcement, and surface deterioration of its concrete. Surface deterioration can be minimized by proper control of mixing, placing, and curing of the concrete. If the surface is exposed to potentially damaging chemicals, such as in a chemical factory or a sewage plant, a special type of cement with appropriate additives should be used in the concrete mix. Use of adequate cover is recommended in Chapters 4 and 5, proper quality control of the materials, and the application of proper crack control and deflection control criteria to the design can minimize and in most cases eliminate these problems.

This chapter deals with the evaluation of deflection and cracking behavior of beams and one-way slabs in some detail. It is intended to give the designer adequate basic background on the effect of cracking on the stiffness of the member, the short- and long-term deflection performance, and the manner in which the cracked concrete beam element can still perform adequately and economically without loss of reliability in its performance. Deflection of two-way action slabs and plates is given in Chapter 11 with numerical examples of deflection calculations for both short- and long-term loading.

Photo 8.1 Kennedy International airport TWA terminal, New York. (Courtesy of Ammann & Whitney)
8.2 SIGNIFICANCE OF DEFLECTION OBSERVATION

The working stress method of design and analysis used prior to the 1970s limited the stress in concrete to about 45% of its compressive strength and the stress in the steel to less than 50% of its yield strength. Elastic analysis was applied to the design of structural frames as well as reinforced concrete sections. The structural elements were proportioned to carry the highest service-level moment along the span of the member, with redistribution of moment effect often largely neglected. As a result, heavier sections with higher reserve strength resulted as compared to those obtained by the current ultimate strength approach.

Higher-strength concretes having \( f_c' \) values in excess of 12,000 psi (83 MPa) and higher-strength steels are being used in strength design, and expanding knowledge of the properties of the materials has resulted in lower values of load factors and reduced reserve strength. Hence more slender and efficient members are specified, with deflection becoming a more pronounced controlling criteria.

Beams and slabs are rarely built as isolated members, but a monolithic part of an integrated system. Excessive deflection of a floor slab may cause dislocations in the partitions it supports. Excessive deflection of a beam can damage a partition below, and excessive deflection of a lintel beam above a window opening could crack the glass panels. In the case of open floors and roofs such as top garage floors, ponding of water can result.

For these reasons, deflection control criteria are necessary, such as those given in Table 11.3.

8.3 DEFLECTION BEHAVIOR OF BEAMS

The load–deflection relationship of a reinforced concrete beam is basically trilinear, as idealized in Figure 8.1. It is composed of three regions prior to rupture:

- **Region I**: precracking stage, where a structural member is crack-free (Figure 8.2)
- **Region II**: postcracking stage, where the structural member develops acceptable controlled cracking both in distribution and width
- **Region III**: postservicability cracking stage, where the stress in the tension reinforcement reaches the limit state of yielding

![Figure 8.1](image)
8.3.1 Precracking Stage: Region I

The precracking segment of the load-deflection curve is essentially a straight line defining full elastic behavior. The maximum tensile stress in the beam in this region is less than its tensile strength in flexure, that is, less than the modulus of rupture \( f_c \) of concrete. The flexural stiffness \( EI \) of the beam can be estimated using Young’s modulus \( E \) of concrete and the moment of inertia of the uncracked reinforced concrete cross-section. The load-deflection behavior depends on the stress-strain relationship of the concrete as a significant factor. A typical stress-strain diagram for concrete is shown in Figure 8.3.

The value of \( E_c \) can be estimated using the ACI empirical expression given in Chapter 3:

\[
E_c = 33,000 \sqrt{f_c}
\]

or

\[
E_c = 57,000 \sqrt{f_c}
\]

for normal-weight concrete

An accurate estimation of the moment of inertia \( I \) necessitates consideration of the contribution of the steel reinforcement \( A_s \). This can be done by replacing the steel area by an equivalent concrete area \( (E_c/E) A_s \), since the value of Young's modulus \( E \) of the reinforcement is higher than \( E_c \). One can transform the steel area to an equivalent concrete area, calculate the center of gravity of the transformed section, and obtain the transformed moment of inertia \( I_e \).

Example 8.1 presents a typical calculation of \( I_e \) for a transformed rectangular section. Most designers, however, use a gross moment of inertia \( I_g \) based on the uncracked
concrete section, disregarding the additional stiffness contributed by the steel reinforcement as insignificant.

The precracking region stops at the initiation of the first flexural crack when the concrete stress reaches its modulus of rupture strength $f_r$. Similarly to the direct tensile splitting strength, the modulus of rupture of concrete is proportional to the square root of its compressive strength. For design purposes, the value of the modulus for normal-weight concrete may be taken as

$$ f_r = 7.5 \sqrt{f_c} \quad (8.1) $$

If lightweight concrete is used, the value of $f_r$ from Eq. 8.1 is multiplied by 0.75 for all lightweight concrete and by 0.85 for sand-lightweight concrete.

If the distance of the extreme tension fiber from the center of gravity of the section is $y$, and the cracking moment is $M_{cr}$,

$$ M_{cr} = \frac{b f_r y}{2} \quad (8.2) $$

For a rectangular section

$$ y = \frac{h}{2} \quad (8.3) $$

where $h$ is the total thickness of the beam. Equation 8.2 is derived from the classical bending equation $\sigma = M/C$ for elastic and homogeneous materials.

Calculations of deflection for this region are not important since very few reinforced concrete beams remain uncracked under actual loading. However, mathematical knowledge of the variation in stiffness properties is important since segments of the beam along the span in the actual structure can remain uncracked.

### 8.3.1 Example 8.1: Alternative Methods of Cracking Moment Evaluation

Calculate the cracking moment $M_{cr}$ for the beam cross-section shown in Figure 8.4, using both (a) transformed and (b) gross cross-section alternatives in the solution. Given:

- $f_c' = 4000$ psi (27.6 MPa)
- $f_y = 60,000$ psi (414 MPa)
- $E = 29 \times 10^6$ psi (200,000 MPa), normal-weight concrete

Reinforcement: four No. 9 bars (four bars, 28.6-mm diameter) placed in two bundles.

![Figure 8.4 Cross-section transformation in Ex. 8.1: (a) midspan section; (b) transformed section.](image)
Chapter 8  Serviceability of Beams and One-Way Slabs

Solution: (a) Transformed section solution: Depth of center-of-gravity axis, \( \bar{y} \), can be obtained using the first moment of area:

\[
\bar{y} = \frac{bh^2}{6} + \left( \frac{E_s}{E_c} - 1 \right) \frac{A_d}{A_r} \cdot \frac{bh}{2} + \left( \frac{E_s}{E_c} - 1 \right) \frac{A_d}{A_c} \delta
\]

Note that \( \left( \frac{E_s}{E_c} - 1 \right) \) is used instead of \( E_s/E_c \) to account for the concrete displaced by the reinforcing bars.

It is customary to denote \( n = E_s/E_c \) as the modular ratio. Taking moments about the top extreme fibers of the section,

\[
\bar{y} = \frac{bh^2/2}{bh + (n-1) A_d} = \frac{bh^2/2}{bh + (n-1) A_c}
\]

For normal-weight 4000 psi concrete,

\[
E_c = 57,000 \sqrt{4000}
\]

\[
= 3.6 \times 10^6 \text{ psi (24.8 kMPa)}
\]

\[
n = \frac{29 \times 10^6}{3.6 \times 10^6} = 8.1
\]

\[
\bar{y} = \frac{2}{12 \times 24 + (8.1 - 1)4.0 \times 21} = 12.8 \text{ in. (325 mm)}
\]

If the moment of inertia of steel reinforcement about its own axis is neglected as insignificant,

transformed section \( I_p = \frac{bh^4}{12} + bh(12.8 - 12.0)^2 + (n-1) A_d \delta - \bar{y}^2 \)

or

\[
I_p = \frac{12 \times 24^3}{12} + 12 \times 24 \times 0.8^2 + 7.1 \times 4.0(21 - 12.8)^2
\]

\[
= 15,918 \text{ in.}^4 (66.22 \text{ kN-m}^2)
\]

The distance of the center of gravity of the transformed section from the lower extreme fibers is

\[
y_c = 24 - 12.8 = 11.2 \text{ in. (285 mm)}
\]

\[
f_c = 7.5 \sqrt{4000} = 474.3 \text{ psi (3.27 MPa)}
\]

\[
M_{cr} = \frac{y_c f_c}{11.2} = 674,100 \text{ in.-lb (76.2 kN-m)}
\]

(b) Gross section solution

\[
\bar{y} = \frac{6}{2} = 3 \text{ in.}
\]

gross section \( I_g = \frac{bh^4}{12} + \frac{12 \times 24^3}{12} = 13,824 \text{ in.}^4
\]

\[
y_g = 12 \text{ in. (305 mm)}
\]

\[
f_g = 474 \text{ psi}
\]

\[
M_{cr} = 546,394 \text{ in.-lb (64,707 N-m)}
\]

There is a difference of about 15% in the value of \( f_c \) and 19% in the value of \( M_{cr} \). Even though this percentage difference in the values of \( M_{cr} \) is caused by the two methods of computation being different in the fact that \( f_c \) values in the


8.3 Deflection Behavior of Beams

significance and in most cases does not justify using the transformed-section method for evaluating \( M_c \).

8.3.2 Postcracking Service Load Stage: Region II

The precracking region ends at the initiation of the first crack and moves into region II of the load-deflection diagram in Figure 8.1. Most beams lie in this region at service loads. A beam undergoes varying degrees of cracking along the span corresponding to the stress and deflection levels at each section. Hence cracks are widest and deeper at midspan, whereas only narrow minor cracks develop near the supports in a simple beam.

When flexural cracking develops, the contribution of the concrete in the tension zone reduces substantially. Hence the flexural rigidity of the section is reduced, making the load-deflection curve less steep in this region than in the precracking stage segment. As the magnitude of cracking increases, stiffness continues to decrease, reaching a lower-bound value corresponding to the reduced moment of inertia of the cracked section. At this limit state of service load cracking, the contribution of tension-zone concrete to the stiffness is neglected. The moment of inertia of the cracked section designated as \( I_c \) can be calculated from the basic principles of mechanics.

Strain and stress distributions across the depth of a typical cracked rectangular concrete section are shown in Figure 8.5. The following assumptions are made with respect to deflection computation based on extensive testing verification: (1) the strain distribution across the depth is assumed to be linear; (2) concrete does not resist any tension; (3) both concrete and steel are within the elastic limit; and (4) strain distribution is similar to that assumed for strength design, but the magnitudes of strains, stresses, and stress distribution are different.

To calculate the moment of inertia, the value of the neutral axis depth, \( c \), should be determined from horizontal force equilibrium.

\[
A_c f_c = bc f_c \quad (8.4a)
\]

Since the steel stress \( f_c = E_s \varepsilon_s \), and concrete stress \( f_c = E_c \varepsilon_c \), Eq. 8.4a can be rewritten as

\[
A_c f_c = bc f_c = b c E_s \varepsilon_s = b c E_c \varepsilon_c
\]

Figure 8.5 Elastic strain and stress distributions across a cracked reinforced concrete section: (a) cross-section; (b) strain; (c) elastic stress and force; (d) cracked beam prior to failure in flexure.
\[ A, E, \sigma = \frac{bc}{2} E \varepsilon, \]  
(8.4b)

From similar triangles in Figure 8.3h,
\[ \frac{\varepsilon}{c} = \frac{\varepsilon}{d - c} \]  
(8.5a)
or
\[ \sigma = \varepsilon \left( \frac{d}{c} - 1 \right) \]  
(8.5b)

From Eqs. 8.4b and 8.5b,
\[ A, E, \sigma \left( \frac{d}{c} - 1 \right) = \frac{bc}{2} E \varepsilon \]  
(8.5a)
or
\[ \frac{A, E, \sigma}{E} \left( \frac{d}{c} - 1 \right) = \frac{bc}{2} \]  
(8.5b)

Replacing the modular ratio \( E/E \) by \( n \), Eq. 8.6b can be rewritten as
\[ \frac{bc}{2} + nA, E - nA, d = 0 \]  
(8.6c)

The value of \( c \) can be obtained by solving the quadratic equation, 8.6c. The moment of inertia \( I_c \) can be obtained from
\[ I_c = \frac{bc^3}{3} + nA, (d - c) \]  
(8.7)

where the term \( bc^3/3 \) in Eq. 8.7 denotes the moment of inertia of the compressive area about the neutral axis, that is, the base of the compression rectangle, neglecting the section area in tension below the neutral axis. The reinforcing area is multiplied by \( n \) to transform it to its equivalent in concrete for contribution to the section stiffness. The moment of inertia of the steel about its own axis is disregarded as negligible.

Only part of the beam cross-section is cracked in the case under discussion. As seen from Figure 8.5d, the uncracked segments below the neutral axis along the beam span possess some degree of stiffness, which contributes to the overall beam rigidity. The actual stiffness of the beam lies between \( E, I_c \) and \( E, I_s \), depending on such other factors as: (1) extent of cracking, (2) distribution of loading, and (3) contribution of the concrete, as seen in Figure 8.5d between the cracks. Generally, as the load approaches the steel yield load level, the stiffness value approaches \( E, I_s \).

Branson developed simplified expressions for calculating the effective stiffness \( E, I_e \) for design. The Branson equation, verified as applicable to most cases of reinforced and prestressed beams and universally adopted for deflection calculations, defines the effective moment of inertia as
\[ I_e = \left( \frac{M_{cty, s}}{M_{cty, c}} \right) I_c + \left[ 1 - \left( \frac{M_{cty, s}}{M_{cty, c}} \right) \right] I_s \approx I_s \]  
(8.8a)

Equation 8.8a is also written in the form
\[ I_e = I_c + \left( \frac{M_{cty, s}}{M_{cty, c}} \right) (I_s - I_c) \approx I_s \]  
(8.8b)

The effective moment of inertia \( I_e \) as shown in Eq. 8.8b depends on the maximum moment \( M_{cty, s} \) and the cracking moment \( M_{cty, c} \).
8.3 Deflection Behavior of Beams

8.3.2.1 Example 8.2: Effective Moment of Inertia of Cracked Beam Sections

Calculate the moment of inertia $I_e$ and $Z$ effective moment of inertia $I'$ of the beam cross-section in Ex. 8.1 if the external maximum service load moment is 2,000,000 in-lb (226 kN-m), given (Ex. 8.1):

$h = 12$ in. (305 mm)
$d = 21$ in. (533 mm)
$b = 24$ in. (600 mm);
$A_c = 4.0$ in.$^2$ (259 mm$^2$)
$f_c = 4000$ psi (27.6 MPa)
$f_y = 60,000$ psi (413.7 MPa)
$E_c = 28 	imes 10^6$ psi (200 x 10$^6$ MPa)
$E_y = 3.6 	imes 10^6$ psi (24.8 x 10$^6$ MPa)
$t = 8.1$

Solution: From Eq. 8.6a.

$$\frac{12c}{2} + 8.1 \times 4.0 - 8.1 \times 21 = 0$$

Hence neutral axis depth $c = 8.3$ in. (210.8 mm). From Eq. 8.7,

$$I_e = \frac{12 \times 8.3^2}{2} + 8.1 \times 4.0 \cdot (21.0 - 8.3)^3 = 7835 in.^4 (31.75 \times 10^6 mm^4)$$

Using the $I_c$ and $M_c$ values of Ex. 8.1, which include the effect of the transformed steel area,

$$I_e = 7513 \left( \frac{674.109}{3.000.000} \right) = (15.018 - 7133)$$

$$= 7835 in.^4 (32.59 \times 10^6 mm^4) < I_e \text{ as expected}$$

If the gross cross-section values for $I_c$ and $M_c$ are used without including the effect of transformed steel, the effective moments of inertia becomes

$$I_e = 7513 \left( \frac{546.394}{3.000.000} \right) (13.824 - 7133)$$

$$= 7642 in.^4 (31.79 \times 10^6 mm^4) < I_e$$

Comparison of the two values of effective $I_e$ calculated by the two methods (7835 in.$^4$ versus 7642 in.$^4$) shows a significant difference. Hence, use of the cross-section properties in Eq. 8.6a, with $A_c$, in more cases, adequate, particularly when one considers the variability in the loads and the randomness in the properties of concrete.

8.3.3 Postservicability Cracking Stage and Limit State of Deflection Behavior at Failure: Region III

The load-deflection diagram of Figure 8.1 is considerably flatter in region III than in the preceding regions. This is due to substantial loss in stiffness of the section because of extensive cracking and considerable widening of the stabilized cracks throughout the span. As the load continues to increase, the strain $\varepsilon$ in the steel bars at the tension side continues to increase beyond the yield strain $\varepsilon_y$ with no additional stress. The beam is considered at this stage to have structurally failed by initial yielding of the tension steel. It continues to deflect without additional loading, the cracks continue to open, and the neutral axis continues to rise toward the outer compression fibers. Finally, a secondary compression failure develops, leading to total crushing of the concrete in the maximum moment region, followed by rupture.

The increase in the beam load level between first yielding of the tension reinforcement in a simple beam and the rupture load level varies between 4% and .0%. The de-
flection value before rupture, however, can be several times that at the steel yield level, depending on the beam span/depth ratio, the steel percentage, the type of loading, and the degree of confinement of the beam section. An ultimate deflection value 8 to 12 times the first yield deflection has frequently been observed in tests.

Postyield deflection and limit deflection at failure are not of major significance in design and hence are not being discussed in any detail in this text. It is important, however, to recognize the reserve deflection capacity as a measure of ductility in structures in earthquake zones and in other areas where the probability of overloads is high.

8.4 LONG-TERM DEFLECTION

Time-dependent factors magnify the magnitude of deflection with time. Consequently, the design engineer has to evaluate immediate as well as long-term deflection in order to ensure that their values satisfy the minimum permissible criteria for the particular structure and its particular use.

Time-dependent effects are caused by the superimposed creep, shrinkage, and temperature strains. These additional strains induce a change in the distribution of stresses in the concrete and the steel, resulting in an increase in the curvature of the structural element for the same external load.

The calculation of creep and shrinkage strains at a given time is a complex process, as discussed in Chapter 5. One has to consider how these time-dependent concrete strains affect the stress in the steel and the curvature of the concrete element. In addition, consideration has to be given to the effect of progressive cracking on the change in stiffness factors, considerably complicating the analysis and design process. Consequently, an empirical approach to evaluate deflection under sustained loading is in many cases more practical.

The additional deflection under sustained loading and long-term shrinkage in accordance with the ACI procedure can be calculated using a multiplying factor:

\[
\lambda = \frac{T}{1 + 50p}
\]

(8.5)

![Photo 8.2 Deflected simply supported beam prior to failure. (Tests by Navy et al.)](image-url)
where \( p' \) is the compression reinforcement ratio calculated at midspan for simple and continuous beams and \( T \) is a factor that is taken as 1.0 for loading time duration of 3 months, 1.2 for 6 months, 1.4 for 12 months, and 2.0 for 5 years or more.

If the instantaneous deflection is \( \Delta_s \), the additional time-dependent deflection becomes \( \lambda \Delta_s \), and the total long-term deflection would be \( (1 + \lambda) \Delta_s \). Since live loads are not present at all times, only part of the live load in addition to the more permanent dead load is considered as the sustained load. Figure 8.6 gives the relationship between the load duration in months and the multiplier \( T \) in Eq. 8.6. It is seen from this plot that the maximum multiplier value \( T = 2.0 \) represents a nominal limiting time-dependent factor for 5 years' duration of loading. In effect, the expression for the long-term factor \( \lambda \) in Eq.
8.9 has similar characteristics as the stiffness EI of a section in that it is a function of the material property T and the section property (1 + 50%).

The total long-term deflection is

$$\Delta_{LT} = \Delta_{i} + \lambda_{w} \Delta_{s} + \lambda_{d} \Delta_{s}$$  \hspace{1cm} (8.10)

where

- $\Delta_{i}$ = initial live-load deflection
- $\Delta_{d}$ = initial dead-load deflection
- $\Delta_{s}$ = initial sustained live-load deflection (a percentage of the immediate $\Delta_{i}$ determined by expected duration of sustained load)
- $\lambda_{w}$ = time-dependent multiplier for infinite duration of sustained load
- $\lambda_{d}$ = time-dependent multiplier for limited load duration

The value of the multiplier $\lambda$ is the same for normal-weight or lightweight concrete.

### 8.5 Permissible Deflections in Beams and One-Way Slabs

Permissible deflections in a structural system are governed primarily by the amount that can be sustained by the interacting components of a structure without loss of esthetic appearance and without detriment to the deflecting member. The level of acceptability of deflection values is a function of such factors as the type of building, the use or purpose of partitions, the presence of plastered ceilings, or the sensitivity of equipment or vehicular systems that are being supported by the floor. Since deflection limitations have to be placed at service load levels, structures designed conservatively for low concrete and steel stresses would normally have no deflection problems. Present-day structures, however, are designed by ultimate load procedures efficiently utilizing high-strength concretes and steels. More slender members resulting from such designs would have to be better controlled for serviceability deflection performance, both immediate and long term.

#### 8.5.1 Empirical Method of Minimum Thickness Evaluation for Deflection Control

The ACI Code recommends in Table 8.1 minimum thickness for beams as a function of the span length, where no deflection computations are necessary if the member is not supporting or attached to construction likely to be damaged by large deflections. Other

| Table 8.1 Minimum Thickness of Beams and One-Way Slabs Unless Deflections are Computed
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Member</td>
<td>Simply Supported</td>
<td>One End Continuous</td>
<td>Both Ends Continuous</td>
</tr>
<tr>
<td>Solid one-way slabs</td>
<td>1/28</td>
<td>1/34</td>
<td>1/28</td>
</tr>
<tr>
<td>Beams or ribbed one-way slabs</td>
<td>1/6</td>
<td>1/8</td>
<td>1/10</td>
</tr>
</tbody>
</table>

*Cu” span length “L” in inches. Values given should be used directly for members with normal-weight concrete by = 14.5 p/2 and grade 60 reinforcement. For other conditions, the values should be multiplied as follows: (1) For structural lightweight concrete having unit weight in the range from 90 to 120 lb/ft³, the values should be multiplied by (1.65 - 0.005t). But not less than 1.0, where t is the unit weight in lb/ft³. (2) For floor slabs with deflection requirements in the range of 0.002”/L, the values can be multiplied by (1.5 - 0.003L).
8.6 Computation of Deflections

Table 8.2 Minimum Permissible Ratios of Span (l) to Deflection (Δ) (l = longer span)

<table>
<thead>
<tr>
<th>Type of Member</th>
<th>Deflection, Δ, to be Considered</th>
<th>(l/Δ)lim</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load L</td>
<td>180°</td>
</tr>
<tr>
<td>Floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load L</td>
<td>360°</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections</td>
<td>That part of total deflection occurring after attachment of nonstructural elements: sum of long-term deflection due to all sustained loads (cloud load plus any sustained portion of live load) and immediate deflection due to any additional live load</td>
<td>482°</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections</td>
<td></td>
<td>240°</td>
</tr>
</tbody>
</table>

*Limit not intended to safeguard against ponding. Ponding should be checked by suitable calculations of deflection, including added deflections due to ponded water, and considering long-term effects of all sustained loads, weight, construction tolerances, and reliability of provisions for drainage.

*Long-term deflection has to be determined but may be reduced by the amount of deflection calculated to occur before attachment of nonstructural elements. This reduction is made on the basis of accepted engineering data relating to deflection characteristics of members similar to those being considered.

*Ratio limit may be lower if adequate measures are taken to prevent damage in supported or attached elements, but should not be lower than those of nonstructural elements.

Deflections would have to be calculated and controlled as in Table 8.2. If the total beam thickness is less than required by the table, the designer should verify the deflection serviceability performance of the beam through detailed computations of the immediate and long-term deflections.

8.5.2 Permissible Limits of Calculated Deflection

The ACI Code requires that the calculated deflection for a beam or one-way slab has to satisfy the serviceability requirement of minimum permissible deflection for the various structural conditions listed in Table 8.2 of Section 8.5.1 if Table 8.1 is not used. However, long-term effects cause measurable increases in deflection with time and result sometimes in excessive overstress in the steel and concrete. Hence it is always advisable to calculate the total time-dependent deflection ΔT in Eq. 8.18 and design the beam size based on the permissible span/deflection ratios of Table 8.2.

8.6 COMPUTATION OF DEFLECTIONS

Deflection of structural members is a function of the span length, support, or end conditions, such as simple support or restraint due to continuity, the type of loading, such as concentrated or distributed load, and the flexural stiffness EI of the member.

The general expression for the maximum deflection Δmax in an elastic member can be expressed from the basis of mechanics as:

\[ Δ_{\text{max}} = K \frac{wL^4}{4EI} \]  

(8.11)
where \( W \) = total load on the span
\( l_c \) = clear span length
\( E \) = modulus of elasticity of concrete
\( J_c \) = moment of inertia of the section
\( K \) = a factor depending on the degree of fixity of the support

Equation (8.11) can also be written in terms of moment such that the deflection at any point in a beam is

\[
\Delta = k \frac{Ml^2}{EJ}
\]  

where \( k \) = a factor depending on support fixity and load conditions
\( M \) = moment acting on the section
\( l \) = effective moment of inertia

Table 8.3 gives the maximum elastic deflection values in terms of the gravity load for typical beams loaded with uniform or concentrated load.

### 8.6.1 Example 8.2: Deflection Behavior of a Uniformly Loaded Simple Span Beam

A simply supported uniformly loaded beam has a clear span \( l_c = 27.0 \) ft (8.23 m), a width \( b = 10 \) in. (254 mm), and a total depth \( h = 16 \) in. (406 mm), \( d = 12 \) in. (305 mm), and \( A_o = 1.32 \text{ in.}^2 \) (85 mm\(^2\)). It is subjected to a service dead-load moment \( M_d = 215,000 \) in.-lb (24.3 kN·m), and a service live-load moment \( M_l = 250,000 \) in.-lb (28.3 kN·m). Determine if the beam satisfies the various deflection criteria for short- and long-term loading. Assume that 90% of the live load is continuously applied for 24 months. Given:

\( f_c = 5000 \text{ psi (34.5 MPa)} \), normal-weight concrete

where \( w \) = unit weight of reinforced concrete = 150 lb/ft\(^2\)

\( f_l = 60,000 \text{ psi (413.7 MPa)} \)

**Solution:**

\[
E_c = 3300(1/\sqrt{5000}) = 4.29 \times 10^6 \text{ psi (29.700 MPa)}
\]

\[
E_l = 29 \times 10^6 \text{ psi (200,000 MPa)}
\]

modular ratio \( n = \frac{E_l}{E_c} = \frac{29 \times 10^6}{4.29 \times 10^6} = 6.75 \)

\[
l = 7.5 \sqrt{f_l} = 7.5 \sqrt{5000} = 530 \text{ psi (3.7 MPa)}
\]

**Minimum required depth**

From Table 8.1,

\[
h_{min} = \frac{l_c}{16} = \frac{27.0 \times 12}{16} = 17 \text{ in. (432 mm)} > \text{actual } h = 16 \text{ in.}
\]

Hence deflection calculations have to be made.

**Effective moment of inertia \( I_e \):**

\[
l_e = \frac{bh^3}{12} = \frac{10(16)^3}{12} = 5440 \text{ in.}^4
\]

\[
y_e = \frac{10h}{2} = 8.0 \text{ in.}
\]

\[
M_e = \frac{f_l}{n} = \frac{530 \times 3410}{6.75} = 275,000 \text{ in.-lb}
\]
Table 8.3 Maximum Deflection Expressions for Most Common Load and Support Conditions

<table>
<thead>
<tr>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_x$ at $x$ (center)</td>
<td>( \frac{x^2}{2} (L - x) )</td>
</tr>
<tr>
<td>$\Delta_{max}$ at center</td>
<td>( \frac{qx^4}{384EI} )</td>
</tr>
<tr>
<td>$\Delta_{max}$ at $x = L - \sqrt{\frac{h}{2}}$</td>
<td>( 0.0156 \Delta \frac{R^3}{E} )</td>
</tr>
<tr>
<td>$\Delta_x$ at point of load</td>
<td>( \frac{18k}{128EI} \left[ 15x^2 - 12x^2 + 71 \right] )</td>
</tr>
<tr>
<td>$\Delta_{max}$ when $x &lt; \frac{1}{2}$</td>
<td>( \frac{qR^2}{48EI} \left( 25 - 48x \right) )</td>
</tr>
<tr>
<td>$\Delta_x$ at point of load</td>
<td>( \frac{4R^2}{3EI} \left( 25 - 48x \right) )</td>
</tr>
<tr>
<td>$\Delta_{max}$ at point of load</td>
<td>( \frac{R}{2EI} \left( 15x^2 - 12x^2 + 71 \right) )</td>
</tr>
<tr>
<td>$\Delta_{max}$ when $x &gt; \frac{1}{2}$</td>
<td>( \frac{qR^2}{12EI} \left( 25 - 48x \right) )</td>
</tr>
<tr>
<td>$\Delta_x$ at point of load</td>
<td>( \frac{R}{2EI} \left( 15x^2 - 12x^2 + 71 \right) )</td>
</tr>
<tr>
<td>$\Delta_{max}$ when $x &gt; \frac{1}{2}$</td>
<td>( \frac{qR^2}{12EI} \left( 25 - 48x \right) )</td>
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<td>( \frac{qR^2}{12EI} \left( 25 - 48x \right) )</td>
</tr>
<tr>
<td>$\Delta_x$ at point of load</td>
<td>( \frac{R}{2EI} \left( 15x^2 - 12x^2 + 71 \right) )</td>
</tr>
</tbody>
</table>

(cont.)
Table 6.3 Continued

\begin{align*}
\Delta_m &= 0.440\text{ in} \\
\Delta &= 0.00055\text{ in}.
\end{align*}

Depth of neutral axis \(c\):

\begin{align*}
\frac{10c^2}{2} &= \kappa A_c (d - c) \\
10c^2 &= 6.76 \times 1.32 (13.0 - c), \text{ to get } c = 4.05\text{ in.}
\end{align*}

\begin{align*}
I_x &= \frac{10c^3}{3} + 6.76 \times 1.32 (13.0 - c)^2 \frac{(104.10)^2}{3} + 8.923(13.0 - 4.05)^2 \\
&= 941\text{ in.}^4
\end{align*}

Dead load

\begin{align*}
\frac{M_c}{M_x} &= \frac{225,900}{215,000} = 1.05 > 1.0
\end{align*}

Use \(M_c = M_x\) and \(I_x = I_c\) since the dead-load moment is smaller than the trucking moment (the beam will not creak at the dead-load level).

Dead load = 60% live load

\begin{align*}
\frac{M_{cx}}{M_x} &= \frac{225,900}{215,000 + 0.6 \times 250,000} = 0.62
\end{align*}

Dead load = live load

\begin{align*}
\frac{M_c}{M_x} &= \frac{225,900}{215,000 + 250,000} = 0.49 \\
I_x &= \left(\frac{M_{cx}^3}{M_x^3}\right) + \left(1 - \left(\frac{M_{cx}^3}{M_x^3}\right)\right) I_c
\end{align*}

Dead load

\begin{align*}
I_x &= 3410\text{ in.}^4
\end{align*}

Dead load = 0.6 live load

\begin{align*}
I_x &= 0.24 \times 3410 + 0.76 \times 941 = 1530\text{ in.}^4
\end{align*}

Dead load = live load

\begin{align*}
I_x &= 0.12 \times 3410 + 0.88 \times 941 = 1230\text{ in.}^4
\end{align*}

Short-term deflection

\begin{align*}
\Delta &= \frac{M_{cx}}{384EI} \times \frac{546}{48} \times \frac{527.0 \times 12}{10^6} = \frac{0.0025}{M_{cx}}\text{ in.}
\end{align*}
**Chapter 6: Serviceability of Beams and One-Way Slabs**

**Initial live-load deflection**

\[
\Delta_L = \frac{0.0025(13,000 + 230,000)}{1230} = \frac{0.0025(243000)}{3410}
\]

\[= 0.943 \times 0.8 \text{ in.}
\]

**Initial dead-load deflection**

\[
\Delta_d = \frac{0.0025 \times 215,000}{3410} = 0.16 \text{ in.}
\]

**Initial 50% sustained live-load deflection**

\[
\Delta_{sl} = 0.0025 \left( \frac{215,000 - 230,000 \times 0.6}{1530} \right)
\]

\[= 0.60 - 0.10 = 0.44 \text{ in.}
\]

**Long-term deflection**

From Eq. 8.10,

\[
\Delta_T = \Delta_L + \lambda \Delta_d + \lambda \Delta_{sl}
\]

\[
\lambda = \frac{T}{1 + 55p^*}
\]

where \(p^* = 0\) for singly reinforced beam

\(T\) for 5 years or more = 2.0 \(\lambda_n = \frac{2.0}{1 + 0} = 2.0\)

\(T\) for 24 months = 1.65 \(\lambda_n = \frac{1.65}{1} = 1.65\)

\[
\Delta_{sl} = 0.8 + 2.0 \times 0.16 + 1.65 \times 0.44 = 1.5 \text{ in.}
\]

**Deflection requirements (Table 8.2)**

\[
\frac{l}{180} = \frac{27 \times 12}{180} = 1.80 \text{ in.} \geq \Delta_L
\]

\[
\frac{l}{300} = 0.06 \text{ in.} \geq \Delta_I
\]

\[
\frac{l}{60} = 0.68 \text{ in.} < \Delta_{sl}
\]

\[
\frac{l}{240} = 1.35 \text{ in.} < \Delta_T
\]

Hence, the use of this beam is limited to floors or roofs not supporting or attached to non-structural elements such as partitions.

**8.7 DEFLECTION OF CONTINUOUS BEAMS**

As discussed in Chapter 5, a continuous reinforced concrete beam would have a flanged section at midspan, and sometimes a doubly reinforced section at the support if the reinforcement at the bottom fibers of the support section are adequately tied and anchored. Consequently, it is necessary to be able to find the effective moment of inertia \(I\) of T-sections and of doubly reinforced sections. A simple procedure is to use the weighted-average section properties as required by previous ACI code provisions.
1. Beams with both ends continuous:
\[
I_{e} = 0.70I_{c} + 0.15(I_{1e} + I_{2e})
\]  
(8.13)

2. Beams with one end continuous:
\[
I_{e} = 0.85I_{c} + 0.15I_{c}
\]
(8.14)

where  
- \(I_{c}\) = midspan section \(I\),
- \(I_{1e}, I_{2e}\) = \(I\), for the respective beam ends,
- \(I_{e}\) = \(I\), of continuous end.

It is seen from Eqs. 8.13 and 8.14 that the controlling moment of inertia for deflection evaluation is the midspan-section effective moment of inertia. Present code provisions permit using \(I_{e}\) of the midspan section as an approximation.

Moment envelopes have to be used to calculate the positive and negative values of \(I_{e}\). If the continuous beam is subjected to a single heavy concentrated load, only the midspan effective moment of inertia \(I_{e}\) is to be used.

### 8.7.1 Deflection of T Beams

The most common nonrectangular sections are the flanged T and L beams. The same principles used for deflection computations of rectangular sections can be applied to the nonrectangular ones. The contribution of the compressive resisting force can be obtained using the appropriate concrete area, as explained below.

As in the case of rectangular beams, the contribution of steel to the moment of inertia of the uncracked section is disregarded. The cross section of the beam in Figure 8.7a is divided into two areas for the purpose of calculating \(I_{e}\):

- The depth of center of gravity \(y\) is
\[
y = y + \frac{A_{1}y_{1} + A_{2}y_{2}}{A_{1} + A_{2}}
\]  
(8.15a)

- The gross moment of inertia, \(I_{e}\), for the two rectangles is
\[
I_{e} = \frac{bh_{1}^{3}}{12} + bh_{2}\left(y - \frac{h_{2}^{2}}{2}\right) + \frac{b_{1}(h - h_{1})^{3}}{12} + b_{2}(h - h_{2})\left(y - \frac{h - h_{2}^{2}}{2}\right)
\]  
(8.15b)

For the cracked section, the depth \(c\) of the neutral axis is calculated from the horizontal force equilibrium, as in Figure 8.7b and c. If the depth of neutral axis falls within the flange thickness, the beam behaves as a rectangular section having a width \(b\) of the flange and an effective depth \(d\).

---

**Figure 8.7** Stress and strain distribution across depth of flanged sections: (a) geometry; (b) strains; (c) stresses.
When the depth $c$ of neutral axis falls below the flange thickness $h_f$ the appropriate areas of concrete in the flange and the web of the section and corresponding stresses are applied in the calculation of the compression force. The average stress in the flange area $bh_f$ would be \( f_c + f_{c-h} \), where $f_{c-h}$ is the stress at the bottom of the flange. Using similar triangles yields

$$ f_{c-h} = f_c \frac{c - h_f}{c} \tag{8.17} $$

The average stress in compression for the web area, $b_h(c - h_f)$, would be $f_c/2$ based on the triangular distribution of stress. Hence the force equilibrium equation can be written as

$$ A_c f_c = b h_f \frac{f_c + f_{c-h}}{2} + b_h(c - h_f) \frac{f_c}{2} \tag{8.18a} $$

Photo 8.4 Deflected simply supported beam at failure. (Tests by Navy et al.)

Photo 8.5 Flexural stabilized cracks at failure. (Tests by Navy et al.)
Using Eqs. 8.17 and 8.18a,

$$2AE, s = bh, E, (\frac{c - h_1}{c}) + b_1(c - h_2)E, A \frac{c - h_1}{c}$$  \tag{8.18b}

Expressing $s$ in terms of $s_1$ and using modular ratio $\alpha$ gives us

$$2\alpha A, \frac{c - e}{c} = bh, 2c - h_1 \frac{c - h_1}{c} + b_1(c - h_2) \frac{c - h_1}{c}$$  \tag{8.18c}

or

$$b_1(c - h_1)^2 - 2nA, (d - c) + bh_1(2c - h_1) = 0$$  \tag{8.18d}

The quadratic equation 8.18d has to be solved to obtain $c$. Once $c$ is known, the moment of inertia $I, c$ of the cracked section can be calculated using the following expression:

$$I, c = \frac{1}{2} h, (c - h_1)^2 + \frac{1}{12} bh_1^2 + bh_1(c - h_1)^2 + nA, (d - c)^2$$  \tag{8.19}

The effective moment of inertia $I, c$ and deflection $\delta$ can be computed as in the case of rectangular sections using Eqs. 8.8a and 8.8b. In the case of L sections, expressions for $I, c$ such as those of Eq. 8.19 can be developed in a similar manner as for T sections.

### 8.7.2 Deflection of Beams with Compression Steel

Beams with compression reinforcement can be treated similarly to singly reinforced sections except that the contribution of the compression reinforcement to the stiffness of the beam should be considered because of its high stiffening effect. For the moment of inertia of the uncracked section, $I, c$ can be used with sufficient accuracy. The contribution of the compression steel $A, c$ to the cracked moment of inertia $I, c$ has to be included. Also, Eq. 8.6c has to be modified for calculating the neutral-axis depth $c$ of the beam. If the compressive force $A, f$, of the steel is added to the compressive force of the concrete, Eq. 8.6a as seen from Figure 8.8 becomes

$$A, f = bh, \frac{c - e}{2} - A, f', \frac{c - e'}{2} + A, f'$$  \tag{8.20a}

where $e'$ is the effective cover of compression reinforcement.

As in the case of singly reinforced concrete beams (Eqs. 8.4 to 8.6), Eq. 8.20a can be written in the form

$$\frac{bc^2}{2} + [(r - 1)A, c - nA, d - (n - 1)A, d'] = 0$$  \tag{8.20b}

![Figure 8.8 Stress and strain distribution at service load in doubly reinforced beam: (a) geometry; (b) strains; (c) stresses.](image)
The moment of inertia $I_o$ of the cracked section can therefore be expressed as

$$I_o = \frac{bc^2}{3} + nA_d(d - c)^2 + (n - 1)A_e(c - d)^2$$  \hspace{1cm} (8.21)

The procedure for calculating the effective moment of inertia $I_o$ and the deflection $\Delta$ is the same as in the case of singly reinforced beams.

### 8.7.3 Bending Moment Deflections in Continuous Beams

The flexural moment envelope has to be constructed for the total continuous beam span in order to evaluate the effective moment of inertia $I_o$. The usual methods of structural analysis are followed in finding the continuity moments at supports and the positive midspan moments for the various spans. Once these moments are determined, the immediate central postelastic (i.e., postcracking) deflection can be evaluated.

As in the case of simply supported beams, the deflection $\Delta$ can be written either in terms of load as in Eq. 8.11 or in terms of moment as in Eq. 8.12. If an interior span $AB$ subjected to a uniform load is isolated as in Figure 8.9, the midspan deflection $\Delta$ is

$$\Delta = \frac{M_o(b/4)(l/4)^2}{4EI} + 3M_{o1}(l/4)^2 + 3M_{o2}(l/4)^2$$

$$+ \frac{M_{o1} + M_{o2}}{4EI}$$

where $M_o$ is positive and $M_{o1}$ and $M_{o2}$ generally negative.

---

**Figure 8.9** Bending moment deflections in continuous beams: (a) loads; (b) moment envelope; (c) deflections.
8.7 Deflection of Continuous Beams

\[ \lambda = \frac{12^2}{48EI} [M_s + 0.5(M_s + M_e)] \]  

(8.22)

where \( M_s \) and \( M_e \) = negative service load bending moments

\( M_s \) = simple span service load static moment

\( M_e \) = midspan moment

Use the correct algebraic sign for the moments in Eq. 8.22, with \( M_e \) and \( M_s \) due to the same loading generally negative. As the exterior span is subjected to the largest positive and negative moments, deflection calculations control for this span in most cases.

8.7.4 Example 8.4: Deflection of a Continuous Four-span Beam

A reinforced concrete beam supporting a 4-in. (100-mm) slab is continuous over four equal spans \( L = 26 \text{ ft} \) (11 m) as shown in Figure 8.10. It is subjected to a uniformly distributed load \( w = 700 \text{ lb/ft} \) (10.22 kN/m), including its self-weight and a service live load \( w_s = 1200 \text{ lb/ft} \) (17.32 kN/m). The beam has the dimensions \( b = 14 \text{ in.} \) (356 mm), \( d = 18.25 \text{ in.} \) (464 mm) at midspan, and a total thickness \( h = 21.9 \text{ in.} \) (553 mm). The first-freedom span is reinforced with four No. 9 bars at midspan (28.6 mm diameter) at the bottom fibers and six No. 3 bars at the top fibers of the support sections.

Calculate the maximum deflection of the continuous beam and determine what code deflection criteria it meets and what limitations, if any, have to be placed on its use. Given:

- Beam properties
- Load conditions
- Reinforcement details

![Diagram of the four-span beam](image)

Figure 8.10 Details of continuous beam in Ex. 8.4: (a) beam elevation; (b) section 1-1; (c) section 2-2.
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\( f' = 4000 \text{ psi (27.6 MPa), normal-weight concrete} \)

\( f_c = 60,000 \text{ psi (414 MPa)} \)

50% of the live load is sustained 36 months on the structure.

Solution:

**Minimum depth requirement**

From Table 8.1,

\[
\text{minimum } h = \frac{f_c}{k} = \frac{36.5 \times 12}{18.5} = 23.35 \text{ in.}
\]

actual \( h = 31.0 \text{ in.} < 23.35 \text{ in.} \)

Deflection calculations have to be made.

**Material properties and bending moment envelope**

\[ E_c = 57,000 \sqrt{f_c} = 57,000 \sqrt{4000} = 3.6 \times 10^6 \text{ psi (24.922 MPa)} \]

\[ E_i = 29 \times 10^6 \text{ psi (200,000 MPa)} \]

modular ratio \( k = \frac{E_c}{E_i} = \frac{29 \times 10^6}{3.6 \times 10^6} = 8.1 \)

modulus of rupture \( f'_c = 7.5 \sqrt{f_c} = 7.5 \sqrt{4000} = 474.3 \text{ psi (3.27 MPa)} \)

From bending moment analysis, the bending moment diagram for the beam is shown in Figure 8.11. For deflection, the largest moments are in spans AB and ED.

Positive moment:

\[
M_{p} = 0.0772 \times 700(36.0)^2 \times 12 = 840,430 \text{ in.-lb}
\]

\[
M_{n} = 0.0772 \times 1200(36.0)^2 \times 12 = 1,440,737 \text{ in.-lb}
\]

\[
(M_p + M_n) = 0.0772 \times 700(36.0)^2 \times 12 = 2,351,167 \text{ in.-lb}
\]

Negative moment:

\[
M_{n} = 0.1071 \times 700(36.0)^2 \times 12 = 1,165,921 \text{ in.-lb}
\]

\[
M_{p} = 0.1071 \times 1200(36.0)^2 \times 12 = 1,998,743 \text{ in.-lb}
\]

\[
(M_n + M_p) = 0.1071 \times 700(36.0)^2 \times 12 = 5,314,856 \text{ in.-lb}
\]

**Effective moment of inertia \( I \)**

Figure 8.12 shows the theoretical midspan and support cross-sections to be used for calculations of the gross moment of inertia \( I \).

\[
I_{gross} = 0.066 \text{ in.}^3 (10 \text{ in. span})
\]

Figure 8.11  Bending moment envelope.
6.7 Deflection of Continuous Beams

![Diagram](image)

**Figure 8.12:** Gross moment of inertia $I_x$ cross sections in Ex. 8.4: (a) midspan section; (b) support section.

1. Midspan section

Width of T-beam flange $= b_x = 100\text{ in.} = 1.0\text{ m}$

Depth from compression flange to the elastic centroid from Eq. 8.15c:

$$y = \frac{A_x \cdot r_1}{A_1 + A_2}$$

$$= \frac{78(4 \times 2) + 14 \times (21 - 4) \times 12.5}{78 \times 4 + 14 \times 17} = 6.54\text{ in.}$$

$$y_1 = h - y = 21.0 - 6.54 = 14.46\text{ in.}$$

From Eq. 8.16,

$$I_x = \frac{78(4)^2}{12} + 78 \times 4 \left(6.54 - \frac{4^2}{4}\right)^2 + \frac{14(21 - 4)^2}{12} + 14(21 - 4) \left(14.46 - \frac{21 - 4^2}{2}\right)^2$$

$$= 21,033\text{ in.}^4$$

$$M_{o} = \frac{M_{o}}{y_1} = \frac{424.3 \times 21,033}{14.46} = 609,900\text{ in.-lb}$$

**Depth of neutral axis from Eq. 8.16:**

$$A_n = \text{lent No. 9 bars } = 4.8\text{ in.}^2$$

From Eq. 8.16,

$$14(c - 4.8)^2 = 2 \times 8.1 \times 14.0(18.25 - c) + 78 \times 4(2c - 4.8) = 0$$
or \( c^2 + 84.17c - 157.0 = 0 \) to give \( c = 3.5 \) in. Hence the neutral axis is inside the flange and the section is analyzed as a rectangular section.

From Eq. 8.6c for rectangular sections:

\[
\frac{76c}{2} = 8.1 \times 4 \times c - 8.1 \times 4 \times 3.25 - 0
\]

Thus, \( c = 3.5 \) in.

\[
I_w = \frac{76(8.35)^3}{3} = 8.1 \times 4(8.35 - 2.5)^2 = 816.3 \text{ in.}^4
\]

Ratio \( M_L/M_w \):

\[
D \text{ ratio} = \frac{689,900}{840,430} = 0.821
\]

\[
D + 50\% \text{ L ratio} = \frac{689,900 - 0.5 \times 1,446,737}{2,811,767} = 0.442
\]

\[
D = L \text{ ratio} = \frac{689,900}{2,811,767} = 0.302
\]

Effective moment of inertia for midspan section

\[
I_e = \left( \frac{M_w}{M_L} \right) I_w + \left[ 1 - \left( \frac{M_w}{M_L} \right) \right] I_w
\]

\( I_w \) for dead load = 0.553 \( \times \) 21.033 \( + \) 0.466 \( \times \) 816.3 \( = \) 15.286 in.\(^4\)

\( I_w \) for \( D + 0.5L \) = 0.055 \( \times \) 21.033 \( + \) 0.026 \( \times \) 816.3 \( = \) 0.926 in.\(^4\)

\( I_w \) for \( D - L \) = 0.025 \( \times \) 21.033 \( - \) 0.9725 \( \times \) 816.3 \( = \) 851.9 in.\(^4\)

2. Support section

\[
I_s = \frac{63^3}{12} = 1421.5 \text{ in.}^4
\]

\[
x_s = \frac{21.0}{2} = 10.5 \text{ in.}
\]

\[
M_s = \frac{254.3 \times 19,804.5}{18.5} = 488,065 \text{ in.-lb}
\]

Depth of neutral axis

\[
A_e = \text{six No. 6} = 6.9 \text{ in.}^2 (3670 \text{ mm}^2)
\]

\[
A_{e1} = \text{two No. 9} = 10.6 \text{ in.}^2 (1290 \text{ mm}^2)
\]

\[
d = 21.0 - 3.75 = 17.25 \text{ in. (438.2 mm)}
\]

From Eq. 8.29b,

\[
\frac{14c^3}{2} = [8.1 \times 6.0 + (8.1 - 1)2.0]c - 8.1 \times 6.0 \times 17.25 - (8.1 - 1) \times 2.0 \times 3.75 = 0
\]

or \( c^2 + 89.7c - 125.34 = 0 \). Hence \( c = 7.58 \) in.

From Eq. 8.21, the torsion moment of inertia is

\[
I_e = \frac{63^3}{3} + ud^3(d + c)^2 - (u - 1)4d^3(c - d)^2
\]

\[
= \frac{1475.88}{3} + 8.1 \times 6.0(17.25 - 7.58)^2 - (8.1 - 1)2.0(7.58 - 3.75)^2
\]

\[= 6968.2 \text{ in.}^3\]
Ratio $M_e/M_s$

\[ D \text{ ratio} = \frac{480.055}{1.185.953} = 0.41 \]

\[ D = 50\% 
\]

\[ L = \frac{480.055}{1.185.953 + 0.5 \times 1.996.743} = 0.215 \]

\[ D = L = \frac{480.055}{3.196.676} = 0.15 \]

Effective moment of inertia for support section

\[ I_e \text{ for dead load} = 0.0741 \times 10.804.5 + 0.0259 \times 6908.2 = 7196.9 \text{ in}^4 \]

\[ I_e \text{ for } D = 0.5 I = 0.0122 \times 10.804.5 + 0.0078 \times 6908.2 = 6955.7 \text{ in}^4 \]

\[ I_e \text{ for } D = L = 0.0034 \times 10.804.5 + 0.0066 \times 6908.2 = 6921.6 \text{ in}^4 \]

Average effective $I_e$ for continuous span

From Eq. 8.14,

\[ \text{average } I_e = 0.85 I_e + 0.15 I_{rel} \]

\[ 
\]

\[ \text{dead load: } I_e = 0.85 \times 15.286 + 0.15 \times 7196.9 = 14.073 \text{ in}^4 \]

\[ D = 0.5 I_e: I_e = 0.85 \times 9.276 + 0.15 \times 6955.7 = 8928 \text{ in}^4 \]

\[ D = L: I_e = 0.85 \times 8.518 + 0.15 \times 6921.6 = 8278 \text{ in}^4 \]

Short-term deflection

From Table 8.3, the maximum deflection for span $AB$ or $DE$ is

\[ \Delta = \frac{0.0065 w L^4}{E I} \]

\[ \text{assumed } I_e \text{ for all practical purposes} \]

\[ \Delta = \frac{0.0065 \times 560 \times 12^4}{3.6 \times 10^6} \times \frac{1}{12} = 5.240 \text{ in} \]

(A more accurate result can be obtained from Eq. 8.2.)

Initial live-load deflection

\[ \Delta_L = \Delta_{L,1} - \Delta_{L,0} \]

\[ \Delta_L = \frac{5.240(1000)}{14.073} = 1.20 - 0.26 = 0.94 \text{ in} \]

Initial dead-load deflection

\[ \Delta_D = \frac{5.240(700)}{14.073} = 0.26 \text{ in} \]

Initial 50% sustained live-load deflection

\[ \Delta_{1/2} = \frac{5.240(1100)}{8928} = \frac{5.240(700)}{14.073} = 0.76 - 0.25 = 0.50 \text{ in} \]

Long-term deflection

\[ \frac{\nu}{L} = \frac{\Delta}{L} = 0 \text{ at middle in this case} \]
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From Eq. 8.9,

multiplier \( \lambda = \frac{T}{1 + 50p'} \)

From Figure 8.6,

\( T = 1.75 \) for 36-month sustained load
\( T = 2.0 \) for 5-year loading

Therefore,

\( \lambda_s = 2.0 \) and \( \lambda = 1.75 \)

From Eq. 8.10, the total sustained load deflection is

\( \Delta_{sf} = \Delta_1 + \lambda_s \Delta_3 + \lambda \Delta_{sf} \)

or

\( \Delta_{sf} = 0.04 \times 2.0 \times 0.26 + 1.75 \times 0.50 = 2.35 \text{ in. (60 mm)} \)

Deflection requirements (Item 4.2)

\( \frac{I}{L} = \frac{16 \times 12}{180} = 2.4 \text{ in.} > \Delta_t \)

\( \frac{I}{h} = 1.2 \text{ in.} > \Delta_t \)

\( \frac{I}{h} = 0.9 \text{ in.} < \Delta_{st} \)

\( \frac{I}{h} = 1.8 \text{ in.} < \Delta_{st} \)

Hence the continuous beam is limited to floors or rood not supporting or attached to non-structural elements such as partitions.

8.8 OPERATIONAL DEFLECTION CALCULATION PROCEDURE
AND FLOWCHART

Deflection of structures affects their esthetic appearance as well as their long-term serviceability. The following step-by-step procedure should be followed after the structural member is designed for flexure.

1. Compare the total design depth of the member with the minimum allowable value obtained from Table 8.1. If it is less than the allowable, proceed to perform a detailed calculation of short- and long-term deflection. It is, however, advisable to perform the detailed calculations regardless of the comparison with Table 8.1.

2. The detailed calculations should establish as a first step:
   (a) The gross moment of inertia \( I \)
   (b) The cracking moment \( M'_{cr} \), which is a function of the modulus of rupture of concrete

3. Calculate the depth \( c \) of the neutral axis of the transformed section. Find the cracking moment of inertia \( I_{cr} \).

4. Find the effective moment of inertia \( I_e \) as follows:

\[ I_e = \left( \frac{M_{cr}}{M} \right)^2 I + \left[ 1 - \left( \frac{M_{cr}}{M} \right)^2 \right] I_{cr} = I_s \]
8.9 Deflection Control in One-Way Slabs

\[ l_e = l_e + \left( \frac{M_{eq}}{M_{eq}} \right) (l_e - l_e) = l_e \]

The effective \( l_e \) has to be calculated for the following service load-level combinations:
(a) Dead load (D)
(b) Dead load + sustained portion of live load (\( D + \alpha L \), where \( \alpha \) is less than 1.5)
(c) Dead load + live load (\( D + L \))

5. Calculate the immediate deflection based on \( f_e \) of the three combinations in step 4, using the elastic deflection expression in Table 8.3. If the beam is continuous over more than two supports, find the average \( f_e \) as follows:
   - Both end continuous: average \( f_e = 0.70l_e + 0.15l_1 + 0.15l_2 \)
   - One end continuous: average \( f_e = 0.85l_e + 0.15l_1 \)

6. Calculate the long-term deflection, finding first the time-dependent multiplier \( \lambda = f_t/(1 + 50\gamma) \) from values in Figure 8.6. The total long-term deflection is
   \[ \Delta_{LT} = \Delta_1 = \lambda_1 \Delta_0 + \lambda_1 \Delta_0 \]

7. If \( \Delta_{LT} \) is maximum permissible \( \Delta \) in Table 8.2, limit the use of the structure to particular loading types or conditions, or replace the section. Figure 8.12 gives a flowchart of the operational sequence of deflection control checks that the designer engineer should use when deflection computations are necessary.

8.9 DEFLECTION CONTROL IN ONE-WAY SLABS

One-way slabs can be treated as rectangular beams of 12-in. (304.8-mm) width. Because floor loads are specified as load intensity per unit area, such intensity on a one-way slab over a 1-ft width becomes pounds per linear foot. Reinforcement is chosen in terms of bar spacing instead of number of bars, and the area of steel for a 12-in. width of slab can be easily calculated for the total number of bars in a 12-in.-wide strip.

8.9.1 Example 8.8: Deflection Calculations for a Simply Supported One-way Slab

A 5-in.-thick \( (h = 127 \text{ mm}) \) one-way slab has a span of 12 ft \( (3.66 \text{ m}) \). It is subjected to a live load of 60 psf \( (2.88 \text{ kPa}) \) in addition to its self-weight. Calculate the immediate and long-term deflections of this slab, assuming that 45% of the live load is sustained over a 24-month period. Determine what type of elements is should support. Given:
   \[ f_y = 3500 \text{ psi (24.1 MPa)} \]
   \[ f_y = 60,000 \text{ psi (414 MPa)} \]
   \[ E_y = 29 \times 10^6 \text{ psi (200,000 MPa)} \]

Steel reinforcement: No. 4 bars at 6 in. center-to-center spacing \( (22.7 \text{ mm diameter at } 152 \text{ mm center to center}) \)

Solution:

Minimum depth requirement

From Table 8.1,

\[ \frac{l}{h} = \frac{12 \times 12}{20} = 7.20 \text{ in.} \]
\[ \text{actual } h = 5 \text{ in.} < 7.20 \text{ in.} \]
Figure 8.13 Deflection evaluation flowchart.
0.9 Deflection Control in One-Way Slabs

Deflection calculations have to be made.

**Material properties and bending moments**

\[ E = 57,000 \sqrt{f_f} = 57,000 \sqrt{3500} = 3.37 \times 10^6 \text{ psi (23.256 MPa)} \]

\[ E = 29 \times 10^6 \text{ psi (200,000 MPa)} \]

modulus ratio \( \mu \) = \( \frac{E}{E' \phi} \) = \( 29 \times 10^6 \) / \( 3.37 \times 10^6 \) = 8.61

modulus of rupture \( f_r = 7.5 \sqrt{f_f} = 7.5 \sqrt{3500} = 443.9 \text{ psi} \)

gross moment of inertia \( I_g = \frac{bh^3}{12} = \frac{125.0^3}{12} = 125.0 \text{ in.}^4 \)

cracking moment \( M_c = \frac{f_r b h^2}{2.5} = \frac{443.7 \times 125.0}{2.5} = 22,185 \text{ in.-lb} \)

service load bending moment \( = \frac{w h^2}{8} = \frac{w(12.0)^2}{8} \times 12 \text{ in.-lb} = 216w \text{ in.-lb} \)

Neutral-axis depth of transformed section

If \( c \) is the depth from the compression fibers to the neutral axis of the transformed section,

\[ A' = \text{No. 4 at 6 in.} = 0.40 \text{ in.}^3 \text{ per 12-in.-wide strip} \]

\[ d = h - 0.75 - \frac{h}{2} = 5.0 - 0.75 - 2.0 = 4.0 \text{ in.} \]

From Eq. 8.6(c) for rectangular sections,

\[ \frac{bc^2}{2} + nA_c - nA_d = 0 \]

\[ 12c^2 + 8.61 \times 0.4c - 8.61 \times 0.40 \times 4.0 = 0 \]

or \( c = 0.574 \) or \( 2.296e = 0 \), giving \( c = 1.255 \text{ in.} \)

**Effective moment of inertia**

**Dead load**

\[ w_{dl} = \text{weight of slab} = \frac{5}{12} \times 150 \text{pcf} = 62.5 \text{ pcf} \]

\[ M_d = 216w = 216 \times 62.5 = 13,500 \text{ in.-lb} < M_c \]

Hence the slab will not crack under dead load and \( I_d = I_c = 125.0 \text{ in.}^4 \)

**45\% live load**

\[ M_d = 216(62.5 + 0.45 \times 60) = 19,322 \text{ in.-lb} < M_c \]

Hence the slab will not crack under dead load and 45\% sustained live load and \( I_d = I_c = 125.0 \text{ in.}^4 \)

**Dead + live load**

\[ M_d = 216(62.5 + 60.0) = 26,460 \text{ in.-lb} > M_c \]

This section is cracked.

\[ I_c = \frac{bc^2}{3} + nA_c(d - c) \text{ from Eq. 8.7} \]
\[ L_s = \frac{12(1.255)^3}{3} = 6.61 \times 0.40(4.0 - 1.255)^2 = 33.80 \text{ in.}^2 \]

\[ \frac{M_s}{M_e} = \frac{22.183}{26.460} = 0.841 \]

\[ I_s = \left( \frac{M_s}{M_e} \right) I_e + \left[ 1 - \left( \frac{M_s}{M_e} \right) \right] I_e = 0.841 \times 125.0 + 0.16 \times 33.80 = 87.63 \text{ in.}^4 \]

Short-term deflection

From Table 8.3,

\[ a = \frac{5w_{pl}}{384E I_s} = \frac{5w(12.0 \times 12)^2}{384 \times 3.37 \times 10^6} \times 12 = 0.1384 \]

Initial live-load deflection

\[ \Delta_0 = \frac{0.1384(62.5 + 66.0)}{87.63} = 0.1384(125.0) = 0.194 - 0.069 = 0.125 \text{ in. (32 mm)} \]

Initial dead-load deflection

\[ \Delta_0 = \frac{0.1384(62.5)}{125.0} = 0.065 \text{ in. (1.8 mm)} \]

Initial 41% sustained L.L. deflection

\[ \Delta_0 = \frac{0.1384(63.5 + 0.45 \times 60)}{125.0} = \frac{0.1184(63.5)}{125.0} \]

\[ = 0.099 - 0.069 = 0.030 \text{ in. (0.8 mm)} \]

Long-term deflection

From Eq. 8.9, multiplier \( \lambda = 7(1 + 0.06) \), from Figure 8.t, \( \beta = 1.65 \) for 24-month sustained load Therefore,

\[ \lambda_{L_s} = 2.0 \text{ and } \lambda_{L_T} = 1.65 \]

From Eq. 8.10, the total sustained load deflection \( \Delta_{L} \)

\[ \Delta_{L} = \Delta_{0} + \lambda_{L_s} \Delta_{s} + \lambda_{L_T} \Delta_{T} \]

or

\[ \Delta_{L} = 0.125 + 2.0 \times 0.069 + 1.65 \times 0.030 = 0.313 \text{ in. (8 mm)} \]

Deflection requirements (Table 8.2)

\[ \frac{1}{180} = 12 \times 12 = 0.83 \text{ in.} > \Delta_i \]

\[ \frac{1}{360} = 0.40 \text{ in.} > \Delta_i \]

\[ \frac{1}{480} = 0.30 \text{ in.} = \Delta_{LT} \]

\[ \frac{1}{240} = 0.60 \text{ in.} > \Delta_{LT} \]

Therefore, the slab can support sensitive attached structural elements that are otherwise damaged by large deflections.
8.10 Flexural Cracking in Beams and One-Way Slabs

8.19 FLEXURAL CRACKING IN BEAMS AND ONE-WAY SLABS

8.10.1 Fundamental Behavior

Concrete cracks at an early stage of its loading history because it is weak in tension. Consequently, it is necessary to study its cracking behavior and control the width of the flexural cracks. Cracking contributes to the corrosion of the reinforcement, surface deterioration, and its long-term decorative effects.

Increased use of high-strength reinforcing steels having 60,000- to 100,000 psi (413.7- to 689.5-MPa) yield strength and with high stresses occurring at low load levels is becoming prevalent. Also, higher-strength concretes in excess of 5000- to 20,000 psi strength in compression (35 to 138 MPa) and optimal utilization of the material in the strength theories of analysis and design are possible today. Hence prediction and control of cracking and crack widths are essential for reliable serviceability performance under long-term loading.

Two types of stresses act on the tensile-stretched zones of the concrete surrounding the tension reinforcement shown in Figure 8.14a. They are longitudinal and lateral sets of stress. As the longitudinal bending stress acts, the tensile zone undergoes a lateral contraction before cracking, resulting in lateral compression between the concrete and the reinforcing bars or wires. As the moment that a flexural crack starts to develop, this biaxial lateral compression has to disappear at the crack because the longitudinal tension in the concrete becomes zero at the crack location.

The longitudinal bond stresses gradually reach their peak at the crack. This causes the tensile stress to localize at the concrete at that location suddenly to reach its maximum value. The concrete can no longer withstand any tension because of the high stress concentration at the moment of incipient fracture; it splits, as seen in Figure 8.14a.

The stress in the concrete is dynamically transferred to the reinforcing steel (Figure 8.14a). As the stress is transferred, the tensile stress in the concrete at the cracked section is relieved, becoming zero at the crack (Figure 8.14c). Laterally, the neutral axis position rises at the cracked section in order to maintain equilibrium at that section.

The distance between two adjacent cracks is the stabilized crack spacing, that is, the distance between two cracks when they continue to widen under load as principal cracks slide. When previously formed cracks tend to close due to redistribution of stress. In other words, cracks stabilize when no new cracks form in the structural member. A schematic plot of the crack width versus crack spacing is given in Figure 8.14a. It illustrates in the almost horizontal plateau of the diagram the load at which the crack spacing becomes stabilized.

The width of each of the two cracks would essentially be a function of the difference in elongation between the reinforcing bars and the surrounding unstressed concrete over a length a. From a practical viewpoint, the elongation of the concrete and the shrinkage strain can be neglected as insignificant. Hence

\[
\text{crack width } w = \left( \frac{a}{2} \right) \alpha \sigma
\]

The value of \( \alpha \) partly depends on whether the reinforced concrete member is one- or two-dimensional, while \( \alpha \) and \( \beta \) are experimental nonlinearity constants.

It has been proven that \( a \) varies with \( (1+\mu)^3 \), \( k_f^c \), and \( d_p/d_h \), where \( \mu \) is the bond stress, \( f_c \) is the tensile strength of the concrete, \( d_p \) is the diameter of the steel bar,
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Figure 8.14  Longitudinal stress distribution between two adjacent cracks when cracks are fully developed: (a) crack opening geometry; (b) ultimate bond stress $\mu$; (c) longitudinal tensile stress $f_t$ in the concrete; (d) longitudinal tensile stress $f_t$ in the steel.

Figure 8.15  Schematic variation of crack width versus crack spacing.
8.10  Flexural Cracking in Beams and One-Way Slabs

\( n = A_s/A_c \) is the ratio of the steel area at the tension side of the section, and \( A_c \) is the area of concrete in tension; \( k_1, k_2, \) and \( k_3 \) are constants.

8.10.2 Crack-width Evaluation

While Eq. 8.23 is the basic mathematical model for the evaluation of the maximum crack width, the large number of variables involved, the randomness of cracking behavior, and the large degree of scatter require extensive idealization and simplification. One simplification based on a statistical study of test data of several investigators is the Gergely-Lutz expression:

\[
\begin{align*}
\varepsilon_{\text{max}} &= 0.076\sqrt{f_e} \sqrt{d/A} \\
\end{align*}
\]

where \( \varepsilon_{\text{max}} \) = crack width in units of 0.001 in. (0.0254 mm)
\( \beta = (k - c)/(d - c) \) = depth factor; average value = 1.20
\( d_c \) = thickness of cover to the center of the first layer of bars (in.)
\( f_e \) = maximum stress (ksi) in the steel at service load level with 0.6\( f_y \) to be used if no computations are available
\( A \) = area of concrete in tension divided by the number of bars (in.\(^2\)) = \( b t \gamma_{oc} \)
where \( \gamma_{oc} \) is defined as the number of bars at the tension side

Note that allowance of \( f_e = 0.6f_y \) in lieu of actual steel stress computations is applicable only to normal structures. Special precautions have to be taken for structures exposed to very aggressive climates, such as chemical factories or offshore structures. Additionally, the depth of the concrete area in tension in reinforced concrete is determined by having the center of gravity of the bars as the centroid of the concrete area in tension. Hence, for a single layer of bars, the depth \( d \) of the concrete area in tension equals \( 2d_c \).

The shaded area in Figure 8.16 gives the total concrete area in tension.

As the number of bars affects the magnitude of \( A \), it is evident that a larger number of smaller diameter bars are better controllers of the crack width provided that the total area of all the bars at the tension side of the section satisfies the flexural requirement of the design.

8.10.3 Example 8.6: Maximum Crack Width in a Reinforced Concrete Beam

Calculate the maximum crack width for a rectangular simply supported beam that has the cross-section shown in Figure 8.16. The beam span is 30 ft (9.14 m). It carries a working uniform load of 1000 lb/ft, including its own weight (14.6 kN/m). Given:

\( f_e = 5000 \) psi, normal-weight concrete (34 MPa)
\( f_y = 60,000 \) psi (414 MPa)
\( E_s = 29 \times 10^6 \) psi (200,000 kPa)

![Figure 8.16 Beam geometry.](image)
Solution:

Alternative using the actual steel area

\[ I_g = \frac{bh^3}{12} = \frac{12(21)^3}{12} = 9261.0 \text{ in.}^2 \]

modulus of rupture \( f_r = 7.5 \sqrt{f_y} = 7.5 \sqrt{5000} = 530.3 \text{ psi (3.66 MPa)} \)

cracking moment \( M_c = \frac{I_g}{b} = \frac{9261.0 \times 530.3}{75.5} = 467,725 \text{ in.-lb} \)

maximum beam moment \( M_b = \frac{1000(30)^3}{8} = 112,500 \text{ ft-lb} \)

\[ = 1,350,000 \text{ in.-lb} \]

\[ \frac{bc^2}{8} + nA_s(c - d) = 0 \]

\[ A_s = 2.37 \text{ in.}^2 \quad (193 \text{ mm}^2) \]

\[ E_s = 50,000 \sqrt{5000} = 4.03 \times 10^6 \text{ psi (27,797 MPa)} \]

\[ n = \frac{E_s}{E_e} = \frac{29 \times 10^6}{4.03 \times 10^6} = 7.20 \]

\[ 6c^3 + 7.2 \times 2.37c - 7.2 	imes 2.37 \times 18.5 = 0 \quad \text{to give } c = 3.97 \text{ in. (100 mm)} \]

From Eq. 8.7, the cracked moment of inertia is

\[ I_c = \frac{bc^2}{3} + nA_s(c - d) \]

\[ = \frac{12(21)^3}{3} + 7.2 \times 2.37(18.5 - 5.97)^2 = 3330 \text{ in.}^4 \]

steel stress \( f_s = \frac{M}{I_c} (c - d) \)

\[ = \frac{1350000}{3330} \times (18.5 - 5.97) \times 7.2 \]

\[ = 34500 \text{ psi (238 MPa)} < 36000 \text{ psi O.K.} \]

steel stress \( f_s = 34.5 \text{ ksi} \) to be used in Eq. 8.24

\[ \beta = \frac{h - c}{d - c} = \frac{21.0 - 5.97}{18.5 - 5.97} = 1.20 \]

\[ A = \frac{hs}{6(2h)} = \frac{32.0 \times 2.5}{3} = 20 \text{ in.}^2 \]

\[ w_{max} = 0.076f_s \sqrt{Ax} \times 10^{-3} \]

\[ = 0.076 \times 1.20 \times 34.5 \sqrt{2.5 \times 20.0} \times 10^{-3} = 0.0116 \text{ in. (0.29 mm)} \]

Alternative using \( f_s = 0.6f_y \)

\[ \beta = 1.21 \quad \text{for beams} \]

\[ f_s = 0.6f_y = 0.6 \times 30.0 = 36.0 \text{ ksi} \]

\[ w_{max} = 0.076 \times 1.20 \times 36.0 \sqrt{2.5 \times 20.0} \times 10^{-3} = 0.0137 \text{ in. (0.35 mm)} \]
8.10.4 Crack-Width Evaluation for Beams Reinforced with Bundled Bars

The bond stress between the reinforcing bars and the surrounding concrete is a major parameter affecting flexural crack spacing and hence crack width. The contact area of bundled bars is less than that of the isolated bars if they act independently. Using the perimetric reduction factor deduced from Figure 8.17, the cracking equation becomes

\[ w_{wcb} = 0.075b c \sqrt{\frac{f_y}{A'}} \]  

(8.25)

where \( w_{wcb} \) is the crack width in units of 0.001 in., and \( A' = \frac{A}{n} \) with the factor for \( A_{wcb} \) shown in Figure 8.17a, \( d' \) is the depth of cover to the center of gravity of the bundle. The steps for calculation of \( w_{wcb} \) are identical to those for beams reinforced with nonbundled bars.

8.10.5 Example 8.7: Maximum Crack Width in a Beam Reinforced with Bundled Bars

Find the maximum flexural crack width for a reinforced concrete beam that has the cross-sectional geometry shown in Figure 8.18. Given:

\[ \gamma_{cb} = \frac{0.85f_{cy}}{f_{cy}} \]

where \( A' = \frac{A}{n} \)

Figure 8.17. Perimetric reduction factors for beams with bundled bars: (a) perimetric factors; (b) section geometry of the concrete area in tension.
Chapter 8  Serviceability of Beams and One-Way Slabs

Concrete area in tension

\[ a = 12^\prime \ (364 \text{ mm}) \]

Figure 8.18 Beam geometry.

\[ f_y = 60,000 \text{ psi} \]
\[ f_t = 0.6f_y = 36,000 \text{ psi} \]

- Two bundles of three No. 8 bars each (25.4-mm diameter)
- Size of stirrups = No. 4 (12.7-mm diameter)

Solution:

\[ a_i = \text{center of gravity of the three bars from the outer tension fibers} \]
\[ = (1.5 + 0.5) + \frac{2 \times 0.5 + 1 \times 1.5}{3} = 2.83 \text{ in.} \]

\[ i = \text{depth of the concrete area in tension} \]
\[ = 2 \times 2.83 = 5.66 \text{ in.} \]

\[ n_w = \text{number of bars if all are of the same diameter, or the total steel area} \]

Photo 8.6 Typical flexural crack formation in beams. (Navy et al.)
8.12 ACI 318 Code Provisions for Control of Flexural Cracking

\[ \gamma_s = 0.65 \gamma_{ck} = 0.65 \times 6 = 3.9 \]

\[ A' = \frac{b t}{\gamma_s} = \frac{10 \times 5.66}{3.9} = 14.31 \text{ in}^2 \]

\[ w_{ew} = 0.078 \times 2.0 \times 26.0 \times 14.31 \times 10^{-3} = 0.011 \text{ in. (0.3 mm)} \]

8.11 TOLERABLE CRACK WIDTHS

The maximum crack width that a structural element should be permitted to develop depends on the particular function of the element and the environmental conditions to which the structure is liable to be subjected to. Table 8.4 from the ACI Committee 224 report on cracking serves as a reasonable guide on tolerable crack widths in concrete structures under the various environmental conditions encountered. Engineering judgment has to be exercised in determining the maximum crack width that can be tolerated. When the computed crack width exceeds the value in Table 8.4, the designer can use a larger number of smaller diameter bars. Otherwise, increase of the size of bars required by fracture becomes necessary.

8.12 ACI 318 CODE PROVISIONS FOR CONTROL OF FLEXURAL CRACKING

As indicated in Section 8.10 and also in the author's extensive work reported in Section 11.9, the spacing of the reinforcement is a major parameter in limiting the crack width. As the spacing is decreased through the use of larger number of bars, the area of the concrete envelopes surrounding the reinforcement increases. This leads to a larger number of narrower cracks. As the crack width becomes narrow enough within the values given in Table 8.4, corrosion effects on the reinforcement are considerably reduced.

The new ACI provisions on crack control through reinforcement distribution limits the spacing in reinforced concrete beams and one-way slabs to the values obtained from the following expression:

\[ s = 15(40,000/f_c) - 2.5 c, \quad (8.25) \]

But not greater than \( 12(36/f_c) \).

<table>
<thead>
<tr>
<th>Table 8.4</th>
<th>Tolerable Crack Widths</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exposure Condition</td>
<td>in.</td>
</tr>
<tr>
<td>Dry air or protective membrane</td>
<td>0.016</td>
</tr>
<tr>
<td>Humidity, moist air, soil</td>
<td>0.012</td>
</tr>
<tr>
<td>Deicing chemicals</td>
<td>0.007</td>
</tr>
<tr>
<td>Seawater and seawater spray: wetting and drying</td>
<td>0.006</td>
</tr>
<tr>
<td>Water-retaining structures (excluding nonpressure pipes)</td>
<td>0.004</td>
</tr>
</tbody>
</table>
where,

\( f_y \) = calculated stress in reinforcement at service load = unfactored moment divided by the steel area and the internal arm moment. Alternately, \( f_y \) can be taken as \( 2/3 f_y' \).

\( c_r \) = clear cover from the nearest surface in tension to the flexural tension reinforcement, inches.

\( z \) = center-to-center spacing of flexural tension reinforcement, inches, closest to the tension face of the section.

For 60k reinforcement, \( f_y = \frac{z}{3} \times 60,000 = 20,000 \) psi.

From these provisions, the maximum spacing for 60,000 psi (414 MPa) reinforcement = 12 [40,000 / 40,000] = 12 in. (305 mm). The maximum spacing of 12 in. is in conformity with the extensive tests by the author of in excess of 100 two-way action slabs discussed in Sec. 11.9. Hence, this limitation on the distribution of flexural reinforcement in one-way slabs and wide-web reinforced concrete beams is appropriate. However, in beams of normal web width in normal buildings, these provisions might not be as workable as controlling the crack width through the process presented in Sec. 8.10 and 8.11.

The SI expression for the value of reinforcing spacing in Eq. 8.26 and \( f_y \) in MPa units is,

\[ s_{(mm)} = \frac{350(280/f_y)}{2} c \]

but not to exceed 300(280/f_y). For the usual case of beams with grade-420 reinforcement and 50-mm clear cover to the main reinforcement, with \( f_y = 252 \) MPa, the maximum bar spacing is 250 mm.

It should be stressed that these provisions are applicable to reinforced concrete beams and one-way slabs in structures subject to normal environmental conditions. For other types of structures subject to aggressive environments such as sanitary structures, the recommendations in Sec. 8.10 and 8.11 are more appropriate (see commentary at end of Example 8.8).

Skin Reinforcement for Deep Beams: In order to control cracking in the web of deep beams or joints, some reinforcement has to be placed near the vertical faces of the tension zone. Without such auxiliary steel, the width of the crack in the web may exceed the crack width at the level of the flexural tension reinforcement. Hence, for beams or joints whose depth \( b \) exceeds 36 in., longitudinal skin reinforcement has to be uniformly distributed along both sides of the beam for a distance \( b/2 \) from the tension face.

The maximum spacing between longitudinal bars or wires of skin reinforcement should not exceed the value obtained from Eqs. 8.26 or 8.27, nor should it exceed 12 in., where \( c_r \) is the least distance from the surface of the reinforcement to the side of the face of the concrete section.

8.12.1 Example 8.8 Reinforcement Spacing Limitation in Beams as Required in the ACI 318 Code

Verify if the reinforcement in Ex. 8.6 satisfies the ACI 318 requirements for crack control through reinforcement distribution.

Solution: Crack width, \( w = 0.0137 \) in. (0.35 mm)

\( f_y = 5,000 \) psi, normal-weight concrete (34 MPa)

\( f_y = 60,000 \) psi (414 MPa)

\( b = 12 \) in. (305 mm)

\( d = 18.5 \) in. (457 mm)

\( c_r = 2.1 \times (5/10 \text{ in}) \)
8.13 SI Conversion Expressions and Example of Deflection Evaluation

\[
\varepsilon_{ci} = 21.0 - 15.5 - 0.5 = 2 \text{ in. (51 mm)}
\]
\[
0.6 f_y = 60,000 \times 0.6 = 36,000 \text{ psi} = 250 \text{ MPa}
\]

From Eq. 8.2b, the maximum allowable bar spacing:
\[
s = \frac{1500 \times 600}{f_y - 2.5} = 15(40,000/40,000) - 2.5 \times 2 = 10 \text{ in. (254 mm)}
\]
But not to exceed 12 in/ft = 12(40,000/40,000) = 12 in. > 0 in.

Please max. s = 30 in. controls.

Actual s = (12 - 2 x 2.0 side cover)/2 spaces = 6 in. < 10 in. O.K.

Therefore, the reinforcement distribution in this beam satisfies the ACI 318 code provisions.

Commentary:
If the beam was a structural member in severe environmental conditions where a crack width of 0.013 in. is not tolerable, satisfying the ACI 318 new provision is not adequate for sustaining the long-term structural integrity of the beam.

8.13 SI CONVERSION EXPRESSIONS AND EXAMPLE OF DEFLECTION EVALUATION

1. \( E_c = w^{1.5} \times 0.043 \sqrt{f_y} \) MPa, where \( w_c = 1500 \) to 2500 kg/m³ (90 to 155 lb/ft³). For standard, normal-weight concrete, \( w_c = 2400 \) kg/m³ to give \( E_c = 29,700 \) MPa.
2. \( E_c = 200,000 \) MPa
3. Modulus of rupture \( f_y = 0.7 \sqrt{f_c} \)
4. For rectangular sections, \( I_x = \frac{bh^3}{12} \) and \( I_y = bc^3/12 \), where \( n = E_c / E_y \)
5. \( M_{cr} = f_y I_y / n_y \), where \( n_y \) is the distance from the neutral axis to the tensile extreme fibers = \( l_y \) for rectangular sections
6. \( I_x = M_{cr} / M_x \), \( I_y = [1 - (M_x / M_{cr})^2] I_y \)
7. Long-term deflection multiplier \( h = 0.77 + 0.5 \rho \)

8.13.1 SI Example on Deflection

Solve Ex. 8.3 using SI units.

Solution:

\( f_y = 34.5 \text{ MPa, normal weight, } A_s = 832 \text{ mm}^2 \)
\( f_y = 414 \text{ MPa (MPa = N/mm}^2) \)
\( E_c = 20,000 \text{ MPa} \)
\( b = 254 \text{ mm} \)
\( h = 406 \text{ mm} \)
\( d = 330 \text{ mm} \)

service \( M_{cr} = 24.3 \text{ kN/m} \)
\( M_x = 25.4 \text{ kN/m} \)

Assume 60% live load sustained for 24 months

\( E_c = w_c^{1.5} \times 0.043 \sqrt{f_y} \) (MPa)

where \( w_c = 1500 \) to 2500 kg/m³ (90 to 155 lb/ft³). For standard normal-weight concrete, \( w_c = 2400 \) kg/m³.
\[ E_c = 2400 \times 10^{3} \times 0.043 \times \sqrt{343} = 29,700 \text{ MPa} \]
\[ E_c = 200,000 \text{ MPa} \]
modulus ratio \[ n = \frac{E_c}{E} = \frac{200,000}{29,700} = 6.7 \]
\[ f_c = 0.7 \sqrt{f_e} = 0.7 \sqrt{34.5} = 4.1 \text{ MPa} \]

From Table 8.1.
\[ h_{cr} = \frac{8230}{16} = 520 \text{ mm} > \text{ actual } h = 406 \text{ mm} \]

Hence, deflection calculations have to be made.

Effective moment of inertia
\[ I_e = \frac{bh^3}{12} = \frac{254(496)^3}{12} = 14.2 \times 10^6 \text{ mm}^4 \]
\[ y_e = \frac{b}{2} = \frac{406}{2} = 203 \text{ mm} \]
\[ M_{ce} = \frac{f_c}{y_e} = \frac{4.1 \times 14.2 \times 10^6}{203} = 28.7 \times 10^6 \text{ N-mm} \]
\[ = 28.7 \text{ kN-m} \]

Depth of neutral axis \( c \)
\[ d = 30 \text{ mm} \quad A_s = 852 \text{ mm}^2 \]
\[ \frac{254(10^3)}{2} \cdot nA_s(d - c) \]

or
\[ 127c^2 = 6.7 \times 852(30 - c) \quad \text{to get } c = 102 \text{ mm} \]
\[ I_c = \frac{1}{3} I_e + nA_s(c^2 - y_e^2) \]
\[ = \frac{254(10^3)}{3} \times 6.7 \times 852(30 - 102) \]
\[ = 89.8 \times 10^6 + 296.7 \times 10^6 = 3.86 \times 10^7 \text{ mm}^4 \]

Dead load
\[ M_d = 24.3 \text{ kN-m} \quad \text{(given)} \]
\[ \frac{M_d}{M_{ce}} = \frac{28.7}{24.3} = 1.18 \approx 1.0 \]

Use \( M_s = M_d \) and \( f_s = f_e \) since the dead-load moment is smaller than the cracking moment (the beam will not crack at dead-load level).

Dead load + 60% live load
\[ \left( \frac{M_d}{M_{ce}} \right) = \left( \frac{28.7}{24.3 \times 0.6 \times 28.3} \right) = 0.30 \]

Dead load + live load
\[ \left( \frac{M_s}{M_{ce}} \right) = \left( \frac{28.7}{24.3 + 28.3} \right) = 0.16 \]
\[ L = \left( \frac{M_s}{M_d} \right) f_s + \left[ 1 - \left( \frac{M_s}{M_d} \right) \right] f_e \]
8.13 SI Conversion Expressions and Example of Deflection Evaluation

**Dead load**

\[ I_{d} = 14.2 \times 10^{6} \text{ mm}^{4} \]

**Dead load + 0.6 live load**

\[ I_{L} = 0.3 \times 14.2 \times 10^{8} + 0.7 \times 3.86 \times 10^{8} = 7.0 \times 10^{8} \text{ mm}^{4} \]

**Dead load + live load**

\[ I_{L} = 0.16 \times 14.2 \times 10^{8} + 0.84 \times 3.86 \times 10^{8} = 5.5 \times 10^{8} \text{ mm}^{4} \]

**Short-term deflection**

\[
\Delta = \frac{5M_{d}x^{2}}{384EI} = \frac{5MG}{48\times 29,700} \approx 238 \text{ M mm} \]

**Initial live-load deflection**

\[
\Delta_{l} = \frac{238(24.3 + 20.3) \times 10^{6}}{5.5 \times 10^{6}} - \frac{238(24.3) \times 10^{6}}{14.2 \times 10^{6}} = 23 - 4 = 19 \text{ mm, say 20 mm (0.8 in.)} \]

**Initial dead-load deflection**

\[
\Delta_{d} = \frac{238(24.3) \times 10^{6}}{142 \times 10^{6}} = 4 \text{ mm} \]

**Initial 60% sustained live-load deflection**

\[
\Delta_{1,60} = 238 \left[ \frac{24.3 \times 0.6 \times 20.3 \times 10^{5}}{7.0 \times 10^{5}} - \frac{24.3 \times 10^{5}}{14.2 \times 10^{5}} \right] = 14 - 4 = 10 \text{ mm} \]

**Long-term deflection**

From Eq. 8.10,

\[
\Delta_{l,\text{ls}} = \Delta_{l} + \lambda_{*} \Delta_{l} + \lambda \Delta_{l,\text{ls}} \]

\[
\lambda = \frac{T}{1 + 50\nu} \]

where \( \nu = 0 \) for singly reinforced beam.

- \( T \) for 5 years or more = 2.0, \( \lambda_{*} = 2.0 \)
- \( T \) for 24 months = 1.65, \( \lambda = 1.65 \)
- \( \Delta_{l} = 20 + 2.0 \times 4 + 1.65 \times 10 = 45 \text{ mm} \)

**Deflection requirements** (Table 8.2)

<table>
<thead>
<tr>
<th>( c )</th>
<th>8230</th>
</tr>
</thead>
<tbody>
<tr>
<td>180</td>
<td>180</td>
</tr>
<tr>
<td>240</td>
<td>240</td>
</tr>
<tr>
<td>360</td>
<td>360</td>
</tr>
<tr>
<td>480</td>
<td>480</td>
</tr>
<tr>
<td>( \frac{c}{2h} )</td>
<td>( \frac{8230}{240} )</td>
</tr>
</tbody>
</table>

\[ c = 8230 < 34 \text{ mm} < \Delta_{l,\text{ls}} \]

\[ c = 8230 < 17 \text{ mm} < \Delta_{l,\text{ls}} \]
Hence, the use of the beam is limited to floors or roofs not supporting or attached to non-structural elements such as partitions.

SELECTED REFERENCES


PROBLEMS FOR SOLUTION

8.1. Calculate $f$ and $l_x$ for cross section (a) through (f) in Figure 8.19. Given:

- $f'_c = 4000$ psi (27.6 MPa), normal-weight concrete
- $f = 69.000$ psi (484 MPa)
- $E = 29 \times 10^6$ psi (200,000 MPa)

8.2. Calculate the maximum immediate and long-term deflection for a 6-in-thick slab on simple supports spanning over 15 ft. The service dead and live loads are 70 psf (33.3 kPa) and 120 psf (57.3 kPa), respectively. The reinforcement consists of No. 5 bars (0.38-in diameter) spaced 6 in. on center (154 mm center to center). Also check whether all beams are adequate in accordance with the ACI Building Code. Assume that 60% of the live load is sustained over a 16-month period. Given:

- $f'_c = 4500$ psi (31 MPa)
- $f = (8,000 - 1.4(15))$ psi (41.4 MPa)
- $E = 29 \times 10^6$ psi (200,000 MPa)

8.3. Calculate the deflection due to dead load and dead load plus live load for the following cases in Problem 8.1: cross sections (a), (d), and (e). Use for service-load levels 0.3M, as maximum dead-load moment and 0.35 M, as maximum live-load moment. Assume that all beams are simply supported and have a span of 22 ft (6.71 m).
8.4. Repeat Problem 8.2 assuming the slab to be continuous over four supports. The top tension reinforcement at the support consists of No. 3 bars at 4-in. center-to-center, and the compression reinforcement consists of No. 5 bars at 12-in. centers.

8.5. A beam supporting a 4-in. slab is continuous over four supports. The center-to-center spans are 26 ft with the end span resting on an outer wall. It has a web width, b, = 12 in. and a total thickness, h = 18 in. and carries a service live load, W = 4000 lb/ft and a service dead load, W = 1800 lb/ft, including its self-weight. The midspan tension reinforcement consists of A_t = 5 No. 8 bars (32.6 mm), and the support reinforcement is comprised of A_s = six No. 10 bars (25.3 mm) using A_s = three No. 8 bars. Calculate the maximum immediate and long-term deflections of this beam assuming that 55% of the sustained live load acts over a 54-month period. Also check what deflection servability criteria the beam satisfies and whether it can support attached partitions and other elements that can be damaged by large deflections.

Given:

\[ f' = 5000 \text{ psi (34.5 MPa)} \]
\[ f'' = 60,000 \text{ psi (413.7 MPa)} \]
\[ f = 29 \times 10^6 \text{ psi (200,000 MPa)} \]

8.6. Calculate the maximum expected flexural crack width in the beam of Problem 8.5 and verify if it satisfies the serviceability criteria for crack control if it is subjected to (a) interior exposure and (b) freeze-thaw and deicing cycles.
8.7. A rectangular beam under simple bending has the dimensions shown in Fig. 8.20. It is subjected to an aggressive chemical environment. Calculate the maximum expected tensile crack width and whether the beam satisfies the serviceability criteria for crack control. Given:

\[ f' = 5000 \text{ psi} \ (34.5 \text{ MPa}) \]
\[ f = 60,000 \text{ psi} \ (414 \text{ MPa}) \]

Minimum clear cover = \( \frac{1}{2} \) in. \((3.2 \text{ mm})\)

![Beam Diagram]

Figure 8.20 Beam geometry.

8.8. Check Problem 8.7 using the ACI bar spacing provisions and determine if it satisfies the serviceability criteria for crack control if it is subjected to severe exposure conditions.

8.9. Find the maximum web of a beam, reinforced with bundled bars to satisfy the crack-control criteria for interior exposure conditions. Given:

\[ f' = 4000 \text{ psi} \ (27.6 \text{ MPa}) \]
\[ f = 60,000 \text{ psi} \ (414 \text{ MPa}) \]
\[ d = \text{two bundles of three No. 9 bars each (three bars of } 28.6 \text{ mm diameter each in a bundle)} \]
\[ \text{No. 4 stirrups used (13-mm diameter)} \]
9

COMBINED COMPRESSION AND BENDING: COLUMNS

9.1 INTRODUCTION

Columns are vertical compression members of a structural frame intended to support the load-carrying beams. They transmit loads from the upper floors to the lower levels and then to the soil through the foundations. Since columns are compression elements, failure of one column in a critical location can cause the progressive collapse of the adjoining floors and the ultimate total collapse of the entire structure.

Structural column failure is of major significance in terms of economic as well as human loss. Thus extreme care needs to be taken in column design, with higher reserve strength than in the case of beams and other horizontal structural elements, particularly since compression failure provides little visual warning.

As will be seen in subsequent sections, the ACI Code requires a considerably lower strength reduction factor 0.6 in the design of compression members than the 0.9 factors in flexure, shear, or tension. The discussion presented in Chapter 4 on the probability of failure and reliability of performance explains and justifies in more detail the reasons for the additional reserve strength needed in proportioning compression members.

Photo 9: High-strength concrete high-rise building at 535 Madison Avenue, New York. (Courtesy of Construction Industry Board, New York.)
The principles of stress and strain compatibility used in the analysis (design) of beams discussed in Chapter 5 are equally applicable to columns. A new factor is introduced, however; the addition of an external axial force to the bending moments acting on the critical section. Consequently, an adjustment becomes necessary to the force and moment equilibrium equations developed for beams to account for combined compression and bending.

The amount of reinforcement in the case of beams was controlled so as to have ductile failure behavior. In the case of columns, the axial load will occasionally dominate; hence compression failure behavior in cases of a large axial load to bending moment ratio cannot be avoided.

As the load on a column continues to increase, cracking becomes more intense along the height of the column at the transverse tie locations. At the limit state of failure, the concrete cover on tied columns or the shell of concrete outside the spirals of spirally confined columns spalls and the longitudinal bars become exposed. Additional load leads to failure and local buckling of the individual longitudinal bars at the unsupported length between the ties. It is noted that at the limit state of failure, the concrete cover to the reinforcement spalls first after the bond is destroyed.

As in the case of beams, the strength of columns is evaluated on the basis of the following principles:

1. A linear strain distribution exists across the thickness of the column.
2. There is no slippage between the concrete and the steel (i.e., the strain in steel and in the adjoining concrete is the same).
3. The maximum allowable concrete strain at failure for the purpose of strength calculations is 0.003 in./in.
4. The tensile resistance of the concrete is negligible and is disregarded in computations.

9.2 TYPES OF COLUMNS

Columns can be classified on the basis of the form and arrangement of reinforcement, the position of the load on the cross-section, and the length of the column in relation to its lateral dimensions.

The form and arrangement of the reinforcement identify three types of columns, as shown in Figure 9.1:

1. Rectangular or square columns reinforced with longitudinal bars and lateral ties (Figure 9.1a).
2. Circular columns reinforced with longitudinal reinforcement and spiral reinforcement, or lateral ties (Figure 9.1b).
3. Composite columns where steel structural shapes are encased in concrete. The structural shapes could be placed inside the reinforcement cage, as shown in Figure 9.1c.

Although tied columns are the most commonly used because of lower construction costs, spirally wound rectangular or circular columns are also used where increased ductility is needed, such as in earthquake zones. The ability of the spiral column to sustain the maximum load at excessive deformations prevents the complete collapse of the structure before local redistribution of moments and stresses is complete. Figure 9.2 shows the large increase in ductility (toughness) due to the effect of spiral binding.
Based on the position of the load on the cross section, columns can be classified as concentrically or eccentrically loaded, as shown in Figure 9.3. Concentrically loaded columns (Figure 9.3a) carry no moment. In practice, however, all columns have to be designed for some unforeseen or accidental eccentricity due to such causes as imperfections in the vertical alignment of formwork.

Eccentrically loaded columns (Figure 9.3b and c) are subjected to moment in addition to the axial force. The moment can be converted to a load $P$ and an eccentricity $e$, as shown in Figure 9.3b and c. The moment can be unilateral, as in the case of an exterior column in a multistory building frame or when two adjacent panels are not similarly loaded, such as columns $A$ and $B$ in Figure 9.4. A column is considered biaxially loaded when bending occurs about both the $X$ and $Y$ axes, such as in the case of corner column $C$ of Figure 9.4b.
Chapter 9  Combined Compression and Bending: Columns

Figure 9.2  Comparison of load-deformation behavior of tied and spirally bound columns.

Figure 9.3  Types of columns based on the position of the load on the cross section: (a) concentrically loaded column; (b) axial load plus uniaxial moment; (c) axial load plus bi-axial moment.
9.3 Strength of Non-Slender Concentrically Loaded Columns

Failure of columns could occur as a result of material failure by initial yielding of the steel at the tension face or initial crushing of the concrete at the compression face, or by loss of lateral structural stability (i.e., through buckling).

If a column fails due to initial material failure, it is classified as a short or non-slender column. As the length of the column increases, the probability that failure will occur by buckling also increases. Therefore, the transition from the short column (material failure) to the long column (failure due to buckling) is defined by using the ratio of the effective length $l_e$ to the radius of gyration $r$. The height, $l_e$, is the unsupported length of the column, and $r$ is a factor that depends on end conditions of the column and whether it is braced or unbraced. For example, in the case of unbraced columns, if $l_e/r$ is less than or equal to 22, such a column is classified as a short column, in accordance with the ACI load criteria. Otherwise, it is defined as a long or a slender column. The ratio $l_e/r$ is called the slenderness ratio.

9.3 STRENGTH OF NON-SLENDER CONCENTRICALLY LOADED COLUMNS

Consider a column of gross cross-sectional area $A_t$ with width $b$ and total depth $d$, reinforced with a total area of steel $A_s$ on all faces of the column. The net cross-sectional area of the concrete is $A_t - A_s$.

Figure 9.3 presents the stress history in the concrete and the steel as the column load is increased. Both the steel and the concrete behave elastically at first. At a strain of
Figure 9.5 Stress-strain behavior of concrete and steel (concentric load).

approximately 0.002 in./in. to 0.003 in./in., the concrete reaches its maximum strength $f'_c$. Theoretically, the maximum load that the column can take occurs when the stress in the concrete reaches $f'_c$. Further increase is possible if strain hardening occurs in the steel at about 0.003-in./in. strain levels.

Therefore, the maximum concentric load capacity of the column can be obtained by adding the contribution of the concrete, which is $(A_s - A_s \times 0.85f'_c)$, and the contribution of the steel, which is $A_s f'_c$, where $A_s$ is the total gross area of the concrete section and $A_s$ is the total steel area = $A_s + A_s$. The value of $0.85f'_c$ instead of $f'_c$ is used in the calculation since it is found that the maximum attainable strength in the actual structure approximates 0.85$f'_c$. Thus, the nominal concentric load capacity, $P_N$, can be expressed as

$$P_N = 0.85f'_c (A_s - A_s) + A_s f'_c$$

(9.1)

It should be noted that concentric load causes uniform compression throughout the cross section. Consequently, at failure, the strain and stress will be uniform across the cross section, as shown in Figure 9.6.

Figure 9.6 Column geometry; strain and stress diagrams (concentric load): (a)
It is highly improbable to attain zero eccentricity in actual structures. Eccentricities could easily develop because of factors such as slight inaccuracies in the layout of columns and unsymmetric loading due to the difference in thicknesses of the slabs in adjacent spans or imperfections in the alignment, as indicated earlier. Hence a minimum eccentricity of 10% of the thickness of the column in the direction perpendicular to its axis of bending is considered as an acceptable assumption for reduction of column load in columns with ties and 5% for the load in spirally reinforced columns.

To reduce the calculations necessary for analysis and design for minimum eccentricity, the ACI Code specifies a reduction of 20% in the axial load for tied columns and a 15% reduction for spiral columns. Using these factors, the maximum nominal axial load capacity of columns cannot be taken greater than

\[ P_{\text{nom}} = 0.8(0.85f'_c(A_y - A_s) + A_s f_y) \]  

(9.2a)

for tied reinforced columns and

\[ P_{\text{nom}} = 0.85(0.85f'_c(A_y - A_s) + A_s f_y) \]  

(9.2b)

for spirally reinforced columns.

Equations 9.2(a) and 9.2(b), respectively, give \( A_y = P_y/(0.68 f'_c + 0.8 p_s f_y) \) and \( A_s = P_s/(0.75 f'_c + 0.83 p_s f_y) \). For a first trial section with acceptable eccentricity, the designer can try equations 9.2(a) and (b) for assuming the gross section area \( A_y \):

\[ A_y = \frac{P_y}{0.68(f'_c + f_y p_s)} \]  

(9.3a)

for tied columns, where \( p_s = \text{total reinforcement percentage} \), and

\[ A_y = \frac{P_y}{0.55(f'_c + f_y p_s)} \]  

(9.3b)

for spirally reinforced columns.

These nominal loads should be reduced further using strength reduction factors \( \phi \), as explained in later sections. Normally, for design purposes, \( (A_y - A_s) \) can be assumed to be equal to \( A_y \) without great loss in accuracy.

### 9.3.1 Example 9.1: Analysis of an Axially Loaded Non-Slender

**Rectangular Tied Column**

A non-slimer tied column is subjected to axial load only. It has the geometry shown in Figure 9.6a and is reinforced with eight No. 9 bars (23.6 mm diameter) one each of the two faces parallel to the x axis of bending. Calculate the maximum nominal axial load strength \( P_{\text{nom}} \).

**Given:**

\[ f'_c = 40,000 \text{ psi (276 MPa)} \]
\[ f_y = 60,000 \text{ psi (414 MPa)} \]

**Solution:** \( A_y = A'_y = 2 \text{ in}^2 \). Therefore, \( A_s = 6 \text{ in}^2 \). Using Eq. 9.2 yields

\[ P_{\text{nom}} = 0.8(0.85 \times 40,000)(12 \times 20 - 6) + 6 \times 60,000) = 92,480 \text{ lb (4110 kN)} \]

If \( A_s = A_y \) is taken as equal to \( A_y \), it results in

\[ P_{\text{nom}} = 0.8(0.85 \times 40,000 \times 12 \times 20 + 6 \times 60,000) = 940,800 \text{ lb (4180 kN)} \]
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Figure 9.7 Column geometry: strain and stress diagrams (concentric load): (a) cross-section; (b) concrete strain; (c) stress (forces).

Note from Figure 9.6b and c that the entire concrete cross-section is subjected to a uniform stress of 0.85 $f_c$, and a uniform strain of 0.003 in./in.

9.3.2 Example 9.2: Analysis of an Axially Loaded Non-Slender Circular Column

A 20-in.-diameter, non-slender, spirally reinforced circular column is symmetrically reinforced with six No. 8 bars, as shown in Figure 9.7. Calculate the strength $P_{\text{allow}}$ of this column if subjected to axial load only. Given:

- $f_c = 4000$ psi (27.5 MPa)
- $f_y = 60,000$ psi (414 MPa)

Solution:

- $A_e = 4.74$ in.$^2$
- $A_t = \frac{\pi}{4} (20)^2 = 314$ in.$^2$

Using Eq. 9.3 yields

$$ P_{\text{allow}} = 0.85[0.85 \times 4000(1.4 - 4.74) + 4.74 \times 60,000] $$

$$ = 1,335,501 \text{ lb (5955 kN)} $$

or, assuming $A_t = A_e - A_r$,

$$ P_{\text{allow}} = 0.85[0.85 \times 4000 \times 314 + 4.74 \times 60,000] $$

$$ = 1,349,200 \text{ lb (5962 kN)} $$

9.4 STRENGTH OF ECCENTRICALLY LOADED COLUMNS: AXIAL LOAD AND BENDING

9.4.1 Behavior of Eccentrically Loaded Non-Slender Columns

The same principles concerning the stress distribution and the equivalent rectangular stress $f_x$ applied to beams are equally applicable to columns. Figure 9.8 shows a typical rectangular column cross-section with strain, stress, and force distribution diagrams. The diagram differs from Figure 5.13 in the introduction of an additional longitudinal nominal force $P_e$ at the limit failure strain acting at an eccentricity $e$ from the plastic (geometric) centroid of the section. The depth of neutral axis primarily determines the strength of the column.

The equilibrium expressions for forces and moments from Figure 9.8 can be expressed as follows for non-slender columns.
Nominal axial resisting force $P_e$ at failure is given by $C_e + C_s - T_e$.

Nominal resisting moment $M_n$ which is equal to $P_e e$ can be obtained by writing the moment equilibrium equation about the plastic centroid. For columns with symmetrical reinforcement, the plastic centroid is the same as the geometric centroid.

$$M_n = P_e e = C_e \left( \frac{a}{2} \right) + C_s \left( \frac{a}{2} + d' \right) + T_e (d - \bar{y})$$

Since:

- $C_e = 0.75 f_y b t$
- $C_s = A_s f_s$
- $T_e = A_s f_s$
Eqs. 9.4 and 9.5 can be rewritten as

\[ P_e = 0.85 \sigma_e b a + A_e f_e - A_e f_t \]  
\[ M_e = P_e d = 0.85 \sigma_e b a \left( \frac{d}{2} - \frac{a}{2} \right) + A_e f_e (d - \frac{a}{2}) + A_e f_t (d - \frac{a}{2}) \]  

(9.6)

(9.7)

where \( \bar{Y} \) for rectangular sections = \( b/2 \)

In Eqs. 9.6, the depth of the neutral axis \( \epsilon \) is assumed to be less than the effective depth \( d \) of the section, and the steel at the tension face is in actual tension. Such a condition changes if the eccentricity \( \epsilon \) of the axial force \( P_e \) is very small. For such small eccentricities, where the total cross-section is in compression, the contribution of the tension steel should be added to the contribution of concrete and compression steel. The term \( A_e f_t \) in Eqs. 9.6 and 9.7 in such a case would have a reverse sign since all the steel is in compression. It is also assumed that \( b - A_e f_e - A_e f_t \) that is, the volume of concrete displaced by compression steel is negligible.

Symmetrical reinforcement is usually used such that \( A_e = A_t \) in order to prevent the possible interchange of the compression reinforcement with the tension reinforcement during bar cage placement. Symmetry of reinforcement is also often necessary where the possibility exists of stress reversal due to change in wind direction.

If the compression steel is assumed to have yielded and \( A_e = A_t \), Eqs. 9.6 and 9.7 can be rewritten as

\[ P_e = 0.85 \sigma_e b a \]  
\[ M_e = P_e d = 0.85 \sigma_e b a \left( \frac{d}{2} - \frac{a}{2} \right) + A_e f_e (d - \frac{a}{2}) + A_e f_t (d - \frac{a}{2}) \]  

(9.8a)

(9.8b)

or

\[ M_e = P_e d = 0.85 \sigma_e b a \left( \frac{d}{2} - \frac{a}{2} \right) + A_e f_e (d - \frac{a}{2}) \]  

(9.8c)

If only one layer of reinforcement at the tension side, \( d \) becomes equal to \( a \).

In Eq. 9.8(c), the geometric centroid is replaced by \( b/2 \) for symmetrical reinforcement and \( A_e \) is replaced by \( A_t \).

Additionally, Eqs. 9.8(a) and 9.8(c) can be combined to obtain a single equation for \( P_e \). Replacing \( 0.85 \sigma_e b a \) in Eq. 9.8(b) by Eq. 9.8(a) gives

\[ P_e = P_e \left( d - \frac{a}{2} \right) + A_e f_e (d - \frac{a}{2}) \]  

(9.9a)

Also, from Eq. 9.6,

\[ a = \beta, c = \frac{A_e f_e - f_t}{0.85 f_e b} \]  

(9.9b)

It should be noted that the axial force \( P_e \) cannot exceed the maximum axial load strength \( P_{\text{axial}} \) calculated using Eq. 5.2. Depending on the magnitude of the eccentricity \( a \), the compression steel \( A_e \) or the tension steel \( A_t \) will reach its yield strength \( f_y \). Stress \( f_x \) reaches \( f_y \) when failure occurs by crushing of the concrete. If failure develops by yielding of the tension steel \( f_t \) should be replaced by \( f_t' \). When the magnitude of \( f_t' \), or \( f_t \) is less than \( f_y \), the axial stresses can be calculated using the stress distribution across the depth of the section (Figure 9.6).

\[ f_x = \frac{P_e}{b \epsilon } = \frac{0.003 (d - \frac{a}{2})}{e} \leq f_y \]  

(9.9d)

or

\[ f_x = \epsilon, E, = \frac{0.003}{e} \left( 1 - \frac{d}{e} \right) \leq f_y \]  

(9.9e)
The stress functions \( f_r = E_0 \delta \), \( f_t = E_0 \delta \), and \( f_s = E_0 \delta \) can be expressed in terms of the depth of neutral axis \( c \) as in Eqs. 9.9 and 9.10 and thus in terms of \( a \). The two remaining unknowns, \( \sigma \) and \( P_0 \), can be solved using Eqs. 9.6 and 9.7. However, combining Eqs. 9.6 and 9.7 to 9.8 leads to a cubic equation in terms of the neutral-axis depth \( c \). We also must check whether the yield strength \( f_y \) is less than the yield strength \( f_y \). Hence the following trial-and-adjustment procedure is suggested for a general case of analysis (design).

For a given section geometry and eccentricity \( e \), assume a value for the distance \( c \) down to the neutral axis. This value is a measure of the compression block depth \( a \) since \( a = b_c \). Using the assumed value of \( c \), calculate the axial load \( P_0 \) using Eq. 9.6 and \( a = b_c \). Calculate the stresses \( f_r \), \( f_s \), and \( f_t \) in compression and tension, respectively, using Eqs. 9.9 and 9.10. Also, calculate the eccentricity corresponding to the calculated load \( P_0 \) using Eq. 9.7. This calculated eccentricity should match the given eccentricity. If not, repeat the steps until a convergence is accomplished. If the calculated eccentricity is larger than the given eccentricity, this indicates that the assumed value for \( c \) and the corresponding depth \( a \) of the compression block are less than the actual depth. In such a case, try another cycle, assuming a larger value of \( c \). This process ensures stress-compatibility across the depth of the section as discussed in Sec. 5.7.

This trial-and-adjustment process converges rapidly and becomes exceedingly simpler if a computer program is used, as explained in Appendix A. This discussion pertains to a general case. Simplifying assumptions can be made in most cases to shorten the iteration process.

9.5.1 Strain Limits Zones

As discussed in Chapter 5, Section 5.3, the strain limits for compression-controlled sections can be represented by the following strain distributions across the depth of the beam section, with \( \varepsilon_r = 0.002 \) for Grade 60 steel, or generally \( \varepsilon_r = f_r/E \).

Figure 5.5 is reproduced here at Fig. 9.10 to illustrate the behavior limits presented in Fig. 9.9, where \( \delta \) is the column section depth to the center of the first layer of the tensile reinforcement.
9.5.2 Stress Limits

(1) Tension-controlled limit case ($\varepsilon_t > 0.005$)
As presented in Equations 9.8 and 9.9

\[
\delta = \frac{\varepsilon_t - \varepsilon_c}{\varepsilon_t \text{ or } \varepsilon_c} = \frac{0.005}{0.003 + 0.005} = 0.375
\]  
\[
\sigma = \beta \delta = 0.375 \beta \delta
\]  
(9.11a)
(9.11b)

From similar triangles
\[
\varepsilon_s^* = 0.003 \left( 1 - \frac{d'_s}{d_s} \right) = 0.003 \left( 1 - 2.67 \frac{d'_s}{d_s} \right)
\]  
(9.12a)

Hence, for 60 ksi reinforcement,

\[
f'_t = \varepsilon_s^* E_s = 37,000 \left( 1 - 2.67 \frac{d'_s}{d_s} \right) \leq f_t
\]  
(9.12b)
9.5 Strain Limits Heurist to Establish Biaxiality Factor $\phi$

(2) Compression-controlled limit case ($\epsilon_c = 0.002$)

The limit strain in the tensile reinforcement in this case, namely, $f_y/E_y$, represents the balanced strain state, where the tensile reinforcement yields simultaneously with the crating of the concrete at the concrete extreme compression fibers. As the neutral axis depth $c$ increases beyond this state, the strain $\epsilon_t$ in the tensile reinforcement would decrease in value below the yield strain. As a result, the stress in the tensile reinforcement becomes smaller than the yield strength $f_y$.

For 60 ksi steel reinforcement, yield strain is

$$
\epsilon_t = \frac{f_y}{E_y} = \frac{60,000}{29 \times 10^6} = 0.002 \text{ in./in.}
$$

This corresponds to ultimate design strain $\epsilon_u = 0.003$ in./in. in the concrete extreme compression fibers, by the ACI-318 Code. Other codes allow higher design compressive strains, such as 0.0035 and 0.0038 (CEB and EuroCode 2).

$$
\begin{align*}
\epsilon_t &= \epsilon_u - \epsilon_c = 0.003 + f_y/E_y = 0.003 + 0.002 = 0.005 \\
\alpha &= \beta_c \cdot c = 0.60 \beta_c d_t
\end{align*}
$$

From similar triangles,

$$
\frac{c}{c-d_t} = \epsilon_t - \epsilon_c
$$

giving $\epsilon_t = 0.003 \left(1 - \frac{d_t}{c}\right)$

or

$$
\epsilon_t = 0.003 \left(1 - \frac{d_t}{\beta_c d_t}\right)
$$

Hence,

$$
f_y = \epsilon_t E_y = 87,000 \left(1 - 1.67 \frac{d_t}{\beta_c d_t}\right) \leq f_y
$$

(3) Transition zone for limit strain with intermediate behavior

This characterizes compression members in which the tensile reinforcement $A_t$ has yielded but the compressive reinforcement $A_c$ has a stress level $f_c \leq f_y$, depending on the geometry of the section. Intermediate $\phi$ values change linearly with $\epsilon_t$ from $\phi = 0.00$ when $\epsilon_t > 0.003$ to $\phi = 0.05$ for tied columns, or $\phi = 0.70$ for spiral columns when $\epsilon_t = 0.002$. It should be noted that for nonpropped flanged members and for nonpropped members with axial load less than $0.10 f_y A_t$, the net tensile strain $\epsilon_t$ should not be less than 0.004. Hence, in the transition zone of Fig. 9.10, the minimum strain value in flanged members for determining the $\phi$ value is 0.004. This limit is necessitated, as a $\phi$ value can otherwise become too low that additional reinforcement would be needed to give the required nominal moment strength.

9.5.3 Summary: Modes of Failure in Columns

Based on the magnitude of strain in the tension face reinforcement (Figure 9.8) the section is subjected to one of the following three conditions:

1. Tension-controlled state, by initial yielding of the reinforcement at the tension side, and strain $\epsilon_t$ greater than 0.006.
2. Transition state, denoted by initial yielding of the reinforcement at the tension side, but with strain $\epsilon_t$ value smaller than 0.005 but greater than the strain balancing state $\epsilon_t = 0.002$ for Grade 60 steel, or $\epsilon_t = f_y/E_y$. 

3. Compression-controlled case. By initial crushing of the concrete at the compression face. As previously stated, the balanced strain state occurs when failure develops simultaneously in tension and in compression. This condition is defined by the limit strain state \( \varepsilon_c = \varepsilon_t = \varepsilon_e \) at the tension side with the strain \( \varepsilon_t = 0.002 \) for 60 Grade steel.

Accordingly, in analysis and design, the following eccentricity limits correspond to the strain limits presented:

\[
\begin{align*}
\varepsilon_e & \geq \text{limit } \varepsilon_{c,\text{max}} (c = 0.375 d) : \text{tension-controlled} \\
\varepsilon_e & = \varepsilon_{\text{trans}} (c = 0.375 d - 0.60 d) : \text{intermediate transition} \\
\varepsilon_e & = \text{limit } \varepsilon_{c,\text{min}} (c = 0.60 d) : \text{compression-controlled}
\end{align*}
\]

In all those cases, the strain-compatibility relationship must be maintained at all times through the computation of the strain \( \varepsilon_e \) in the compression reinforcement on the basis of linearity of distribution of strain across the concrete section depth. It should be noted that for each limit strain case, there is a unique value of nominal thrust \( P_e \) and nominal moment \( M_e \). Consequently, a unique eccentricity \( e = M_e / P_e \) can be determined for each case.

9.5.4 Initial Tension Failure in Rectangular Concrete Compression Members

For initial yielding of the tensile reinforcement, the limit strain \( \varepsilon_e = 0.005 \) in./in., or higher. The analysis and design procedure requires applying the basic equilibrium Eqs. 9.4 through 9.7, using the trial and adjustment procedure and performing strain compatibility checks at all loading stages. This procedure is summarized in Section 9.4.2. The limit strain in this mode of failure is defined by Equations 9.11a and 9.11b for the depth of the compressive block having a value of \( a = 0.375 b_1 \), \( d \) at the limit strain of 0.005 in./in. in the tension-controlled zone and lower in the transition zone illustrated in Fig. 9.11. The strain in the reinforcement at the tension side is equal or higher than the yield strain, and the stress \( f = f_y \). The column eccentricity \( e \geq e_{\text{min}} \) for tension-controlled and \( e < e_{\text{max}} \) for intermediate cases. It should be noted that if \( P_e > 0.10 f_y A_1 \) or less, the column essentially behaves as a flexural beam because of the low magnitude of the axial force, resulting in a large eccentricity into a strain greater than 0.004. The following examples illustrate the use of the trial and adjustment procedure for the analysis and design of tension-controlled and intermediate compression members.

9.5.5 Example 9.3: Analysis of a Column Subjected to Limit Strain in Compression and in Tension

Calculate the nominal axial load, \( P_e \), in Example 9.1 for the compression- and tension-controlled strain limits, if the column shown in Figure 9.11 is subjected to combined axial load and bending. Given:

- \( b = 12 \) in. 
- \( A_t = A_t' = 3.0 \) in.
- \( d = 17.5 \) in. 
- \( f_y = 60,000 \) psi
- \( n = 20 \) in. 
- \( f_c = 4000 \) psi

Solution:

(a) Limit compression-controlled case

This is the balanced strain condition where the steel reinforcement yields simultaneously with the crushing of the concrete at the external compression fibers.

Yield strain \( \varepsilon_e = \frac{f_y}{E_y} = \frac{60,000}{29,000} = 0.002 \) in./in.

From Equation 9.14b,

\[
\varepsilon = \varepsilon_y - \frac{f_y}{E_y} (\frac{1}{A_t} - \frac{1}{A_t'}) = \varepsilon_y (1 - \frac{A_t'}{A_t}) = \varepsilon_y (1 - \frac{1.57}{3.0}) = 0.002 \times 0.57 = 0.001 \text{ in.}
\]
Figure 9.11 Column geometry: strain and stress diagrams (balanced failure); (a) cross-section; (b) balanced strains; (c) stress.

Therefore, \( f_e = f_s = 60,000 \, \text{psi} \)

\( \frac{S}{d} = 0.60 \) for limit strain in compression \( (\epsilon_s = 0.002 \, \text{in./in.}) \)

From Equation 9.12b,

\[ a = 0.60 \, B_d = 0.60 \times 0.85 \times 17.5 = 8.95 \, \text{in.} \]

From Equation 9.6,

\[ P_e = 0.85 f_e; b = a + A I' = A f_s \]

where, \( f_s = f_e \) for the transition zone limit of \( \epsilon_e = 0.002 \, \text{in./in.} \)

\[ P_e = 0.85 	imes 4.000 \times 12 \times 8.93 = 3.0 \times 60,000 = 364,340 \, \text{lb.} \]

This is synonymous with the balanced condition used in ACI-318, Appendix B.

From Equation 9.7,

For rectangular sections, the geometric centroid \( y = A/2 \).

\[ M_e = P_e \epsilon_e = 0.85 f_e h (\frac{2}{2} - \frac{2}{2}) + A f_s (\frac{2}{2} - \epsilon_e) = A f_s \left( 2 - \frac{2}{2} \right) \]

\[ = 364,340 \left( \frac{20}{2} - \frac{8.93}{2} \right) + 3.0 \times 60,000 \left( \frac{20}{2} - 2.5 \right) + 3.0 \times 60,000 \left( 17.5 - 20 \right) \]

\[ = 364,340 \times 5.54 + 180,000 \times 7.5 + 180,000 \times 7.5 = 4,718,444 \, \text{in.-lb} \]

Eccentricity \( \epsilon_e = \frac{4,718,444}{364,340} = 13.0 \, \text{in.} \) \( (330 \, \text{mm}) \)

(b) Limit tension-controlled case

Yield strain \( \epsilon_s = 0.005 \, \text{in./in.} \)

From Equation 9.12b,

\[ f_e = 4; E_s = 87,000 \left( \frac{1 - 2.07 \frac{d}{d_s}}{2} \right) = 87,000 \left( \frac{1 - 2.07 \times 2.55}{17.5} \right) = 33,816 \, \text{psi} < f_e \]

\[ \frac{S}{d} = 0.375 \, \text{for limit strain in tension} \ (\epsilon_s = 0.005 \, \text{in./in.}) \]

From Equation 9.11b,

\[ a = 0.375 \, B_d = 0.375 \times 0.85 \times 17.5 = 5.58 \, \text{in.} \]
Photo 9.2 Eccentrically loaded column at ultimate state of failure. (Tests by Nowy et al.)

From Equation 9.6,

\[ P_u = 0.85 f_u b d - A_t f_t \]

\[ = 0.85 \times 4000 \times 12 \times 5.58 + 30 \times 53,816 - 30 \times 60,000 \]

\[ = 227,664 + 161,445 - 180,000 = 209,112 \text{ lb} \]

From Equation 9.7,

\[ M_u = P_u \frac{d}{2} - A_t f_t \left( \frac{d}{2} - t \right) - A_t f_t \left( d - t - \frac{d}{2} \right) \]

\[ = 227,664 \left( \frac{20}{2} - \frac{5.58}{2} \right) + 161,445 \left( \frac{20}{2} - 2.56 \right) + 180,000 \left( 17.5 - \frac{20}{2} \right) \]

\[ = 227,664 \times 7.21 + 161,445 \times 7.5 + 180,000 \times 7.5 = 4,202,317 \text{ in.-lb} \]

Eccentricity \( e = \frac{4,202,317}{209,112} = 30 \text{ in.} \) (510 mm)

Note that as the column strains move towards the tension-controlled zone, the eccentricity increases, resulting in a tensile mode of failure.

9.5.6 Example 9.4: Analysis of a Column Controlled by Initial Tension Failure; Stress in Compression Steel Near Yield Strength

Calculate the nominal axial load strength \( P_u \) of the section in Ex. 9.1 (see Figure 9.12) if the load acts at an eccentricity \( e = 16 \text{ in.} \) (400 mm). Given:

\[ b = 12 \text{ in.} \]

\[ d = 12.5 \text{ in.} \]
Figure 9.12. Column geometry: strain and stress diagrams (tension failure): (a) cross section; (b) strains; (c) stresses.

\( h = 20 \text{ in.} \)
\( d' = 2.5 \text{ in.} \)
\( A_e = A'_e = 3 \text{ in.}^2 \)
\( f_c = 4000 \text{ psi} \)
\( f_y = 60,000 \text{ psi} \)

Solution:
Required \( c = 15 \text{ in.} \)
\( d = 17.5 \text{ in.} \)
From Example 9.3, limit \( c_e = 13.0 \text{ in.} \) and \( e = 20.1 \text{ in.} \)
Hence, the strain in the column is in the transition zone of Figure 9.10.
At the tension side, \( f_y = f_y = 60,000 \text{ psi} \).

First trial
Assume for the first trial and adjustment procedure that \( \phi = 0.44 \) to be subsequently verified by demonstrating that the resulting eccentricity is equal to the required eccentricity.
\( e = 0.44 \times 17.5 = 7.70 \text{ in.} \); hence, \( e = 7.70 \) in.
\( c_e = 0.03 \left( \frac{d}{e} - 1 \right) = 0.03 \left( \frac{17.5}{7.70} - 1 \right) = 0.00382 \), hence this case is in the transition zone, with the tension face yielding.

From Equation 9.12a,
\( f_c' = c \cdot f_y = 87,000 \left( 1 - \frac{e}{c} \right) = 87,000 \left( 1 - \frac{7.70}{17.5} \right) = 58,753 \text{ psi} < f_c \)

Also, from Figure 9.8,
\( C_o = 0.45 f_y' = 0.45 \times 4000 \times 12 \times 6.55 = 267,240 \text{ lb.} \)
\( C_i = A_i f_c = 3.0 \times 58,753 = 176,259 \text{ lb.} \)
\( T_y = A_e f_y = 60,000 \times 20.1 = 1,206,000 \text{ lb.} \)

From Equation 9.4,
\( P_e = C_o + C_i - T_y = 267,240 + 176,259 - 1,206,000 = 263,499 \text{ lb.} \)
From Equation 9.5,

\[ M_e = C_e \left( \frac{y - d}{2} \right) + C_s (y - d') + T_e (d - \bar{d}) \]

where \( y = \frac{b}{6} = \frac{20}{6} = 3.33 \) in.

Therefore,

\[ M_e = 267.240 \left( 10 - 6.5 \frac{3}{2} \right) + 176.259 (10 - 2.5) + 180.000 (17.5 - 10) = 4,469.132 \text{ in.-lb} \]
\[ e = \frac{4,469.132}{263.449} = 17.0 \text{ in.} > r = 16 \text{ in.} \]

Hence revise solution, assuming a larger \( e / \bar{d} \) value for a second cycle, to increase the compression area in the section, hence, a lower eccentricity.

**Second trial**

Assume \( e / \bar{d} = 0.47 \)

\[ r = 0.47 \times 17.5 = 8.23 \text{ in.} \]

\[ a = 11, c = 0.85 \times 8.23 = 7.0 \text{ in.} \]

From Equation 9.12a,

\[ f_c = E / E_c = 87,000 \left( 1 - \frac{d'}{c} \right) = 87,000 \left( 1 - \frac{2.5}{8.23} \right) = 60,872 \text{ psi} > f_c \]

Hence, \( f_c = f_c = 60,000 \text{ psi} \)

\[ C_s = 0.85 f_c / b = 0.85 \times 6,000 \times 12 \times 7.0 = 285,600 \text{ lb} \]

\[ P_c = C_s + C_e - T_e = C_s \text{ since } C_e = T_e = 0 \]

\[ P_c = 285,600 \text{ lb} \]

\[ M_e = 285,600 \left( 10 - 7.0 \frac{3}{2} \right) + 180,000 (10 - 2.5) + 180,000 (17.5 - 10) = 4,556,400 \text{ in.-lb} \]
\[ e = \frac{4,556,400}{285,600} = 16.29 \text{ in.} > e = 16 \text{ in.} \text{ O.K.} \]

\( e < e_{\text{lim}} > e_{\text{trans}} \text{ since } c > 0.375 d, \text{ hence tension-controlled (transition zone).} \)

Compatibility of strain is satisfied, using the applicable \( f_c / b \) in the compression reinforcement.

Therefore, the nominal axial load for this column is \( P_c = 285,600 \text{ lb} \) (1,270 kN).

**Example 9.5: Analysis of a Column Controlled by Initial Tension Failure; Stress in Compression Steel Less Than Yield Strength**

A non-sleender, rectangular, reinforced concrete column is 12 in. x 15 in. (305 mm x 381 mm), as shown in Figure 9.13, and is subjected to a load eccentricity \( e = 10 \text{ in.} \) (254 mm). Calculate the safe nominal load strength \( P_c \) and the nominal moment strength \( M_c \) of the column section. Given:

\[ f_c = 6,000 \text{ psi} (41.4 \text{ MPa}) \text{, normal-weight concrete} \]

\[ f_s = 60,000 \text{ psi} (414 \text{ MPa}) \]

three No. 9 bars (28.7 mm diameter) for each of the compression and tension reinforcements.

**Solution**

**Required** \( e = 10 \text{ in.} \)

\( d_e = d = 12.5 \text{ in.} \)

**First trial**

Assume the first trial and adjustment procedure that \( e / d = 0.50 < 0.60 > 0.375, \text{ hence the} \)

column is in the transition zone, with the stress in the tensile reinforcement is hence

\[ f_s = 0.6 \]
Figure 9.13 Column geometry: strain and stress diagrams (tension failure).

- $f_t < f_e$ (c) cross section; (d) strains; (e) stresses (Example 9.5).

- $c = 0.30 \times 12.5 = 6.25$ in.
- $b_t = 0.85 - 0.03 \left( \frac{6.000 - 4.000}{1.000} \right) = 0.75$
- $\varepsilon = b_t, c = 0.75 \times 6.25 = 4.69$ in. to be subsequently verified.

From Equation 9.12a,

\[ f' = \varepsilon E, \quad E = 87,000 \left( 1 - \frac{\varepsilon}{c} \right) = 87,000 \left( 1 - \frac{2.5}{6.25} \right) = 52,200 \text{ psi} \]

From Figure 9.8,

\[ C_t = 0.85 f' b; a = 0.85 \times 6.000 \times 12 \times 4.69 = 287,628 \text{ lb} \]
\[ C_t = A_t f'; c = 3.0 \times 52,200 = 156,600 \text{ lb} \]
\[ T_t = A_t f'; d = 3.0 \times 60,000 = 180,000 \text{ lb} \]

From Equation 9.4,

\[ P_e = C_t + C_t - T_t = 287,628 + 156,600 - 180,000 = 263,628 \text{ lb} \]
\[ \frac{d}{2} = \frac{15}{2} = 7.5 \text{ in.} \]

From Equation 9.5,

\[ M_t = C_t \left( \frac{d}{2} \right) + C_t (d - \frac{d}{2}) + T_t (d - \frac{d}{2}) \]
\[ = 287,628 \left( 7.5 - \frac{4.69}{2} \right) + 156,600 (7.5 - 2.5) + 180,000 (12.5 - 7.5) \]
\[ = 287,628 \times 5.16 + 156,600 \times 5.0 + 180,000 \times 5.0 = 3,164,064 \text{ in.-lb} \]

\[ r = 3,164,064 \times 12 = 37,968,768 \text{ in.-lb} \]
\[ = 263,628 \text{ in.-lb} \]

From Equation 9.6,

\[ S = \frac{1}{2} d \times 12 \times 10 = \frac{1}{2} \times 10 = 50 \text{ in.} \]

From 9.5 Strain Limits Method to Establish Reliability Factor. 

Assume $c/d = 0.50$, hence column is in the tensile-controlled transition zone, with the tensile reinforcement stress $f_t = 60,000$ psi.
\[ e = 0.59 \times 12.5 = 7.38 \text{ in.} \]
\[ a = b = 0.75 \times 7.38 = 5.54 \text{ in. to be subsequently verified.} \]
\[ f' = \frac{E}{E'} = \frac{67,000}{67,000} = 67,000 \left(1 - \frac{d'}{c}\right) = 75,328 \text{ psi} < f. \]
\[ C_e = 0.85 \frac{f' e}{2} = 0.85 \times 6000 \times 12 \times 5.54 = 339,048 \text{ lb.} \]
\[ C_e = A' f' = 3.0 \times 57,328 = 171,984 \text{ lb.} \]
\[ T_e = A' f' = 2.5 \times 6000 = 150,000 \text{ lb.} \]
\[ P_e = C_e + T_e = 339,048 + 171,984 = 180,000 = 331,642 \text{ lb.} \]
\[ M_e = C_e \left(\frac{2 - d}{2}\right) + C_e (c - d') + T_e (d - c) \]
\[ = 339,048 \left(7.5 - \frac{5.54}{2}\right) + 171,984 (7.5 - 2.5) + 180,000 (12.5 - 7.5) \]
\[ = 339,048 \times 4.73 + 171,984 \times 5.0 + 180,000 \times 5.0 = 3,396,097 \text{ in.-lb} \]
\[ e = \frac{3,396,097}{331,642} = 10.1 \text{ in.} = \text{required} \]

Note that the eccentricity of the axial load as defined in Eq. 9.15 is less than \( e_{	ext{res}} \) but larger than \( e_{	ext{cr}} \), since the depth of the neutral axis is smaller than \( d_e \), hence this is a tension (tension) case with initial yield of the tension face stress. Strain compatibility is satisfied, using the stress value \( f' \) of the compression reinforcement compatible with the depth \( e \) of the neutral axis. Therefore, nominal axial load \( P_e = 331,642 \text{ lb} \) (474 kN), and nominal moment \( M_e = 3,396,097 \text{ in.-lb} \) (381 kN\( \cdot \text{m} \)).

### 9.5.8 Initial Compression Failure in Rectangular Concrete Compression Members

For initial crushing of the concrete, the limit strain in the tensile reinforcement has to be \( e_c = 0.002 \text{ in.} / \text{in.} \) or lower. This results in a stress in the tensile reinforcement to be below the yield strength, namely, \( f < f_c \). The analysis (design) process necessitates applying the basic equilibrium Eqs. 9.4 through 9.7, using the trial and adjustment procedure and ensuring strain compatibility checks at all loading stages. The procedure is summarized in Section 9.4.2. The limit strain in this mode of failure is defined by Equations 9.12a and 9.12b for the depth of the compressive block for a minimum value of \( a = 0.61 \text{ in.} \). For larger values of \( a \), the strain in the tensile reinforcement becomes less than the yield strain, and the stress \( f < f_c \). The stress \( f_c \) for the first trial is obtained from Equation 9.10b, where

\[ f_c = \frac{E}{E'} \left(\frac{d}{c} - 1\right) < f. \]

If only one layer of tension reinforcement is used, \( d \) becomes equal to \( d_e \) for strain in the extreme tension reinforcement. The following examples illustrate the use of the trial and adjustment procedure for the analysis and design of compression-controlled compression members.

### 9.5.9 Example 9.6: Analysis of a Column Controlled by Compression Failure;

Calculate the nominal load \( P_0 \) of the section in Ex. 9.1 (see Figure 9.14) if the column is subjected to a load eccentricity \( e = 10 \text{ in.} \) (254 mm). Given:

\[ h = 12 \text{ in.} (305 \text{ mm}) \]
\[ d = 17.5 \text{ in.} (445 \text{ mm}) \]
\[ h = 20 \text{ in.} (508 \text{ mm}) \]
9.5 Strain Limits Method to Establish Reliability Factor \( \Phi \)

\[ d' = 2.5 \text{ in.} \]
\[ A_t = A_c = 3.0 \text{ in.}^2 (1940 \text{ mm}^2) \]
\[ f'_c = 4000 \text{ psi (27.6 MPa)} \]
\[ f_y = 60,000 \text{ psi (414 MPa)} \]

**Solution:** Using the results of Eq. 9.3, eccentricity for the limit compression-controlled behavior is larger than the given eccentricity of 10 in. Therefore, failure will occur by initial crushing of concrete at the compression face, as the depth \( c \) of the neutral axis has to be larger than 0.60 \( d' \).

**First trial**

Assume \( \alpha = 0.66 \)
\[ c = 0.66 \times 17.5 = 11.55 \text{ in.} \]
\[ a = \beta, \ e = 0.35 \times 11.55 = 9.82 \text{ in.} \text{ to be subsequently verified.} \]

From Equation 9.14b
\[ f'_c = f_y \]
\[ E_t, E_c = 57,000 \left(1 - \frac{n}{d'}\right) = 57,000 \left(1 - \frac{1.67 \times 2.5}{17.5}\right) = 62,244 \text{ psi} \]

Therefore \( f'_c = f_y \)

Since the behavior is compression-controlled, the strain in the tension reinforcement is below yield strain.

Use Eq. 9.17c for strain compatibility to find \( f'_c \) in the tensile reinforcement.
\[ f'_c = E_t, \varepsilon_t = 0.003 \times E_t, \left(\frac{d'}{d} - 1\right) = 0.003 \times 29 \times 10^3 \left(\frac{17.5}{11.5} - 1\right) = 45.391 \text{ psi} \]

\[ P_t = 0.65 \times f' c \times b d + A_t, f'_c = A_t, f_y \quad (b d - \frac{d}{2}) \]
\[ = 0.85 \times 4000 \times 12 \times 9.82 + 3.0 \times 60,000 - 3.0 \times 45.391 \]
\[ = 480.656 + 180,000 - 136,173 = 444,483 \text{ lb} \]

\[ M_t = P_t, \varepsilon_t = 0.65 \times f' c \times b d + A_t, f'_c \left(\frac{d}{2} - \frac{d}{2}\right) + A_t, f_y \left(\frac{d}{2} - \frac{d}{2}\right) \]
\[ = 480.656 \left(\frac{20}{2} - \frac{9.82}{2}\right) + 180,000 \left(\frac{20}{2} - 2.5\right) + 136,173 \left(17.5 - \frac{20}{2}\right) = 4,106,937 \text{ in.-lb} \]
Chapter 9  Combined Compression and Bending: Columns

Photo 9.3  Compression side of eccentrically loaded column at failure. (Tests by Navvy et al.)

Excentricity $e = 4,410.637/444.483 = 9.92$ in. $< 10$ in. $e = 10$ in. Hence decrease the neutral axis depth in order to increase the excentricity, through decreasing the volume of the concrete compressive block.

Second trial
Assume $c/d_t = 0.655$

$e = 0.655 \times 17.5 = 11.46$ in.

$a = b_t e = 0.85 \times 11.46 = 9.74$ in.

$f' = 60,000$ psi from trial 1.

$f_t = 87,000 \left( \frac{b_t}{c} - \frac{1}{2} \right) = 87,000 \left( \frac{17.5}{11.46} - \frac{1}{2} \right) = 45,853$ psi

$P_a = 0.85 f' b_t a = 45,853 \times 9.74 \times 12 \times 9.74 + 3 \times 400.00 = 45,853$ psi

$M_a = 397,392 \left( \frac{12}{2} \right) = 236,010 \left( \frac{20}{2} - 2.5 \right) + 137,599 \left( 17.3 - \frac{20}{2} \right) = 4,420,313$ in.-lb

Excentricity $e = \frac{4,420,313}{440,000} = 10.05$ in. $e = 10$ in. O.K.
9.5 Strain Limits Method to Establish Reliability Factor $b$

9.5.10 General Case of Columns Reinforced on All Faces: Exact Solution

In cases where columns are reinforced with bars on all faces and those where the reinforcement in the parallel faces is nonsymmetrical, solutions have to be based on using first principles. Eqs. 9.6 and 9.7 have to be adjusted for this purpose and the trial-and-adjustment procedure adhered to. 

Strain-compatibility checks for strain in each reinforcing bar layer have to be performed at all load levels.

Figure 9.15 illustrates the case of a column reinforced on all four faces. Assume that

- $G_c$: center of gravity of the steel compressive force
- $G_y$: center of gravity of the steel tensile force
- $F_c$: resultant steel compressive force $= \Sigma A_c f_c$
- $F_y$: resultant steel tensile force $= \Sigma A_y f_y$

Equilibrium of the internal and external forces and moments requires that:

$$P_0 = 0.85 fh_b G_c + F_{cc} - F_c$$  \hspace{1cm} (9.16a)

$$F_y - 0.85 fh_b G_y = F_{yc} + \frac{1}{2} b_c c$$  \hspace{1cm} (9.16b)

for moments about the geometric centroid.

Trial and adjustment is applied assuming a neutral-axis depth $c$ and consequently a depth $\delta$ of the equivalent rectangular block. The strain values in each bar layer are determined by the linear strain distribution in Figure 9.15b to ensure strain compatibility. The stress in each reinforcing bar is obtained using the expression

$$f_n = E_{sy} = E_s \frac{f_y}{c} - 87,000 \frac{\delta}{c}$$  \hspace{1cm} (9.16c)

where $f_n$ has to be $\leq f_y$.

Find $P_0$ corresponding to the assumed $c$ in Eq. 9.16a. Substitute into Eq. 9.16b the $P_0$ value thus obtained with the parameter $c$ as the unknown. If the resulting $c$ is not close to the assumed value, proceed to another trial. The nominal resisting load $P_0$, of the section would be the one corresponding to the trial depth $c$ of the last trial cycle.

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Figure 9.15 Column reinforced with steel on all faces: (a) cross-section; (b) strain; (c) forces.
9.5.11 Circular Columns

The angle $\theta$ subtended by the compressive block chord shown in Figure 9.16(b) is

**Case 1:**

\[ a \leq \frac{b}{2}, \theta < 90^\circ \]

\[ \theta = \cos^{-1} \left( \frac{b/2 - a}{b/2} \right) \]  \hspace{1cm} (9.17a)

**Case 2:**

\[ a > \frac{b}{2}, \theta > 90^\circ \]

\[ \theta = \cos^{-1} \left( \frac{b/2 - a}{b/2} \right) \quad \text{and} \quad \phi = \cos^{-1} \left( \frac{a - h/2}{h/2} \right) \]  \hspace{1cm} (9.17b)

The area of the compressive segment of the circular column in Figure 9.16(b), is

\[ A_c = \pi \left( \frac{h^2}{2} - \frac{a^2}{4} \cos \theta \right) \]  \hspace{1cm} (9.18a)

where $\theta$ is in radians (1 radian = 180°/π = 57.3°)
9.5 Strain Limit Method to Establish Reliability Factor $\phi$

$$\phi = \frac{\sigma}{\sigma_y}$$

![Diagram of circular column with strain, stress, and compressive block segment]

(a) Centeroid $G_{CP}$ of compression zone

(b) Case 1: $a \leq \frac{h}{2}, \theta < 90^\circ$

$$a = \cos^{-1} \left( \frac{h/2 - a}{h/2} \right)$$

Case 2: $a > h/2, \theta > 90^\circ$

$$\theta = 180^\circ - a$$

$$\phi = \cos^{-1} \left( \frac{h/2 - a}{h/2} \right)$$

Figure 9.16 Circular column (a) strain, stress, and compressive block segment; (b) compressive segment chord $x$-axis geometry

The moment of area of the compressive segment about the center of the column is

$$M = \frac{h^3}{12} \int_{a}^{b} y \, dA$$

where $y$ = distance of the centroid of the compressive block to the section centroid

$$d_i = \frac{h}{2} \gamma \frac{h_i}{2} \cos \theta_{CP}$$

where $\gamma = (h - 2x^2)/h$

$$f' = 87,000 \left( 1 - \frac{d_i}{c} \right) \leq f$$

where $f'$ = stress in bars within the compressive zone.
\[ f_a = 87,000 \left( \frac{d}{c} - 1 \right) \leq f_y \]  
\hspace{2cm} (9.1.9c)

where \( f_a \) = stress in bars within the tension zone below the neutral axis

\[ P_a = 0.85 \bar{f}_d \bar{A} + \bar{f}_d \bar{A}_d \]  
\hspace{2cm} (9.2.0a)

\[ M_a = 0.85 \bar{f}_d \bar{A} \bar{y} + 2 \bar{f}_d \bar{A}_d \left( \frac{d}{2} - d' \right) \]  
\hspace{2cm} (9.2.0b)

(moment taken about the circular column center).

The ACI-318 code requires that at least six bars be used in spiral columns. A useful model for any even number of bars in circular column sections can be derived with six basic locations of bars, 60° apart, as seen in the ensuing design example.

Note that in order to simplify the strain-compatibility computations, and the equilibrium of forces and moments, is both the rectangular sections with bars on all faces and the circular sections, the individual stress, force and moment for each bar has to be computed separately and tabulated (see Example 9.9).

9.6 WHITNEY’S APPROXIMATE SOLUTION IN LIEU OF EXACT SOLUTIONS

Empirical expressions proposed by Whitney can be used for rapidity in lieu of the trial-and-adjustment method, although with some loss in accuracy.

9.6.1 Rectangular Concrete Columns

These expressions are presented particularly for circular columns, since longhand trial-and-adjustment procedures for their analysis or design can be time consuming. Strain-compatibility checks for the reinforcement require that the strain in the bars be evaluated at each level across the depth of the section; hence the use of an empirical one-step expression becomes useful for a quick first trial analysis. The use of hand-held or desk-top computers reduces or preempts the need for the use of the Whitney empirical approach for designers familiar with computer use. Appendix A applies the exact method of analysis and design of circular and rectangular columns with strain-compatibility checks at all stages through the use of personal computer programs developed for this purpose.

Whitney’s solution is based on the following assumptions:

1. Reinforcement is symmetrically placed in single layers parallel to the axis of bending in rectangular sections.
2. Compression steel has yielded.
3. Concrete displaced by the compression steel is negligible compared to the total concrete area in compression; hence no correction is made for the concrete displaced by the compression steel.
4. For the purpose of calculating the contribution \( C_c \) of the concrete, the depth of the stress block is assumed to be 0.5 \( c \), corresponding to an average value of \( c \) for balanced conditions in rectangular sections.
5. The interaction curve in the compression zone is a straight line.

For most cases, Whitney’s method leads to a conservative solution, except when the factored load \( \bar{P}_f \) has a value higher than the balanced case divined by the limit strain for
tension-controlled states, as in Ex. 9.7(b), and the external eccentricity ε is very small. Otherwise, the method leads to a nonconservative solution.

Essentially, the Whitney Approach is presented here for its intrinsic classical value and for the choice of a first trial section before proceeding with an exact analysis. If compression controls, the equation for rectangular sections can be written as

\[ P_c = \frac{A_f}{\varepsilon + \frac{bf}{M(e^2 + 1.18)}} \]  

(3.21)

The following example illustrates the use of this equation.

9.6.2 Example 9.7: Analysis of a Rectangular Column Controlled by Compression Failure; Whitney’s Equation

Calculate the nominal strength load \( P_c \) for the section in Ex. 9.6 using Whitney’s equation if the load eccentricity is \( (r_e = 6 \text{ in.}) \) and \( (r = 10 \text{ in.}) \). \( (254 \text{ mm}) \). \( (254 \text{ mm}) \).

Solution: \( (3) r_e = 6 \text{ in.} \)

\[ P_c = \frac{3 \times 60,000}{(3/17.5 - 3.5) \times 0.5} + \frac{12 \times 29 \times 4000}{(13 \times 29 \times 41.5) + 1.18} = 697,555 \text{ lb} \ (2734 \text{ kN}) \]

Exact solution, with trial and adjustment and including the displaced concrete gives \( P_c = 698,000 \text{ lb} \ (2738 \text{ kN}) \). The approximate solution is conservative.

(9) \( r = 10 \text{ in.} \) Using Eq. 9.21.

\[ P_c = \frac{3 \times 60,000}{(10/17.5 - 2.5) \times 0.5} + \frac{12 \times 20 \times 4000}{(13 \times 20 \times 41.5) + 1.18} = 469,098 \text{ lb} \ (2070 \text{ kN}) \]

The exact solution, using the trial and adjustment procedure and including the effect of the displaced concrete, gives \( P_c = 433,138 \text{ lb} \ (1900 \text{ kN}) \), showing that the approximate solution is not always conservative, as discussed above.

9.6.3 Circular Concrete Columns

As in the case of rectangular columns, force and moment equilibrium equations can be used to solve for the unknown nominal axial load \( P_c \) for any given eccentricity. The equilibrium equations are similar to Eqs. 9.8 and 9.7 except that (1) the shape of the area under compressive stress will be a segment of a circle, and (2) reinforcing bars are not grouped together parallel to the compression side tension side. Therefore, the force and stress in each bar should be considered separately. The area of the segment of a circle in compression should be calculated using the appropriate mathematical expressions. This accurate approach can be easily adapted if we choose to use hand-held or desktop computers. The following simplified empirical Whitney’s approach can be used for longhand calculations, as a first trial.

9.6.4 Empirical Method of Analysis of Circular Columns

Transform the circular column to an idealized equivalent rectangular column as shown in Ex. 9.8 and Figure 9.17. For compression failure, the equivalent rectangular column would have (1) the thickness in the direction of bending equal to 9.86, where it is the outside diameter of the circular column (Figure 9.17(a)); (2) the width of the idealized rectangular column to be obtained from the same gross area \( A_f \) of the circular column such
Figure 9.17 Equivalent column section: (a) given circular section ($A_u$, total reinforcement area), (b) equivalent rectangular section (compression failure), (c) equivalent column (tension failure).

that $b = A_t/0.8t$, and (3) the total area of reinforcement $A_t$ to be equally divided in two parallel layers and placed at a distance of $2D_t/3$ in the direction of bending, where $D_t$ is the diameter of the cage measured center to center of the inner vertical bars. For tension failure, use the actual column for evaluating $C_t$, but place 40% of the steel $A_t$ in parallel at a distance of $0.75D_t$ as shown in Fig. 9.17. The equivalent column method provides satisfactory results for most cases.
9.6 Whitney’s Approximate Solution in Lieu of Exact Solutions

Once the dimensions of the equivalent rectangular column are established, the analysis (design) can be made as for rectangular columns.

The equations for tension and compression failure can also be expressed in terms of the dimensions of the circular column, as follows:

For tension failure,

\[ P_t = 0.85 f_t \left[ \sqrt{\left( \frac{0.85 e}{h} - 0.38 \right)^2 + \frac{0.67 D_t}{2.5 h} - \left( \frac{0.85 e}{h} - 0.38 \right)} \right] \]  \hspace{1cm} (9.22)

For compression failure,

\[ P_c = \frac{A_{sf}}{(3e/D_t) + 1.00 + \left[ \frac{9.6 e(h/0.8 h) + 0.67 D_t}{2} \right]^2 + 1.18} \]  \hspace{1cm} (9.23)

where \( h \) = diameter of section

\( D_t \) = diameter of the reinforcement cage center to center of the outer vertical bars

\( e \) = eccentricity to the plastic centroid of section

\( A_{gs} = \frac{A_g}{A_t} \) = gross steel area

\( A_{gc} = \frac{A_g}{A_c} \) = gross concrete area

\( m = \frac{f_c}{0.85 f_t} \)

9.6.5 Example 9.8: Calculation for Equivalent Rectangular Cross Section for a Circular Column

Obtain an equivalent rectangular cross section for the circular column shown in Figure 9.17(a). Assume that \( D_t = 15 \text{ in.} \)

Solution:

- thickness of the rectangular section = \( 0.8 \times 20 = 16 \text{ in.} \)
- width of the rectangular section = \( \frac{(20)^2}{16} = 19.6 \text{ in.} \)
- \( d - d' = \frac{2}{3} \times 15 = 10 \text{ in.} \)
- \( A_t = A_{t'} = \frac{A_t}{2} \)

9.6.6 Example 9.9: Analysis of a Circular Column

A concrete circular column 20 in. (508 mm) in diameter is reinforced with eight No. 8 equally spaced bars, as shown in Figures 9.18 and 9.19. Compute using the Whitney approach the nominal axial load \( P_t \) for (a) eccentricity \( e = 16.0 \text{ in.} \) (406 mm) and (b) eccentricity \( e = 5.0 \text{ in.} \) (127 mm). Additionally, solve as part (c) of this example, the nominal axial load \( P_n \) for part (a) using the limit strain compatibility method, and compare the resulting design axial load with the value obtained from the Whitney solution.

Given: \( f_t = 4000 \text{ psi (27.6 MPa)} \)

\( f_c = 60,000 \text{ psi (414 MPa)} \)
Figure 9.18 Column geometry: strain and stress diagrams (balanced failure): (a) equivalent section; (b) strains; (c) stresses (Example 9.8).

Solution:

(a) Whitney solution: large eccentricity

\[ e = 16 \text{ in.} \text{, with axial load 6 in. outside the circular section} \]

It can be assumed that the section is tension-controlled, because of the large eccentricity compared to the section depth; to be subsequently verified as in part (c) of the solution

\[ P_t = 20 - 2 \times 2.5 = 15 \text{ in.} \]

\[ \varepsilon = \frac{A_t}{A_s} = \frac{2 \times 2.37}{\pi / 4} = 4.74 \frac{4.74}{312} = 0.15 \]

Using Equation 9.25:

\[ m = \frac{f}{0.85} = \frac{60,000}{0.85 \times 4000} = 17.65 \]

\[ P_e = 0.85 \times 4000 \times 400 \left( \sqrt{\frac{70.85 \times 16}{20} - 0.38} \right) + 0.015 \times 47.65 \times 15 \]

\[ - \left( \frac{0.85 \times 15}{20} - 0.38 \right) \]

\[ = 151,793 \text{ lb (672 kN)} \]

(b) Whitney solution: small eccentricity

For \( e = 5.0 \text{ in.} \lt e_t \), compression failure controls since the axial load is within the section; (\( e_t \) = eccentricity for balanced strain, namely \( e_t = 0.09 \), or \( e_t = 0.001 \)). Using Eq. 9.23, we have:

Total steel area \( A_s = A_{ss} + A_{ss} = 2 \times 3.37 = 4.74 \text{ in.}^2 (30.57 \text{ mm}^2) \)

Gross concrete area \( A_c = \frac{m(20.67)^2}{4} = 314.3 \text{ in.}^2 (2013 \text{ mm}^2) \)

\[ P_e = \frac{4.74 \times 60,000}{3 \times 5.0 \times 15} = \frac{314.2 \times 4000}{(0.8 \times 20 \times 3) + 1.18} \]

\[ = 650,777 \text{ lb (2910 kN)} \]
9.6 Whitney's Approximate Solution in lieu of Exact Solutions

(Under strain compatibility \( P_e = 621.653 \text{ lb}, \) indicating that the Whitney solution is in this case not conservative.)

(d) Exact strain-compatibility analysis
Column diameter \( h = 20 \text{ in.} \)
Cover to center of the bars centerline. \( a' = 3.5 \text{ in.} \)
\( A_e = A_e' = 3 \text{ in}^2 \text{.} \) 5 bars. \( e = 2.37 \text{ in.} \)

Since six bars are used equally spaced in the circular section, the angle subtended by each bar is:

\[
\bar{\theta} = \frac{360}{6} = 60^\circ
\]

Trial and adjustment is applied to determine the correct neutral axis depth and develop the strain-compatibility solution.

**First Trial**
Assuming \( e = 7.4 \text{ in.} \), resulted in \( P_e = 215.700 \text{ lb} \),

\[
M_e = 2,442,693 \text{ in.-lb}, \quad e = 13.10 \text{ in.} < \text{required } e = 16.0 \text{ in.}
\]

In order to get a larger eccentricity, the compressive block area has to be reduced by assuming a smaller neutral axis depth \( e \).

**Second Trial**
Assuming \( e = 6.9 \text{ in.} \).

\[
\sigma = \sigma_0, \quad e = 0.05 \times 6.9 = 5.865 \text{ in.}
\]

From Equation 9.17a,

\[
\theta = \cos^{-1} \left( \frac{h/2 - a}{h/2} \right)
\]

\[
= \cos^{-1} \left( \frac{10 - 5.865}{10} \right) = \cos^{-1} 0.4235 = 65.58^\circ
\]

\[
= \frac{65.58}{57.3} = 1.144 \text{ radians.}
\]

From Equation 9.18a,

\[
A_e = h' \left( \frac{h - 2 - \sin \theta \cos \theta}{4} \right)
\]

\[
= (20)^2 \left( \frac{1.144 - 0.736}{4} \right) = 76.5 \text{ in.}^2
\]

From Equation 9.18b,

\[
A_e, y = h' \left( \frac{\sin \theta}{12} \right) = \left( \frac{20}{12} \right)^2 \times 0.736 = 50.3 \text{ in.}^2
\]

<table>
<thead>
<tr>
<th>Bar No.</th>
<th>( d' ) (in.)</th>
<th>( h/2 - d' ) (in.)</th>
<th>( f_a ) (psi)</th>
<th>( \Sigma F_a ) (lb)</th>
<th>( \Sigma M_e ) (in.-lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 2</td>
<td>3.5</td>
<td>6.5</td>
<td>42,870</td>
<td>647,753</td>
<td>439,628</td>
</tr>
<tr>
<td>3 + 4</td>
<td>10.0</td>
<td>0</td>
<td>30,087</td>
<td>61,378</td>
<td>0</td>
</tr>
<tr>
<td>5 + 6</td>
<td>16.5</td>
<td>6.5</td>
<td>66,000</td>
<td>-94,800</td>
<td>616,290</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>( \Sigma ) = -685,823</td>
<td>1,055,928</td>
</tr>
</tbody>
</table>
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Figure 9.19  Column geometry in Example 9.9 (a) strain distributions; (b) column cross section.

From Equation 9.20(a),

\[ P_0 = 0.85 f_e A_e + \sum f_i A_i \]

\[ = 0.85 \times 4,000 \times 71.8 = 88,823 \text{ lb.} \]

From Equation 9.20(b),

\[ \frac{M_x}{P_0} = 0.85 f_e A_e + \sum f_i A_i \left( \frac{1}{\lambda} - \sigma \right) \]

\[ = 0.85 \times 4,000 \times 503.3 + 1,555,829 = 2,767,048 \text{ in.-lb} \]

\[ \epsilon = \frac{M_x}{P_0} = \frac{2,767,048}{172,297} = 16.06 \text{ in.} \text{ as required.} \]

Therefore, strain compatibility is satisfied.

Adopt \( P_e = 172,297 \text{ lb (766 kN)} \)

**Design Load \( P_e \)**

Actual, \( \frac{\Delta}{d} = 0.418 > 0.415 \) limit tension-controlled \( \frac{\Delta}{d} = 2.375 < \) limit compression-controlled

\[ \frac{\Delta}{d} = 0.60 \]

Hence, section is in the transition zone of Figure 9.10.

From Equation 9.25a,

Spiral \( \phi = 0.37 \left( \frac{c}{d} \right) \)

\[ = 0.37 \times \frac{0.20}{0.418} = 0.248 \]

Design \( P_s = 0.848 \times 172,297 = 146,107 \text{ lb.} \)

Whichever \( P_e \) from part (a) of the solution = 0.848 \times 151,793 = 128,720 \text{ lb, which is usually more conservative if tension-controlled.} \)

Comparison of the strain compatibility solution with the solution of the \( c = 1.3 \) in Example 9.14 min. the \( F_c = \cdots \text{section} \cdots \text{is} \cdots \text{in} \cdots \text{the} \cdots \text{tension} \cdots \text{domain.} \)
9.7 COLUMN STRENGTH REDUCTION FACTOR $\phi$

For members subject to flexure and relatively small axial loads, failure is initiated by yielding of the tension reinforcement and takes place in an increasingly ductile manner. Hence for small axial loads, it is reasonable to permit an increase in the $\phi$ factor from that required for pure compression members. When the axial load vanishes, the member is subjected to pure flexure, and the strength reduction factor $\phi$ becomes 0.70. Figure 9.10 shows the zone in which the value of $\phi$ can be increased from 0.65 to 0.70 for tied columns and 0.70 to 0.9 for spiral columns.

A compression value of $0.10f_A_s$ can be mostly considered as the design axial load value $P_d$ below which the $\phi$ factor could safely be increased to 0.9 for most compression members.

The strength reduction factor, $\phi$, from Figure 9.10 can be linearly interpolated between the compression-controlled state and the tension-controlled states, either as a function of the strain $\epsilon_r$ of the tension reinforcement, or as a function of the neutral axis depth ratio $c/c_t$.

As shown in Fig. 9.10 the transition expressions in terms of strain $\epsilon_r$ are

\[
\text{Spiral: } \quad \phi = 0.70 + (\epsilon_r - 0.002) \left( \frac{200}{3} \right) \\
\text{Tied: } \quad \phi = 0.65 + (\epsilon_r - 0.002) \left( \frac{250}{3} \right)
\]

(9.24a, 9.24b)

The transition expressions in terms of the neutral axis depth ratio are

\[
\text{Spiral: } \quad \phi = 0.70 + 0.2 \left( \frac{1}{c/c_t} - \frac{5}{3} \right) \\
\text{Tied: Other: } \quad \phi = 0.65 + 0.2 \left( \frac{1}{c/c_t} - \frac{5}{3} \right)
\]

(9.25a, 9.25b)

9.7.1 Example 9.10 Calculation of the Design Load Strength $P_d$ from the Nominal Resisting Load $P_n$

Calculate the design loads $P_d$ in Examples 9.2 to 9.7 and 9.9 using the appropriate $\phi$ strength reduction factors.

Solution:

Example 9.2

$P_n = 1,135,501$ lb: spirally reinforced column in axial compression.

Therefore, $\phi = 0.70$

$P_d = 0.70 \times 1,135,501 = 794,851$ lb.

Example 9.3

Tied rectangular column:

(a) Compression limit strain case: $\phi = 0.65$

$P_n = 364,340$ lb

$P_d = 0.65 \times 364,340 = 236,287$ lb
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9(c) Limit tension-controlled strut case: \( \phi = 0.90 \)

\[
\begin{align*}
P_c &= 209,112 \text{ lb} \\
P_s &= 0.90 \times 209,112 = 188,200 \text{ lb}
\end{align*}
\]

**Example 9.4**

Transition controlled strut case: \( \epsilon = 10 \text{ in.} \)

\[
P_s = 285,600 \text{ lb, } \frac{\epsilon}{d_i} = 0.47
\]

From Eq. 9.25(b), with \( P_s > 0.10f/L_{A_s} \), no strain check for computing \( \phi \) is needed.

\[
\begin{align*}
\phi &= 0.65 + 0.25 \left( \frac{0.18}{0.47} - \frac{5}{4} \right) \\
&= 0.65 + 0.25 \left( \frac{1}{4.47} - \frac{5}{4} \right) \\
&= 0.765
\end{align*}
\]

Hence, \( P_s = \phi P_c = 0.765 \times 285,600 = 218,484 \text{ lb} \)

**Example 9.5**

Tension-controlled rectangular tied column in the transition zone, \( \epsilon = 10 \text{ in.} \)

\[
P_s = 331,642 \text{ lb, } \frac{\epsilon}{d_i} = 0.59
\]

From Eq. 9.25(b),

\[
\begin{align*}
\phi &= 0.65 + 0.25 \left( \frac{1}{0.59} - \frac{5}{4} \right) \\
&= 0.657
\end{align*}
\]

Hence, \( P_s = \phi P_c = 0.657 \times 331,642 = 217,899 \text{ lb} \)

**Example 9.6**

Compression-controlled rectangular tied column, \( \epsilon = 10 \text{ in.} \)

\[
P_c = 439,853 \text{ lb, } \frac{\epsilon}{d_i} = 0.655
\]

From Eq. 9.25(b),

\[
P_s = 0.65 \times 439,853 = 285,891 \text{ lb}
\]

**Example 9.7**

Compression-controlled rectangular tied column

(a) \( \epsilon = 6 \text{ in.} \quad P_s = 607,555 \text{ lb} \quad \phi = 0.65 \)

Hence, \( P_s = 0.65 \times 607,555 = 394,911 \text{ lb} \)

(b) \( \epsilon = 10 \text{ in.} \quad P_s = 460,098 \text{ lb} \quad \phi = 0.65 \)

\[
P_s = 0.65 \times 460,098 = 299,064 \text{ lb}
\]

**Example 9.9**

(a) Transition zone strain, Whitney circular column solution

The Whitney transformed column section is here analyzed by the strain limits compatibility approach.

\[ \epsilon = 6.11 \text{ in.} \quad d_i = 13.0 \text{ in.} \] (Fig. 9.16); hence, the equivalent rectangular column is in the transition zone, with \( \epsilon \) being less than 0.020 and greater than 0.002.

\[
\frac{\epsilon}{d_i} = \frac{6.11}{13.0} = 0.47, \text{ and } P_s = 107,195 \text{ lb} \text{ by strain compatibility.}
\]
9.8 Load-Moment Strength Interaction Diagrams (P-M Diagrams) for Columns Controlled by Material Failure

From Equation 9.23, for the equivalent tied rectangular section,

\[
\frac{P}{A} = 0.65 + 0.25 \left( \frac{1}{\phi} - \frac{2}{3.47} \right)
\]

\[
P_e = 0.765 \times 387.35 = 143,172 \text{ lb as compared to the Whitney approximate solution that gives}
\]

\[
P_e = 151,793 \text{ lb.}
\]

(b) Compression-controlled state, circular column

\[\phi = 0.70 \text{ for circular columns in compression, having a small eccentricity of 5 in., namely, that the}
\]

axial load is inside the section close to its geometrical centroid.

Whitney's \(P_e = 626,577\) lb giving

\[
P_e = 0.70 \times 626,577 = 438,604 \text{ lb.}
\]

(c) Transition zone state, circular column (Strain Limits approach)

\[\phi \text{ can range between 0.70 to 0.90. As shown in the solution of Example 9.9, the section is in the}
\]

transition zone of Figure 9.10, with the tension fuse steel in initial yielding.

Strain-continuity exact solution gave \(P_e = 172,297\) lb.

\[\phi = 0.848 \text{ from before giving}
\]

\[P_e = 0.848 \times 172,297 = 146,107 \text{ lb.}
\]

9.8 LOAD-MOMENT STRENGTH INTERACTION DIAGRAMS (P-M DIAGRAMS) FOR COLUMNS CONTROLLED BY MATERIAL FAILURE

From the discussion in Sections 9.3 and 9.4 and the numerical examples presented, we can postulate that the capacity of reinforced concrete sections to resist combined axial and bending loads can be expressed by P-M interaction diagrams to relate the axial load to the bending moment in compression members.

Each point on the curve represents one combination of nominal load strength \(P_e\) and nominal moment strength \(M_e\) corresponding to a particular neutral-axis location. The interaction diagram is separated into the tension control region and the compression control region by the balanced condition. The following example illustrates the construction of the P-M diagram for a typical rectangular section.

9.8.1 Construction of P-M Interaction Diagrams

Controlling coordinates for major points on the interaction diagram are determined by the strain level in the tension reinforcement. The strain level is set by the position of the neutral axis depth, \(c\) for the strain and stress states shown in Figure 9.20. Seven states for depth \(c\) are indicated, as an example, in Figure 9.20. The neutral axis depth, \(c_n\) in case 4 denotes the balanced strain condition, namely, when the tensile reinforcement yields simultaneously with the crushing of the concrete at the extreme compression fibres. This state represents the coordinates \(P_e\) and \(M_e\) in Figure 9.21.

9.8.2 Example 9.11: Load-Moment Interaction Diagram for Rectangular Columns

Construct a P-M diagram for the rectangular column shown in Figure 9.22, having the geometry: width \(b = 12\) in. (305 mm); thickness \(t = 14\) in. (356 mm); steel reinforcement = 4 No. 11 bars (38.8 mm diameter), for the following conditions, given \(f' = 6000\) psi (414 MPa), \(f_e = 60,000\) psi (414 MPa), \(d' = 3.0\) in. (76 mm):

(i) Concentrated load,
(ii) Limit compression-controlled strain state (balanced strain condition),
(iii) Limit tension-controlled strain state,
(iv) Axial load \(P_e = 0.10f' A_y\),
(v) \(c = 10\) in.,
(vi) Pure bending, \(M_e\).
\[ a = 0.003 \]

1. \( e = \frac{P_0}{A_f' A_f + A_d' A_d} \) (strain in tension)
2. \( e_2 = 0 \) (strain in compression)
3. \( c_2 = 870(97 + f_c') \) \( e_2 = f_c'/E_2' \)
4. \( c_2 = 870(97 + f_c') \) \( e_2 = f_c'/E_2' \)
5. \( c_2 = 1740(170 + f_c') \) \( e_2 = 0.005 \)
6. \( \varepsilon = 30^{\circ} \)
7. \( c_2 = 0.3d \)
8. \( c_2 > 0.005 \)

Figure 9.20: Strain distribution across section depth for controlling neutral axis positions on the P-M interaction diagram.

Solution:
1. **Concrete Load**
   - \( c = 60,000 \text{ psi} \), \( \varepsilon = 0.003 \text{ in./in.} \). Also \( b = 0.75 \text{ for } f'_c = 6000 \text{ psi.} \)
   - \( A' = A_1 = 3.12 \)
   - \( P_{\text{concrete}} = 0.80(0.85 f'_c/A' + A_d' f_d) \)
     - \( = 984 (0.85 \times 6000 \times 14 \times 12 + 2 \times 3.12 \times 60,000) = 984,960 \text{ lb.} \)
   - \( \varphi P_{\text{concrete}} = 2.65 (P_{\text{concrete}}) = 660,224 \text{ lb.} \)

2. **Limit compression-controlled strain state**
   - This is the balanced strain condition, with the tensile reinforcement yielding simultaneously with the concrete crushing at the extreme compression fibers (\( f_c = 60,000 \text{ psi} \)).
   - \( \varepsilon = 0.003 \text{ in./in.} \)
   - \( \varepsilon_1 = 0.002 \text{ in./in.} \)
   - \( d = 14.0 - 3.0 = 11.0 \text{ in.} \)
   - \( d' = 3.0 \text{ in.} \)
   - \( e = 0.003 \times 11.0 = 0.060 \text{ in.} \)
   - \( c = 6.0 \times 3.0 = 18.0 \text{ in.} \)
   - \( c' = 0.003 \left( \frac{c - d'}{e} \right) = 0.003 \left( \frac{6.00 - 3.0}{0.060} \right) = 0.0076 \text{ in./in.} \)
Figure 9.21 Typical load-moment P-M interaction diagram in compression members (Ref. 9.8).

\[ f' = \frac{E}{E} \times 29 \times 10^6 = 46,400 \text{ psi} \leq f' \]

\[ f' = f' = 60,000 \text{ psi} \]

\[ C = 0.85 f' \times \frac{b}{a} = 0.85 \times 6.000 \times 12 \times 4.95 = 332,940 \text{ lb} \]

\[ C = A; f' = 3.12 \times 46,400 = 144,708 \text{ lb} \]

\[ T' = A; f' = 3.12 \times 60,000 = 187,200 \text{ lb} \]

From Equation 9.4,

\[ P_n = C_1 + C_2 = 302,940 + 144,708 = 447,648 \text{ lb} \]

From Equation 9.5,

\[ M_n = C_1 \left( \frac{b}{2} - \frac{d}{2} \right) + C_2 \left( \frac{d}{2} - \frac{a}{2} \right) + T_1 \left( d - \frac{b}{2} \right) \]

\[ = 302,940 \left( \frac{14}{2} - \frac{4.95}{2} \right) + 144,708 \left( \frac{14}{2} - 3.0 \right) + 187,200 \left( 11 - \frac{14}{2} \right) = 269,876 \text{ in.-lb} \]

\[ e = \frac{2,698,876}{269,508} = 10.36 \text{ in.} \]

Design \( P_n = 0.65 \times 269,876 = 170,320 \text{ lb} \)

Design \( M_n = 0.65 \times 2,698,876 = 1,754,159 \text{ in.-lb} \)

(iii) Limit tension-controlled strain state

\[ C' = 0.05 \times 0.075 \times 110 \text{ in.} \]

\[ a = 0.75 \times 1.125 \text{ in.} \]

\[ a = 0.75 \times 4.139 \text{ in.} \]

\[ f' = 87,000 \left( \frac{1.125 - 3.07}{4.139} \right) = 23,727 \text{ psi} \leq f' \]

\[ C' = 0.85 f' \times \frac{b}{a} = 0.85 \times 6.000 \times 12 \times 3.06 = 189,116 \text{ lb} \]
\[ C_r = A_r f_r = 3.12 \times 23.727 = 74.028 \text{ lb} \]
\[ T_r = A_r f_r = 3.12 \times 60.000 = 187.200 \text{ lb} \]
\[ P_r = C_r - T_r = 189.108 - 187.200 = 7.908 \text{ lb} \]
\[ M_r = 189.108 \left( \frac{14}{2} - 3.0 \right) + 74.028 \left( \frac{14}{2} - 3.0 \right) + 187.200 \left( 11 - \frac{14}{2} \right) = 2076.496 \text{ in.-lb} \]
\[ e_r = \frac{2076.496}{75.936} = 27.3 \text{ in.} \]

From Figure 9.10, \( \phi_{\text{con}} = 0.90 \), therefore.
Design \( P_r = 0.90 \times 75.936 = 68.342 \text{ lb} \)
Design \( M_r = 0.90 \times 2076.496 = 1868.846 \text{ in.-lb} \)

(iv) \( \phi P_n = 0.10 f_r ; A_n \) case.

\[ \phi P_n = 0.10 \times 6000 \times 12 \times 14 = 100,800 \text{ lb} \]

By trial and adjustment, assume \( c = 4.64 \)
\[ \sigma = 0.75 \times 4.64 \times 3.48 \text{ in.} \]
\[ f_r = 87.000 \left( \frac{4.64 - 3.0}{4.64} \right) = 30.730 \text{ psi} < f_r \]
\[ C_r = 0.85 f_r a = 0.85 \times 6.000 \times 12 \times 3.48 = 212.976 \text{ lb} \]
\[ C_r = A_r f_r = 3.12 \times 30.730 = 95.940 \text{ lb} \]
\[ T_r = A_r f_r = 3.12 \times 60.000 = 187.200 \text{ lb} \]
\[ P_r = C_r - T_r = 212.976 + 95.940 - 187.200 = 121.716 \text{ lb} \]
\[ M_r = 212.976 \left( \frac{7.0 - 3.0}{2} \right) + 95.940 \left( 7.0 - 3.0 \right) + 187.200 \left( 11.0 - 7.0 \right) = 2252.814 \text{ in.-lb} \]
\[ e_r = \frac{2252.814}{121.716} = 18.5 \text{ in.} \]

\[ \frac{\sigma - 4.64}{0.42} > 0.80 \text{ ; } \phi \text{ factor not valid.} \]

Hence, the section is in the transition stage of Fig. 9.10. From Eq. 9.25(b) for tied columns,
\[ \phi = 0.65 + 9.25 \left( \frac{1}{0.42} - \frac{3}{3} \right) = 0.326 \]

Therefore, \( P_r = 0.826 \times 121.716 = 100,651 \text{ lb} = 100,537 \text{ lb} \); hence assumed \( c = 4.64 \) is valid.
\[ M_r = \phi M_n = 0.826 \times 2252.814 = 1860.224 \text{ in.-lb} \]

Comparing this load case with the limit tensile-controlled case shows that the \( P_r - M_r \) intercept below the \( P_n = 0.10 f_r ; A_n \) load level is expected to be in the transitional zone of the interaction diagram, where \( e_r \) is less than 0.025 and the tension face steel yielded.

(v) \( c = 10 \text{ in. case:} \)
\[ \frac{C_r}{d} = \frac{10}{11} = 0.91 > 0.60 \text{; hence, from Figure 9.10, strength reduction factor } \phi = 0.65 \]

From Equation 9.9(b)
\[ f_r = 0.105 P_n \left( 1 - \frac{C_r}{d} \right) = 87.000 \left( 1 - \frac{3.0}{10.0} \right) = 60.000 > f_r \]
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Use $f' = 60,000$ psi.
From Equation 9.10(b),

$$f_c = 0.003 E_c \left( \frac{d - 1}{k} \right) = 0.003 \times 29 \times 10^6 \left( \frac{11}{10} \frac{- 1}{1} \right) = 8,700 \text{ psi}$$

$$a = 0.75 \times 10 = 7.5 \text{ in.}$$

$$C_0 = 0.65 f' / b = 0.65 \times 60,000 / 12 = 3500 \text{ lb}.$$  

$$C_0 = A_f / f_c = 3.12 \times 60,000 = 187,200 \text{ lb}.$$  

$$T_n = A_f f' = 3.12 \times 60,000 = 187,200 \text{ lb}.$$  

$$P_n = C_0 + C_0 - T_n = 495,000 + 187,200 = 27,144 \text{ lb}.$$  

$$M_n = C_0 \left( \frac{a}{2} \right) + C_0 (2 - a) + T_n (a - d)$$

$$= 495,000 \left( \frac{7}{10} - \frac{2.5}{2} \right) + 187,200 (7.8 - 3.0) + 27,144 (11.0 - 7.0) = 2,349,126 \text{ in.-lb}.$$  

$$e_r = \frac{M_n}{P_n} = \frac{2,349,126}{419,056} = 3.76 \text{ in.}$$  

$$P_r = \phi P_n = 0.65 \times 419,056 = 272,236 \text{ lb}.$$  

$$M_r = \phi M_n = 0.65 \times 2,349,126 = 1,526,992 \text{ in.-lb}.$$  

(ii) Pure bending $M_n$:
Neglect $A_f$, when $P_n = 0$ as sufficiently accurate when $A_f, f_c$ as is the case in most columns. Otherwise a negative value of $f_c$ or a negligible value would result:

$$a = \frac{A_f f'}{b} = \frac{3.12 \times 60,000}{0.65 \times 60,000 / 12} = 187.200$$

$$e = \frac{a}{2} = \frac{187.200}{2} = 93.6 \text{ in.}$$

For this neutral axis depth, $f_c = 32,410$ psi.

$$P_r = 0, \quad e = 0, \quad \phi = 0.90$$

$$M_r = A_f f' \left( \frac{a}{2} \right) = 187,200 \left( \frac{11.0 - 3.0}{2} \right) = 1,772,784 \text{ in.-lb}.$$  

$$M_r = \phi M_n = 0.90 \times 1,772,784 = 1,595,556 \text{ in.-lb}.$$  

Construct the $P_r - M_r$ and $P_r - M_n$ interaction diagrams as shown in Figure 9.23 for the following neutral axis depth, $c$ (in.):

- $c = \infty$ - Compression-controlled
  - $c = 10.0$ in - Compression-controlled
  - $c = 6.00$ in - Limit compression-controlled strain case
- $e = 4.64$ in - $0.10 f' / f_c$, $\phi$, load level
  - $e = 4.12$ in - Limit tension-controlled strain case
- $e = 4.08$ in - Pure bending case, $M_n$

A systematic selection of coordinates for plotting the $P_r - M_r$ interaction diagram for the various loading stages is given in Figs. 9.20, 9.21 through the selection of neutral axis depth, $c$, and such interaction diagram coordinates so as to cover the complete range of behavior. Typical charts from the ACI-318 2017 Handbook are shown in Fig. 9.24. Example 9.14 to follow.
9.9 PRACTICAL DESIGN CONSIDERATIONS

The following guidelines should be followed in the design and arrangement of reinforcement to arrive at a practical design.

9.9.1 Longitudinal or Main Reinforcement

Most columns are subjected to bending moment in addition to axial force. For this reason and to ensure some ductility, a minimum of 1% reinforcement should be provided in the columns. A reasonable reinforcement ratio is between 1.5% and 3.4%. Occasionally, in high-rise buildings where column loads are very large, 4% reinforcement is not unreasonable. Even though the code allows a maximum of 6% for longitudinal reinforcement in columns, it is not advisable to use more than 4% in order to avoid reinforcement congestion, especially at beam-column junctions.

A minimum of four longitudinal bars should be used in the case of tied columns. For spiral columns, at least six longitudinal bars should be used to provide hoop action in the spirals; see the ACI Code for further discussion.

9.9.2 Lateral Reinforcement for Columns

9.9.2.1 Lateral Ties. Lateral reinforcement is required to prevent spalling of the concrete cover or local buckling of the longitudinal bars. The lateral reinforcement could be in the form of ties evenly distributed along the height of the column at specified intervals. Longitudinal bars spaced more than 6 in. apart should be supported by lateral ties, as shown in Figure 9.25.

The following guidelines are to be followed for the selection of the size and spacing of ties, except in the case of earthquake regions (see Chapter 16).
Figure 9.74 Typical nondimensional column interaction charts: (a) for $f_y = 4,000$ psi; (b) for $f_y = 5,000$ psi (Parts 9.8, 9.10, 9.11).
1. The size of the tie should not be less than a No. 3 (9.5-mm) bar. If the longitudinal bar size is larger than No. 10 (32-mm), then No. 4 (12-mm) bars at least should be used as ties.

2. The vertical spacing of the ties must not exceed:
   (a) Forty-eight times the diameter of the tie
   (b) Sixteen times the diameter of the longitudinal bar
   (c) Least lateral dimension of the column

Figure 9.25 shows a typical arrangement of ties for four, six and eight longitudinal bars in a column cross-section.

9.9.2.2 Spirals. The other type of lateral reinforcement is spirals or helical lateral reinforcement, as shown in Figure 9.26. They are particularly useful in increasing ductility or member toughness and hence are mandatory in high-earthquake-risk regions. Normally, concrete outside the confined core of the spirally reinforced column can totally spall under unusual and sudden lateral forces such as earthquake-induced forces. The columns have to be able to sustain most of the load even after the spalling of the cover in order to prevent the collapse of the building. Hence, the spacing and size of spirals are designed to maintain most of the load-carrying capacity of the column, even under such severe load conditions.

Closely spaced spiral reinforcement increases the ultimate load capacity of columns. The spacing or pitch of the spiral is so chosen that the load capacity due to the confining spiral action compensates for the loss due to spalling of the concrete cover.

$$A_s = \frac{\pi d^2}{4}$$

$$A_t = \frac{\pi d^2}{2}$$

Figure 9.26 Helical or spiral reinforcement for columns.
Equating the increase in strength due to confinement and the loss of capacity in spalling and incorporating a safety factor of 1.2, the following minimum spiral reinforcement ratio \( p_s \) is obtained:

\[
p_s = 0.45 \left( \frac{A_s}{A_{c,s}} - 1 \right) \frac{f'_p}{f_y}
\]

where \( p_s = \frac{\text{volume of the spiral steel in one revolution}}{\text{volume of concrete core contained in one revolution}} \)

\[ A_s = \frac{\pi D_s^2}{4} \]

\[ A_{c,s} = \frac{\pi h^2}{4} \]

\[ h = \text{diameter of the column} \]

\[ a_s = \text{cross-sectional area of the spiral} \]

\[ d_s = \text{nominal diameter of the spiral wire} \]

\[ D_s = \text{diameter of the concrete core out-to-out of the spiral} \]

\[ f'_p = \text{yield strength of the spiral reinforcement} \]

To determine the pitch \( s \) of the spiral, calculate \( p_s \) using Eq. 9.26, choose a bar diameter \( d_s \) for the spiral, and calculate \( a_s \); then obtain pitch \( s \) using Eq. 9.29(b).

The spiral reinforcement ratio \( p_s \) can be written as

\[
p_s = \frac{a_s(\pi D_s - d_s)}{(\pi/4)D_s^2} \]

Therefore,

\[
\text{pitch } s = \frac{a_s(\pi D_s - d_s)}{(\pi/4)D_s^2} \]

or

\[
s = \frac{4a_s(\pi D_s^2 - d_s^2)}{D_s^2 p_s} \]

The spacing or pitch of spirals is limited to a range of 1 to 3 in. (25.4 to 76.2 mm) and the diameter should be at least 1 in. (25.4 mm). The spirals should be well anchored by providing at least 12 extra turns when splicing of spirals rather than welding is used.

### 9.10 OPERATIONAL PROCEDURE FOR THE DESIGN OF NONSLENDER COLUMNS

The following steps can be used for the design of non-sleender (short) columns where the behavior is controlled by material failure.

1. Check whether the column is non-sleender. A non-sleender column satisfies a height ratio value \( h_L/h_D \) less than 12 (Furlong, Ref. 9.17).
2. Evaluate the factored external axial load \( P_e \) and factored moment \( M_e \). Calculate the eccentricity \( e = M_e/P_e \).
3. Assume a cross-section and the type of vertical reinforcement to be used. Fractional dimensions are to be avoided in selecting column sizes.

4. Assume a reinforcement ratio \( \rho \) between 1 and 4% and obtain the reinforcement area.

5. Assume by trial and adjustment a neutral axis depth ratio \( \frac{c}{d} \) for a limit compression-controlled state or a transition state as described in the limit-strain hypothesis.
6. Check for the adequacy of the assumed section. If the section cannot support the factored load or it is oversized, hence uneconomical, revise the cross-section and (or) the reinforcement and repeat steps 4 and 5.

7. Design the lateral reinforcement.

Figure 9.27 presents a flowchart for the sequence of calculations.

9.11 NUMERICAL EXAMPLES FOR ANALYSIS AND DESIGN OF NONSLENDER COLUMNS

9.11.1 Example 9.12: Design of a Column with Large Eccentricity

Initial Tension Failure

The tied reinforced concrete column in Figure 9.28 is subjected to a service axial force due to dead load = 85,000 lb (376 kN) and a service axial force due to live load = 160,000 lb (566 kN). Eccentricity to the geometric centroid is e = 15 in. (406 mm).

Design the longitudinal and lateral reinforcement for this column, assuming a non-sleender column with a total reinforcement ratio between 2 and 3%. Given:

- $f_c' = 4000$ psi (27.6 MPa), normal-weight concrete
- $f_y = 60,000$ psi (414 MPa)

Solution: Calculate the factored external load and moment (step 1)

\[ P' = 1.2D + 1.6L = 1.2 \times 85,000 + 1.6 \times 160,000 = 358,000 \text{ lb (1595 kN)} \]

\[ P'e = 358,000 \times 16 = 5,728,000 \text{ in.-lb (648 kN-m)} \]

Assume a section 20 in. x 20 in. and a total reinforcement ratio of 1% (steps 2 and 3)

Assume that $p = v' = A_{st} / A_c = 0.015$ and $d' = 2.5$ in.

\[ A_{st} = A_t' = 0.015 \times 20 (20 - 2.5) = 5.25 \text{ in.}^2 \]

Try five No. 9 bars, 5.00 in.² on each face (3225 mm²) parallel to the axis of bending.

\[ P = \frac{5.00}{20 \times 17.5} = 0.0043 \text{ on each face} \]

![Diagram](image)

Figure 9.28: Column geometry, strain and stress diagrams in Ex. 9.12 (balanced strain state): (a) cross-section; (b) strains; (c) stresses.
9.11 Numerical Examples for Analysis and Design of Non-linear Column

Limit compression-controlled state (Step 4)

\[
\frac{c}{d_i} = 0.60 \quad \epsilon_i = 0.003 \text{ in./in.} \quad d_i = 17.5 \text{ in.} \quad d = 2.5 \text{ in.}
\]
\[
c = 0.60 \times 17.5 = 10.5 \text{ in.}
\]
\[
a = 0.003 \times 10.5 = 0.030 \text{ in.}
\]
\[
f'_c = E_c \left(1 - \frac{d}{c}\right) = 87,000 \left(1 - \frac{2.5}{10.5}\right) = 66,286 \text{ > } f_c = 60,000 \text{ psi.}
\]

Hence, \( f'_c = 60,000 \text{ psi.} \)

\[
P_a = C_a \left(\frac{b}{2}\right)
\]
\[
C_a = 0.85 f' \quad a = 0.85 \times 4.000 \times 20 \times 8.925 = 606,900 \text{ lb.}
\]
\[
C_a = A'_f f'_c = 5.0 \times 60,000 = 300,000 \text{ lb.}
\]
\[
T_c = A, f_c = 5.0 \times 60,000 = 300,000 \text{ lb.}
\]
\[
P_c = 606,900 \text{ lb. as } A'_f f'_c = A, f_c.
\]

From Equation 9.8(b)

\[
M_c = P_c \epsilon = 606,900 \left(10 - \frac{8.925}{2}\right) + 300,000 \left(10 - 1.5\right) + 300,000 \left(17.5 - 10\right) = 7,960,709 \text{ in.-lb.}
\]

Limit \( \epsilon_c = \frac{M_c}{P_c} = 7,960,709 \frac{2}{606,900} = 12.95 \text{ in.} \) < actual \( \epsilon = 16 \text{ in.} \)

Therefore, this column is either in the compressive-controlled zone or in the transition zone, with \( f'_c = 60,000 \text{ psi.} \)

Limit tension-controlled state (Step 4)

\[
\frac{c}{d_i} = 0.575 \quad \epsilon_i = 0.005 \text{ in./in.} \quad d_i = 175 \text{ in.}
\]
\[
c = 0.575 \times 17.5 = 6.563 \text{ in.}
\]
\[
a = 0.005 \times 6.563 = 0.033 \text{ in.}
\]
\[
f'_c = 87,000 \left(1 - \frac{2.5}{6.563}\right) = 55,860 \text{ psi.}
\]
\[
P'_c = C_c \left(\frac{b}{2}\right)
\]
\[
C_c = 0.85 f'_c \quad a = 0.85 \times 4.000 \times 20 \times 5.57 = 378,760 \text{ lb.}
\]
\[
C_c = A'_f f'_c = 5.0 \times 55,860 = 279,300 \text{ lb.}
\]
\[
T_c = A, f_c = 5.0 \times 60,000 = 300,000 \text{ lb.}
\]
\[
P_c = 378,760 + 279,300 + 300,000 = 348,000 \text{ lb.}
\]
\[
M_c = P_c \epsilon = 378,760 \left(10 - \frac{5.57}{2}\right) + 279,300 \left(10 - 2.5\right) + 300,000 \left(17.5 - 10\right) = 7,002,513 \text{ in.-lb.}
\]
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Limit $\epsilon = \frac{M_y}{P_c} = \frac{7,002,903}{348,000} = 20.1$ in. $> \epsilon = 16$ in., hence assume larger neutral axis depth value in order to decrease the eccentricity.

**Analysis of assumed section (Step 5)**
The column is within the transition zone of Figure 9.10. By trial and adjustment, assume a larger value of $\epsilon = 8.20$ in., giving

$$\frac{c}{d'} = \frac{8.20}{17.5} < 0.469 < 0.60.$$  

Thus, $\epsilon > 0.002$, with the tension face idealized $f' = 60,000$ psi.  

$\sigma = \frac{270,000}{825,000} = 0.32$ in.

$$f' = 87,000 \left(1 - \frac{2.5}{3.5}\right) = 60,476 \text{ psi} > f'$$, thus 60,000 psi.

$$C_h = 0.65 \times 400 \times 20 \times 0.97 = 43,960 \text{ lb}.$$  

$$C_t = 5.0 \times 60,000 = 300,000 \text{ lb}.$$  

$$T = 5.0 \times 60,000 = 300,000 \text{ lb}.$$  

$$P_c = C_h - T = 43,960 \text{ lb}.$$  

$$P_t = 7,582,459 \left(10 - \frac{0.25}{2}\right) + 300,000 \left(10 - 2.5\right) + 300,000 \left(17.5 - 10\right) = 7,587,800 \text{ in-lb}.$$  

$$P = 43,960 \times \frac{1}{2} = 22,000 \text{ lb}.$$  

$$P_t > P$$, hence no limitation on strain $\epsilon$ in the transition zone.

From Equation 9.26(b) for tied columns,

$$\frac{c}{d'} = \frac{8.20}{17.5} < 0.469$$

$$\phi \leq 0.65 + 0.25 \left(\frac{1}{0.469} - \frac{5}{3}\right) = 0.706$$

Design axial load $P = \phi P_t = 0.706 \times 43,960 = 30,653 \text{ lb} > \text{Required} P_c = 338,000 \text{ lb}.$$

Therefore, adopt the $20 \times 20$ in. section with 5 No. 9 reinforcing bars in one layer at each of the facets parallel to the bending axis.

9.11.2 Example 9.12: Design of a Column with Small Eccentricity:

**Initial Compression Failure**

The non slender column shown in Figure 9.29 is subjected to a factored $P_c = 365,000$ lb (1620 kN) and a factored $M_y = 1,640,000$ in-lb (185 kN). Assume that the gross reinforcement ratio $\rho = 1.5$% and that the effective cover to the center of the longitudinal steel is $d'' = 3.5$ in. (88.9 mm). Design the column section and the necessary longitudinal and transverse reinforcement. Given:

$$f' = 4500 \text{ psi (31.6 MPa)}, \text{ normal weight concrete}$$  

$$f_s = 60,000 \text{ psi (414 MPa)}$$

**Solution:** Calculation of factored design loads (step 1)

$$P_c = 365,000 \text{ lb}$$

$$\epsilon = \frac{1,640,000}{3.5 \times 20} = 4.5 \text{ in. (114 mm)}$$
Figure 9.29  Column geometry: (a) cross section; (b) strain (balanced case); (c) stresses.

Assume a 15 in. x 15 in. (c = 12.5 in.) section (steps 2 and 3)
Assume that the reinforcement ratio \( \rho = \rho' = 0.01 \).

Provide two No. 9 bars on each side:

\[ A_s = A_s' = 2.0 \text{ in.}^2 \]

The eccentricity is relatively small, hence the section is in all probability compression-controlled.

**Limit compression-controlled state (Step 4)**

\[ d' = 2.5 \text{ in.} \]
\[ d = 15 - 2.5 = 12.5 \text{ in.} \]
\[ \frac{c}{d'} = 0.60, \text{ hence } c = 0.60 \times 12.5 = 7.5 \text{ in.} \]

\[ f_s = 0.85 \times 0.05 \times \frac{4500 - 4000}{1000} = 0.825 \]
\[ \sigma = f_s c = 0.825 \times 7.5 = 6.19 \text{ in.} \]

From Equation 9.9(b),

\[ f_c' = 87,000 \left( 1 - \frac{d'}{c} \right) \]

\[ = 87,000 \left( 1 - \frac{2.5}{7.5} \right) = 58,300 \text{ psi} \],

\[ f_c = f_s E_s = 0.05 \times 30 \times 10^6 = 50,000 \text{ psi for this limit case in tension} \]

\[ P_c = C_c + C_c' \]

\[ C_c = 0.85 f_s b a = 0.85 \times 4500 \times 15 \times 6.19 = 355,151 \text{ lb} \]

\[ C_c' = A_s' f_s' = 2.0 \times 58,300 = 116,600 \text{ lb} \]

\[ T_c = A_s f_s = 2.0 \times 60,000 = 120,000 \text{ lb} \]

\[ P_c = 355,151 + 116,600 = 120,000 = 355,151 \text{ lb} \]

\[ M_c = C_c \left( \frac{c - d'}{2} \right) + C_c' \left( d - d' \right) + T_c (d - \frac{c}{2}) \]

\[ = 355,151 \left( 7.5 - 2.5 \right) + 116,600 \left( 7.5 - 2.5 \right) + 120,000 \left( 12.5 - 2.5 \right) = 2,744,440 \text{ lb} \]
Limit $\epsilon = \frac{M_L}{P_L} = \frac{2,744,440}{351,151} = 7.82$ in. > $\epsilon = 4.5$ in.

Therefore, increase the inelastic axis depth, $c$, in order to enlarge the volume of the compressive block.

**Trial and adjustment analysis of column section (Step 5)**

Assume $c = 9.9$ in.

\[ a = \frac{c}{2} = \frac{9.9}{2} = 4.95 \text{ in.} \]
\[ f_c = 0.05 f' = 0.05 \times 500 = 25 \text{ psi} \]
\[ f_y = 0.05 f_c = 0.05 \times 25 = 1.25 \text{ psi} \]
\[ f_e = 0.05 f_c = 0.05 \times 500 = 25 \text{ psi} \]
\[ P_c = C_c + C_e = T \]
\[ C_c = f_y A_c = 0.85 \times 4500 \times 15 \times 8.168 = 468,639 \text{ lb} \]
\[ C_e = f_e A_e = 2.0 \times 60,000 = 120,000 \text{ lb} \]
\[ T = A_i f_c = 2.0 \times 22.848 = 45,696 \text{ lb} \]
\[ P_n = 468,639 + 120,000 = 588,639 \text{ lb} \]
\[ M_e = 468,639 \left( 7.5 - \frac{8.168}{2} \right) + 120,000 \left( 7.5 - 2.5 \right) + 45,696 \left( 7.5 - 7.5 \right) = 2,439,351 \text{ in.-lb} \]
\[ \epsilon = \frac{M_e}{P_n} = \frac{2,439,351}{542,943} = 4.47 \text{ in.} \quad \text{actual} \epsilon = 4.5 \text{ in.} \]

Therefore, strain-compatibility analysis is validated. There is no need to compute the limit tension-controlled eccentricity. By inspection, the small value of the actual eccentricity, $\epsilon = 4.5$ in., indicates that the axial load is close to the centroid of the column cross section, and that it is not possible to develop tensile failure or transitional failure.

$P_e = 542,943 \geq 0.10 P_n$, hence no limitation on strain $\epsilon$ in the transition zone.

From Equation 9.25(b) for tied columns,

\[ c = \frac{9.9}{12.5} = 0.79 > 0.60, \text{ compression controlled} \]

If the transition term is used,

\[ \phi = 0.65 + 0.25 \left( \frac{1}{0.79} - 1 \right) = 0.76 < 0.85 \]

Use $\phi = 0.65$

**Design of column ties (Step 6)**

Use No. 3 ties at the least of the following spacings,

(1) $16 \times \frac{8}{3} = 18$ in.
(2) $48 \times \frac{8}{3} = 18$ in.
(3) Least dimension = 16 in.

Therefore, provide No. 3 ties at 15 in. spacing (9.53 mm diameter at 381-mm spacing).

**9.11.3 Example 9.14: Design of a Circular Spirally Reinforced Column**

A spirally reinforced circular column is subjected to an external factored load $P_f = 145,000$ lb (645 kN) acting at an eccentricity to the geometric centroid of magnitude $e = 16$ in. (406 mm).

Design the column cross section and the longitudinal and spiral reinforcement necessary, assuming a nonshaded column with a total reinforcement ratio of about 1%. Use the $P-M$ interaction chart in Appendix D, Fig. 9.22.
9.11 Numerical Examples for Analysis and Design of Nonhanger Columns

\[ f_c = 4000 \text{ psi (27.6 MPa), normal-weight concrete} \]
\[ f_y = 60,000 \text{ psi (414 MPa)} \]
\[ f_t = 60,000 \text{ psi (414 MPa)} \]

**Solution:** Calculation of factored external loads (Step 1)

- **Given:**
  \[ P_e = 145,000 \text{ lb} \]
  \[ f_c' = 4,000 \text{ psi} \]
  \[ c = 16 \text{ in.} \]
  \[ f_y = 60,000 \text{ psi} \]
  \[ d' = 2.5 \text{ in.} \]

**First Trial (Step 1)**

Assume a column diameter = 18 in. Because eccentricity is large, with the axial load even falling outside the section, the column would be either tension-controlled or is transitional (see Figure 9.13). Spiral φ for compression-controlled = 0.20. Spiral φ for tension-controlled = 0.90. Try at this stage using φ = 0.85.

\[ \frac{P_e}{\phi} = \frac{145,000}{0.85} = 170,588 \text{ lb} \]
\[ M_e = \phi P_e = 16 \times 170,588 = 2,729,408 \text{ in.-lb} \]
\[ \gamma = \frac{\phi}{\phi} = \frac{18}{2} = 9 \text{ in.} \]
\[ A_e = \sqrt{\frac{18}{2}} = 3.14 \times \sqrt{\frac{18}{2}} = 253 \text{ in.}^2 \]
\[ K_e = P_e \frac{f_y}{f_c'} = 170,588 \times \frac{4,000}{4,000} = 170,588 \]
\[ R_e = \frac{M_e}{A_e} = \frac{2,729,408}{253} = 10,800 \]

From Appendix A Intersection chart No. A23,
\[ \rho_e = 0.025 > 0.015 \text{ required in the problem statement.} \]

Hence, enlarge the section to \( h = 20 \text{ in.} \)
\[ \gamma = \frac{h}{h} = \frac{20}{2} = 2 \times 2.5 = 0.75 \]
\[ A_e = \sqrt{\frac{20}{2}} = \sqrt{20} = 314 \text{ in.}^2 \]
\[ K_e = P_e \frac{f_y}{f_c'} = 170,588 \times \frac{4,000}{4,000} = 170,588 \]
\[ R_e = \frac{M_e}{A_e} = \frac{2,729,408}{314} = 8,690 \]

Interpolating from Intersection charts A23 and A24, \( \rho_e = 0.015 \)
\[ A_e = 0.015 \times 314 = 4.71 \text{ in.}^2 \]

For No. 8 bars, \( n = 4.71 = 5.96 \text{ bars.} \)
Use 6 No. 8 bars equally spaced. \( A_e = 4.84 \text{ in.}^2 \)

**Check φ (Step 4)**

From the chart, \( \xi = 0.0003 \text{ in./in.} \)
From Equation 9.24(a)
Spiral \( a = 0.70 = (4\sigma_v - 0.002) \left( \frac{200}{2} \right) \)

\[ a = 0.70 + (0.0043 - 0.002) \left( \frac{200}{2} \right) = 0.853 \]

\( P_e = 0.853 \times 179.288 = 145.512 \text{ lb} > \text{required} P_e = 145,000 \text{ lb. (645 kN).} \)

Therefore, adopt the 20-in. (508-mm) diameter section with 6 No. 8 bars equally spaced.

**Design the spiral reinforcement (Step 6)**

Using Eq. 9.26.

Required \( p_0 = 0.45 \left( \frac{A_s}{A_{sa}} - 1 \right) \frac{f_y}{f_v} \)

Using No. 3 spirals with a yield strength \( f_y = 60,000 \text{ psi} \):
- clear concrete cover \( d_c = 1.5 \text{ in. (38 mm)} \)
- \( f_v = 60,000 \text{ psi} \)
- \( D_s = b + 2d_c = 10.0 - 2 	imes 1.5 = 17.0 \text{ in. (432 mm)} \)
- \( A_{sa} = \frac{m(17.0)^2}{4} = 226.96 \text{ in.}^2 \)
- \( A_s = 314.0 \text{ in.}^2 \)
- \( p_0 = 0.45 \left( \frac{314}{226.96} - 1 \right) \frac{60,000}{60,000} = 0.011 \)

For No. 3 spirals, \( a = 0.11 \text{ in.}^2 \). Using Eq. 9.26(b).

Pitch \( s = \frac{44(A_s - d_c)}{E_d \nu} = \frac{4 \times 0.11(17.0 - 0.075)}{(17.0)^2 \times 0.00125} = 2.20 \text{ in. (55 mm)} \)

Provide No. 3 spirals at 2-in. pitch (9.53-mm-diameter spiral at 5.4-mm pitch).

### 9.12 LIMIT STATE AT BUCKLING FAILURE (SLENDER OR LONG COLUMNS)

Considerable literature exists on the behavior of columns subjected to stability considerations. If the column slenderness ratio exceeds the limits for short columns, the compression member will buckle prior to reaching its limit state of material failure. The strain in the compression face of the concrete at buckling load will be less than the 0.003 in./in. shown in Figure 9.8. Such a column would be a slender member subjected to combined axial load and bending, deforming laterally and developing additional moment due to the \( P-\Delta \) effect, where \( P \) is the axial load and \( \Delta \) is the deflection of the column's buckled shape at the section being considered, as seen in Figure 9.30.

\( k \) is the column length factor, as shown in Fig. 9.31. \( M_1 \) and \( M_2 \) are the moments at the opposite ends of the compression section. \( M_1 \), \( M_2 \) are larger than \( M_v \), and the ratio \( M_v/M_1 \) is taken as positive for single curvature and negative for double curvature, as shown in Fig. 9.32(a).

The effective length \( kL \) is used as the modified length of the column to account for end restraints other than pinned. \( kL \) represents the length of an auxiliary pin-ended column, which has an Euler buckling load equal to that of the column under consideration. Alternatively, it is the distance between the points of contraflexure of the member in its buckled form.
9.12 Limit State Bending Failure (Slender or Long Columns)

Figure 9.30 Loading moment (P-M) magnification interaction diagram.

The value of the end restraint effective length factor \( k \) varies between 0.5 and 2.0.

- Both column ends fixed \( k = 0.5 \)
- Both column ends fixed, lateral motion exists \( k = 1.0 \)
- Both column ends pinned, no lateral motion \( k = 1.0 \)
- One end fixed, other end free \( k = 2.0 \)

Typical cases illustrating the buckled shape of the column for several end conditions and the corresponding length factors \( k \) are shown in Figure 9.31.

Figure 9.31 Values of column length factor \( k \) for typical end conditions: (a) fixed-fixed; (b) fixed-fixed with lateral motion; (c) pinned; (d) fixed-free.
For members in a structural frame, the end restraint lies between the hinged and fixed conditions. The actual $k$ value can be estimated from the Jackson and Moreland alignment charts in Figure 9.32. In lieu of these charts, the following equations suggested in the ACI Code commentary can also be used for calculating $k$:

1. **Compression members in non- travelers**: An upper bound to the effective length factor may be taken as the smaller of the following two expressions:

   \[ k = 0.7 + 0.05(\psi_a + \psi_b) \leq 1.0 \]  
   \[ k = 0.85 + 0.05\psi_{max} \leq 1.0 \]  

   where $\psi_a$ and $\psi_b$ are the values of $\psi$ at the two ends of the column and $\psi_{max}$ is the smaller of the two values. $\psi$ is the ratio of the stiffness of all compression members to the stiffness of all flexural members in a plane at one end of the column. Or

   \[ \psi = -\frac{\sum \frac{EI}{l_i}}{\sum \frac{EJ_i}{l_i}} \text{ moments} \]  

   where $l_i$ is the unsupported length of the column; $l_i$ is the clear beam span.

2. **Compression members in sway frames restrained at both ends**: The effective length may be taken as follows:

   For $\psi_{max} < 2$:  
   \[ k = \frac{20 - \psi_{max}}{20} \]  

   For $\psi_{max} \geq 2$:  
   \[ k = 0.9\sqrt{1 + \psi_{max}} \]  

   where $\psi_{max}$ is the average of the $\psi$ values at the two ends of the compression member.

3. **Compression members in sway frames hinged at one end**: The effective length factor may be taken as

   \[ k = 2.0 + 0.3\psi \]  

   where $\psi$ is the value at the restrained end.

   The radius of gyration $r = \sqrt{I/A}$ can be taken as $r = 0.3h$ for rectangular sections, where $h$ is the column dimension perpendicular to the axis of bending. For circular sections, the radius of gyration $r$ is taken as $0.25h$.

### 9.12.1 Rational Analysis of Buckling Considerations

Consider a slender column subjected to axial load $P_{ax}$, at an eccentricity $e$ in Figure 9.32. The buckling effect produces an additional moment $P_{ax}A$, where $A$ is the maximum lateral displacement of the compression member between its two ends from the vertical plumb position. This additional moment reduces the load capacity from point $C$ to point $B$ in the inelastic diagram (Figure 9.30). The total moment $P_{ax} + P_{ax}A$ is represented by point $B$ in the diagram, and the column should be designed for a larger magnified moment $M_{c}$ as a non-slender column by the usual first-order analysis.

In such an analysis, the moments and axial forces in a frame are obtained by the classical elastic procedures. These procedures do not consider the effects of the lateral displacement $A$ on the axial force $P_{ax}$ and the bending moment $M_{c}$. Consequently, the resulting load-deflection and load-moment relationships are linear. If the $P_{ax}$ effect is taken into account, a second-order analysis becomes necessary with a resulting nonlinear relationship that leads to the total displacement (deflection) and the moment. Frames that do not
Figure 9.32 Effective length factor $k$ for (a) braced and (b) unbraced frames.
have lateral bracing such as shear walls, diaphragms, or diagonal coupling beams are more flexible than those that are braced laterally. Lateral flexibility can cause the mass of a structure to sufficiently displace horizontally above the foundations to that significant additional overturning moments can result leading to loss of stability of the structure. This behavior is particularly critical when non-sleeder columns support the floors.

The ACI 318 Code stipulates three methods for determining the forces on slender columns and members in frames that resist lateral forces in addition to the vertical gravity loads. However, for gravity loading without side-way, a first-order analysis using moment magnification factors is adequate. For combined gravity and sideway forces causing the P-Δ effect, the three code methods are:

(a) Computer programs using a second-order analysis that determines iteratively the magnitudes of the additional overturning moments in a frame.
(b) Moment magnification factor β, computed on the basis of a first-order lateral displacements and the mass above each level in Section 9.13.2.
(c) Moment magnification relationship similar in form to those required for computing the no-way magnifier, βi, for columns in braced frames using a stability index Q. Horizontal displacement in this method need not be evaluated, but moments that resist lateral forces have to be computed. This method is too cumbersome and is the least accurate. The most reliable method is (a) using computer programs such as PCA’s Frame Program, STAAD, RCPC.DH, CSI, SAP2000, RISA, etc.

9.13 MOMENT MAGNIFICATION: FIRST-ORDER ANALYSIS

The factored axial forces P, the factored moments M, and M1 at the column ends, and, where required, the relative story deflections are computed in this method using an elastic first-order analysis with the section properties determined taking into account the influence of axial loads, the presence of cracked regions along the length of the member, and the effects of duration of the load. (Method b in Sec. 9.12.1.)

As discussed later in this section, the moment M1 is magnified by a magnification factor β. The column is subjected to moments M1 and M0 at its ends, where M1 is considered larger than M0. The factored axial force P0 and the factored moments M0 and M1 are resisted by analytically chosen section properties, taking into account the cracked regions along the compression member's length or height and the bond duration. In lieu of these computations, the ACI 318-05 Code allows using the following average values for properties of members in a structure for frame analysis:

1. Modulus of elasticity $E = \frac{3}{394} \sqrt{f_c}$ and for concrete strength $f_c > 5000$ psi $< 12,000$ psi

   $E = (40,000 \sqrt{f_c} - 1 \times 10^3) \left(\frac{w_{100}}{145}\right)^{0.5}$

2. Moment of inertia

   - Beams: 0.35$I_0$
   - Columns: 0.70$I_0$
   - Walls, uncracked: 0.70$I_0$
   - cracked: 0.35$I_0$
   - Flat plates and flat slabs: 0.25$I_0$

3. Area: 1.0A
4. Radius of gyration \( r = 0.30d \) for rectangular members, where \( h \) is in the direction of the compression member.

The moments of inertia should be divided by \( 1 + \beta \), when sustained lateral loads act on the members. The factor \( \beta \) is a creep factor, where for non-sway frames

\[
\beta = \frac{\text{maximum factored sustained axial load}}{\text{total factored axial load}}
\]

Hence, for non-sway frames, \( \beta \) is the ratio of maximum factored sustained axial load to maximum factored axial load. The degree of magnification depends on the slenderness ratio \( \ell / t \), where \( t \) is the effective length factor for compression members, as a function of the relative stiffnesses at the joint of each end of the member.

The magnification factor is controlled by the type of the magnified moments \( BM_1 \) and \( BM_2 \) acting at the respective ends \( 2 \) and \( 1 \) of a column, that is, whether sideways of the structural frame occurs or not. Note the case of compression members subjected to bending about both principal axes, that the moment about such axis should be separately considered on the restraint condition corresponding to that axis. For sideways analysis, the magnification method is approximate, and second-order analysis should be used instead.

### 9.13.1 Moment Magnification

In the case of compression members in non-sway frames, the effective length factor \( k \) can be taken as 1.0 after analysis gives a lower value. In such a case, \( k \) values are calculated on the basis of the \( EJ \) tabulated values and the monograms in Figure 9.32.

The slenderness effects can be disregarded if the following term is

\[
k_{c} = \frac{34 - \frac{12}{M_{1}}}{F_{c}} < 40
\]

and the term \( k_{c} / t \) is taken not greater than 40.

It should be noted that the maximum value of moment in a column at a joint occurs with live loads placed only on the beams supported on the same side or face of the column, both on the levels below and above the column. Such placement of live load forces the column into a reverse curvature deformation mode (double curvature) with \( M_1 / M_2 < 0 \). If live load placed on the level above a column were on the opposite face at the lower level, \( M_1 / M_2 > 0 \), and the column would deform in single curvature mode, as seen in Figure 9.32. Rarely is the magnified amount of single curvature be as large as the initial reverse curvature, hence usually not controlling (Furlong, Ref. 9.17).

\( k_{c} \) = effective length between points of intersection, and \( M_1 / M_2 \) is not taken less than -0.5. Since \( r = 0.30 \), this means that slenderness effect can be disregarded for \( M_1 / M_2 < -0.5 \) if \( k_{c} / t \leq 12.0 \). The term \( M_1 / M_2 \) is positive if the column is bent in a single curvature so that the two terms subtract in Eq. 9.34 and is negative in double curvature so that the two terms add (see Fig. 9.32a). If the non-sway magnification factor is \( h \), and the sway factor \( b = 0 \), the magnified moment becomes

\[
M_r = h_1 M_1, \quad (9.35)
\]
where

\[
\delta_m = \frac{C_m}{1 - \left(\frac{P_e}{0.75P_e}\right)} \geq 1.0
\]
\[
E' = \frac{x}{y}
\]
\[
C_m = \text{factor relating the actual moment diagram to an equivalent uniform moment diagram. For members without transverse loads, that is, subjected to end loads only,}
\]
\[
C_m = 0.6 + 0.4 \frac{M_1}{M_1} \geq 0.4
\]
\[
M_{x,\text{min}} = P_l (0.6 + 0.03h)
\]

where \( P_e \) is the Euler buckling load for pin-ended columns.

Stiffness \( EI \) is to be taken as

\[
EI = \frac{0.1E I_x + E I_y}{1 + \beta}
\]

or conservatively as

\[
EI = \frac{0.4 E I_x}{1 + \beta}
\]

\[
C_m = \text{factor relating the actual moment diagram to an equivalent uniform moment diagram. For members without transverse loads, that is, subjected to end loads only,}
\]

\[
C_m = 0.6 + 0.4 \frac{M_1}{M_1} \geq 0.4
\]

where \( M_1 \geq M_2 \) and \( M_1/M_2 > 0 \) if no inflection point exists between the column ends [Figure 9.32a (single curvature)]. For other conditions, such as members with transverse loads between supports, \( C_m = 1.0 \).

The minimum allowed value of \( M_1 \) is

\[
M_{x,\text{min}} = P_l (0.6 + 0.03h)
\]

where \( h \) is in inches.

In SI units \( M_{x,\text{min}} = P_l (15 + 0.03h) \), where \( h \) is in millimeters. In other words, the minimum eccentricity in the slender columns is \( \epsilon_{\text{min}} = 0.6 + 0.03h \). If \( M_{x,\text{min}} \) exceeded the applied moment \( M_2 \), the value of \( C_m \) in Eq. 9.37 should either be taken as 1.0 or be based on the actual computed end moments \( M_1 \) and \( M_2 \).

A slender rectangular column for which the design is based on reverse curvature moment \( M_1 \) about the major axis must be analyzed also for possible slenderness effects from unintended or minimum eccentricity of load about the minor axis of the section. The minor axis bending from minimum eccentricity ought to be considered as a single curvature condition with \( M_1 = M_1 = P_l \). Since such a column is restrained from rotation at its ends, an effective length for single curvature slenderness can be assumed to be 80% of the clear height of the column, unless effective length factors \( F \) are evaluated (Furlong, Ref. 9.17).

Frames braced against sway or braced with shear walls would normally have a lateral deflection less than total height \( h_l/1500 \). Once this ratio is exceeded, appropriate measure must be taken to minimize the additional moments caused by sway and hence to reduce lateral drift of the frame and its constituent columns.

9.13.2 Moment Magnification in Sway Frames
As stated in the previous section, use of second-order analysis is preferable than moment magnification. For compression members not braced against sway, the effective length factor \( k \) can also be determined from the \( EI \) values presented in Section 9.12 but for \( k \) values would exceed 1.3 the lower effective \( k \) are to be disregarded.
9.13 Moment Magnification: First-Order Analysis

\[ k_{sl} < 22 \]  \hspace{2cm} (9.39a)

The end moments \( M_e \) and \( M_s \) should be magnified as follows, with the nonsway moments unmagmified, provided that \( k_{sl} \) is less than 22N\((P/N/A_s)\):

\[
\begin{align*}
M_e &= M_{e,s} + 6, M_1 \\
M_s &= M_{s,e} + 6, M_1
\end{align*}
\]  \hspace{2cm} (9.40b)

On the assumption that \( M_e > M_s \), the design moment should be

\[ M_0 = M_{e,0} + 6, M_0 \]  \hspace{2cm} (9.41)

where \( M_{e,0} \) = factored end moment at the end of the compression member due to loads that cause no appreciable sway, calculated using a first-order elastic frame analysis.

\[ M_{s,0} = \text{factored end moment at the end of the compression members due to loads that cause appreciable sway, calculated using a first-order elastic frame analysis.} \]

where \( P_e \) is the summation of all the vertical loads in a story and \( P_e \) is the summation of all the vertical loads in a story [\( P_e = w \cdot El (k_{sl}) \) from Eq. 9.36b] with the \( El \) values obtained from Eq. 9.36c or 9.37.

In the case of an individual compression member having

\[ \frac{F_e}{f_e} > \frac{35}{\sqrt{P/N/A_s}} \]  \hspace{2cm} (9.42b)

the member must be designed for a factored axial load \( P_e \) and magnified moment \( M_0 = \delta_{e,0} M_e \). This condition can develop in slender columns with high axial loads when the maximum moment may develop between the end of the column, so the end moments might not necessarily be the maximum moments.

It should be emphasized that \( \delta_{e,0} \) factors apply for each column individually, but \( \delta_{a} \) factors apply to all columns at the same level, and the moment \((\delta_{a} - 1) M_0 \) must be applied to the beams at the column joint.

It is important to summarize that the moment magnification method, originally developed for prismatic columns, should work well for columns of slenderess ratio \( k_{sl} \) less than 100, particularly if the frame is braced. In the case of unbraced frames of comparable slenderess ratios, taking into account the \( P-\delta \) effect on the moments and deflections through a second-order analysis can give more comprehensive results. Such an analysis can be either of the following:

1. Execute several applications of the first-order analysis where the lateral load \( P_a \) in Figure 9.33 is incremented by \( \Sigma P_a, \delta_3 \) in each cycle, and consider the final result a second-order result.
2. Use a real second-order analysis computer program in which the reduction in the relative sway resistance is used in a global stiffness matrix for the elements involved.

9.13.3 Moment Magnification in Sway Frames Using A Stability Index, Q

In this method (method c in section 9.12.1), the code permits assuming a column in a braced structure as nonsway if the increase in column loads and moments due to second-order effects does not exceed 5% of the first-order end moments. A story within a struct-
Chapter 9  Combined Compression and Bending: Columns

ture can be considered nonsway if a stability index $Q$, in the following expression does not exceed a value of 0.05:

$$Q = \frac{\sum P \Delta_s}{V_c l_s}$$  \hspace{1cm} (9.43a)

where:

- $\sum P = \text{total vertical load at a story}$
- $V_c = \text{story shear}$
- $\Delta_s = \text{first order relative deflection between the top and bottom of a particular story due to shear} V_c$
- $l_s = \text{length of compression member in a frame measured from centers of joints}$

The sway magnification factor in terms of $Q$ is:

$$\delta_t = \frac{1}{1 - Q} \approx 1.0$$  \hspace{1cm} (9.43b)

When $Q$ exceeds a value of 0.05, proceed to a second-order analysis using computer programs. Such a computer analysis would make it possible to efficiently compute the interacting values of moments and $\Delta_s$, sway values due to the $P-\Delta$ effect in a reasonably accurate and speedy manner.

As indicated in Section 9.12.1, the stability index $Q$ method, while relatively adaptable to hand computations, is too cumbersome and inaccurate for effective evaluation of the $P-\Delta$ effect on moments at the column joints in a braced frame.

9.14 SECOND-ORDER FRAMES ANALYSIS AND THE $P-\Delta$ EFFECT

A second-order analysis is a frame analysis that includes the internal force effects resulting from lateral displacement (deflection) of a column. When such an analysis is performed in order to evaluate $\delta t M$ in a nonbraced frame, the deflections must be computed on the basis of fully cracked sections with reduced $EI$ stiffness values. Approximations such as the use of several first-order analysis cycles and idealizations of nonprismatic sections can be made in the analysis. But the analysis should verify that the predicted strength of the compression members of a structural frame are in good agreement within a 15% range with results of frame analysis for columns in indeterminate reinforced concrete structures. The structure being analyzed should result in geometry of members similar to the geometry of the sections to be built. If the members in the final structure have cross-sectional dimensions differing by more than 10% from those assumed in the analysis, a new computation cycle has to be performed.

A second-order analysis is an iterative procedure of the $P-\Delta$ effects on the slender column, including shear deformations. Hence, it is reasonable to expect that canned computer programs have to be used rather than long-hand computations in the design of the slender columns of a frame structure. An attempt will be made here to illustrate the iterative procedure involved in the use of several cycles of lateral load increments to the $P-\Delta$ values. It must be stated, however, that the large majority of columns in concrete building frames do not necessitate such an analysis since the $N/\ell_c$ ratio is in most cases below 300.

Consider the columns between the two floors $i-1$ and $i$ in the frame shown in Figure 9.33. Assume that the maximum lateral displacement $\Delta$ at the upper end of the top
column in the frame is \( x_{\text{max}} \), and that the total height of the building is \( h_i \). A large drift or lateral displacement of the building upper floors results in cracking of the masonry and interior finishes. Unless precautions are taken to permit movement of interior partitions without damage, the maximum lateral deflection limitation should be \( h/500 \). Hence a good assumption is to choose \( x_{\text{max}} \) in the range of \( h/350 \) to \( h/500 \), considering that a fully bowed frame has normally a ratio of maximum drift \( x_{\text{max}} \) to frame height \( h_i \) less than 1/1500.

If \( x_i \) is the drift at floor level \( i \), and \( h_i \) is the height of the column between floors \( i \) and \( i+1 \) in Figure 9.33(a), it can be assumed that the proportional horizontal drift for a particular floor is directly proportional to the square of the ratio of the height \( h_i \) of the floor and the total height \( h_i \) of the entire frame.

\[
\Delta_i = x_{\text{max}} \left( \frac{h_i}{h_i} \right)^2
\]  

(9.44)

The procedure can be summarized as follows:

1. Choose geometrical sections of the frame and its columns and their stiffness \( EI \) by approximate procedures.
2. Calculate the drifts, that is, the lateral deflections \( \Delta_i \) and the corresponding ultimate loads \( P_{\text{ult}} \) at joints \( i = 1, \ldots, n \) (Figure 9.33).
3. Find the equivalent horizontal forces \( H_i \) from \( H_i = P \cdot \Delta_i/h_i \) (Figure 9.33b).
4. Add the values obtained in step 3 to the axial lateral loads acting on the frame.
5. Perform a frame analysis using the appropriate computer program.
6. The iterative computer programs, using the stiffnesses \( EI \), chosen for the input data gives \( \Delta_i \) results that have to be compared with the \( x_i \) values allowed.
7. If all \( \Delta_i \) values are \( \leq \) all the \( x_i \) values, accept the solution and the design as a second-order solution. If not, run additional computer cycles with modified stiffnesses until the desired results are achieved.

Any of several computer programs can be utilized to account for the \( P-\Delta \) effects in frame sideways. Stodel, PCA Frame, STAAD PRO, SAP2000 are an example of such general-purpose programs.
9.15 OPERATIONAL PROCEDURE AND FLOWCHART FOR THE DESIGN OF SLENDER COLUMNS

1. Determine whether the frame has an appreciable sideways. If it does, use the magnification factors \( b_i \) and \( \beta \). If the sideways is negligible, assume that \( b_i = 0 \). Assume a cross-section, calculate the eccentricity using the greater of the end moments and check whether it is more than the minimum allowable eccentricity: that is,

\[
\frac{M_e}{P_e} = (0.6 + 0.036) \text{ in.}
\]  

If the given eccentricity is less than the specified minimum, use the minimum value.

2. Calculate \( \psi_e \) and \( \psi \), using Eq. 9.31. Obtain \( \alpha \) using Figure 9.32 or Eqs. 9.30a and b. Calculate \( k_{le}f_c' \) and determine whether the column is a short or long column. If the column is slender and \( k_{le}f_c' \) is less than 100, calculate the magnified moment \( M_e' \).

Using the \( M_e' \) value obtained, calculate the equivalent eccentricity to be used if the column is to be designed as a short column. If \( k_{le}f_c' \) is greater than 100, perform a second-order analysis.

3. Design the equivalent non-slender column. The flowchart (Figure 9.34) presents the sequence of calculations. The necessary equations are provided in Section 9.11 and in the flowchart.

9.15.1 Example 9.15: Design of a Slender (Long) Column

A rectangular tied column is part of a 3 x 3 bay frame building subjected to uniaxial bending. Its clear height is \( h = 18 \) ft (5.55 m) and it is not braced against sideways. The factored external load \( P_e = 726,000 \) lb (325 kN). The factored end moments are \( M_{le} = 750,000 \) in.-lb (1039 kNm) and \( M_e = 1,525,000 \) in.-lb (1723 kNm). The dead load and moment due to gravity are 90% of the total load and moment. Design the column section and the reinforcement necessary for the following two conditions:

1. Consider gravity loads only, assuming lateral sideways due to wind as negligible.

2. Consider sideways wind effects to cause an unfactored \( P_e = 90,000 \) lb (403.3 kN) and an unfactored \( M_{le} = 375,000 \) in.-lb (507 kNm). Also check stability of the columns using the stability \( Q \) index method.

Loads per floor of all columns at that level are:

\[ \sum P_i = 15.5 \times 10^6 \text{ lb (68,444 kN)} \]
\[ \sum V_i = 32.0 \times 10^6 \text{ lb (143,336 kN)} \]
\[ V_c = 145,000 \text{ lb/in. (644 kN/mm)} \]
\[ t = 1.5 \text{ in. (38 mm)} \]

Given:

\[ b_i = 0.5, \quad \beta = 2.0, \quad \psi = 3.0 \]

\[ f_c' = 5000 \text{ psi (34.5 MPa), normal concrete} \]

\[ f_y = 60,000 \text{ psi (413.7 MPa)} \]

\[ d' = 2.5 \text{ in. (64 mm)} \]

Solution for Case 1: Gravity Loading Only, sidesway negligible

Check for sidesway: and minimum eccentricity (step 1)

Since the frame has no appreciable sideways, the entire moment \( M_e \) is taken as \( M_{le} \) and magnification factor \( \beta \) is taken as equal to zero. By trial and adjustment, a column section is assumed and analyzed. Try a section 21 in. x 21 in. (533 mm x 533 mm) as shown in Figure 9.35(a).
Figure 9.34 Flowchart for design of slender columns.
actual eccentricity \( M_{new} = \frac{1.525 \times 10^6}{P_e} = 2.1 \text{ in. (52 mm)} \)

minimum allowable eccentricity \( 0.6 \times 0.63 \times 21 = 1.23 \text{ in. < 2.1 in.} \)

Use \( M_{new} = 1.525 \times 10^6 \text{ in.-lb (172.6 kN-m)} \).

Calculate the eccentricity to be used for equivalent short column (step 2):  
From the chart in Fig. 8.20b, \( k = 1.7 \).

actual slenderness ratio \( \frac{4kL}{r} = \frac{1.7 \times 18 \times 12}{0.3 \times 21} = 58.29 \)

allowable slenderness ratio \( \frac{4k}{r} \) for unbraced column = 22

As 58.29 > 22 and < 100, use the moment magnification method.

\( \frac{L}{r} = \frac{30.36 \times 10^3}{31.4 \times 10^3} = 0.96 \times 10^3 \text{ psi} \times 29 \text{ kPa} \)

\[ kL = \frac{\pi^2 EI}{kL} = \frac{12 \times 10^3}{6.26 \times 10^6} \text{ in.}^4 \]

\[ (kL)^2 = 1.7 \times 18 \times 13] = 134.8 \times 10^3 \text{ in.}^2 \]

Hence

\[ P_e = \text{Euler buckling load} = \frac{\pi^2 EI}{(kL)^2} \]

\[ = \frac{31.4 \times 10^3 \times 10^5 \times 10^9}{134.8 \times 10^3} = 1.356 \times 10^9 \text{ lb = 1536 kips (6032 kN)} \]

\[ C_n = 1.3 \text{ for unbraced column} \]

\[ \text{moment magnifier } b_n = \frac{C_n}{1 - P_e/0.75P_s} = \frac{1.3}{1 - \frac{240}{0.75 \times 10^9}} = 3.493 \]

\( \text{design moment } M_e = b_n M_{new} = 3.493 \times 1.525 \times 10^6 \]

\[ = 5.329 \times 10^6 \text{ in.-lb (581 kN-m)} \]

Assume that the reduction factor \( \Phi = 0.65 \) for the compression-controlled state.

\[ \text{required } P_e = \frac{P_e}{\Phi} = \frac{240}{0.65} = 1.116 \times 10^3 \text{ lb (4968 kN)} \]

\[ \text{required } M_e = \frac{M_e}{0.65} = \frac{5.329 \times 10^6}{0.65} = 8.199 \times 10^6 \text{ in.-lb (1722 kN-m)} \]

Hence design a nonender column section for axial load strength \( P_e = 1.116 \times 10^3 \text{ lb and a } \)

\( \text{bending strength } M_e = 8.199 \times 10^6 \text{ in.-lb} \),

\( \sigma = \frac{8.199 \times 10^6}{1.116 \times 10^3} = 7.34 \text{ in. (190 mm)} \)

Design of an equivalent nonender column (step 3)

Analyze the assumed 21 in. x 21 in. square section. Assume that \( \mu = 1.25 \% \).
\[ A_s = A_t = 0.60(25(21 \times 18.5)) = 4.86 \text{ in.}^2 \]

Provide five No. 9 bars (five of 26-mm diameter) on each face: \( A_s = A_t = 5.0 \text{ in.}^2(1228 \text{ mm}^2) \).

**Limit compression-controlled state (Step 4)**

For the design of an equivalent non-slender column (Step 4)

\[ d_s = 21.88 - 2.5 = 18.5 \text{ in.} \]
\[ c_s = 0.80 \]
\[ c = 0.80 \times 6.0 = 0.80 \times 18.5 = 11.1 \text{ in.} \]
\[ a = 0.80 \times 11.1 = 8.88 \text{ in.} \]

From Equation 9.9(b),

\[ f'_c = 67,405 \left(1 - \frac{d}{c} \right) = 70.000 \left(1 - \frac{21.88}{11.1}\right) = 67,405 \text{ psi} \]

Use \( f'_c = 60,000 \text{ psi} \)

\[ f_e = f' = 60,000 \text{ psi} \]

\[ P_e = C_e \times T_e, \quad T_e = A_e f, \quad A_e = 5.0 \times 60,000 = 300,000 \text{ lb} \]

\[ C_e = 0.85 f'_e, \quad b = 0.85 \times 5.000 \times 21 \times 8.88 = 792,540 \text{ lb} \]

\[ P_e = C_e \times T_e = 792,540 + 300,000 = 792,540 \text{ lb} \]

\[ M_e = C_e \left(\frac{b}{2} - d \right) + C_e \left(\frac{b}{2} - d \right) + T_e \left(\frac{b}{2} - d \right) \]

\[ = 792,540 \left(\frac{21}{2} - 8.88 \right) + 300,000 \left(\frac{21}{2} - 2.5 \right) + 792,540 \left(18.5 - \frac{21}{2}\right) = 9,602,792 \text{ in. lb} \]

Limit \( e = \frac{9,602,792}{792,540} = 12.11 \text{ in.} \) > actual \( e = 7.34 \text{ in.} \)

Hence, columns are compression-controlled.

**Trial and adjustment analysis of the section (Step 5)**

By trial and adjustment, try a larger value of neutral axis depth, \( e \), in order to increase the volume of the compressive block, thereby increasing the axial load \( P_e \).

**Assumed** \( e = 13.52 \text{ in.} \)

\[ a = 0.80 \times 13.92 = 11.136 \text{ in.} \]

\[ f'_c = 67,405 \left(1 - \frac{21.88}{13.52}\right) = 71,375 \text{ psi} \] > \( f_e = 60,000 \text{ psi} \)

\[ f_e = 71,375 \left(\frac{18.5}{13.52} - 1\right) = 28,925 \text{ psi} \]

\[ P_e = C_e \times T_e, \quad T_e = A_e f, \quad A_e = 5.0 \times 60,000 = 300,000 \text{ lb} \]

\[ C_e = 0.85 f'_e, \quad b = 0.85 \times 5.000 \times 21 \times 11.36 = 992,888 \text{ lb} \]

\[ C_e = A_e f'_c, \quad T_e = A_e f, \quad T_e = 5.0 \times 60,000 = 300,000 \text{ lb} \]

\[ T_e = A_e f, \quad T_e = 5.0 \times 26.623 = 133,125 \text{ lb} \]
\[ P_a = N = 903,888 + 300,000 - 143,125 = 1,150,763 \text{ lbs} \]

\[ M_e = \frac{993,888}{2} \left( 15 \text{ in} - \frac{11.136}{2} \right) + 30,000 \left( 0.5 - 0.5 \right) + 143,125(18.5 - 13.5) = 8,446,856 \text{ in} \cdot \text{lb} \]

\[ c = \frac{M_e}{P_a} = \frac{8,446,856}{1,150,763} = 7.34 \text{ in.} \text{ actual } c = 7.34 \text{ in.} \]

Therefore, compatibility analysis is verified, and the assumed section geometry for the slender column is OK.

Since the actual \( c = 14.92 \text{ in.} \text{ limit } c = 11.19 \text{ in.} \text{ 4 face tied column} = 0.65 \]

\[ P_e = 4 \cdot P_a = 0.65 \times 1,150,763 = 747,996 \text{ lbs} > \text{ actual } P_a = 726,000 \text{ lb} \]

Adopt the 21 x 21 in. column section with 5 No. 9 bars on each of the two faces parallel to the axis of bending, as shown in Figure 9.35.

**Design of Ties (Step 6)**

Try No. 3 ties (9.52 mm diameter); The spacing must be the least of:

1. In diameter No. 9 bar = 16 x \( \frac{9}{8} = 18 \text{ in.} \) (457 mm)
2. 48 diameter No. 3 ties = 48 x \( \frac{3}{8} = 18 \text{ in.} \) (457 mm)

Least dimension \( t = 21 \text{ in.} \) (533 mm)

Therefore, use No. 3 closed ties at 18 in. center-to-center (9.52 mm diameter at 457 mm spacing).

**Solution for Case 2: Gravity and Wind Loading (Sideways)**

\[ \frac{f_c}{r} = \frac{18 \times 12}{0.3 \times 21} = 34.3 \]

\[ \frac{55}{\sqrt{P_c/A_c}} = \frac{55}{\sqrt{726,000/5860 \times 21 \times 21}} = 60 > 34.3 \]

Hence, nonsway moments \( M_{n,\text{min}} \) need not be magnified.

From Equation 4.6(4), \( U = 1.22D + 0.5L + 1.6W \)

Hence, proportion the live load and moment of the gravity load by the ratio 0.5/1.6

Factored gravity \( P_g = 0.40 \times 726,000 = 290,400 \text{ lb} \)

Factored gravity \( P_e = 726,000 - 290,400 = 435,600 \text{ lb} \)

Service live \( P = 435,600 \times 1.6 = 722,560 \text{ lb} \)

\( P_g = 290,400 + 0.5 \times 272,550 + 1.6 \times 90,000 = 578,525 \text{ lb} \)

Factored gravity \( M_g = 0.40 \times 1,525,000 = 610,000 \text{ in} \cdot \text{lb} \)

Factored gravity \( M_e = 1,525,000 - 0.10 \times 915,000 = 893,250 \text{ in} \cdot \text{lb} \)

\( M_{g,c} = 0.40 \times (0.5 \times 1.6) \times 915,000 = 892,928 \text{ in} \cdot \text{lb} \)

\( M_{g,c} = 1.6 \times 575,000 = 920,000 \text{ in} \cdot \text{lb} \)

\[ h = \frac{1}{h} \leq \frac{1.0}{0.75} \frac{P_g}{P_e} = \frac{1.0}{0.75} \times \frac{578,525}{290,400} = 2.82 > 2.5 \text{ use } 2.5 \]

\[ \text{hence } h, M_a = M_e \text{ OK} \]

\[ M_a = 893,528 + 2.5 \times 920,000 = 3,195,528 \text{ in} \cdot \text{lb} \]

Required \( P_e = \frac{570,525}{10.5^2} = 877.731 \text{ lb} \)
9.16 Compression Members in Biaxial Bending

**Figure 9.35** Column geometry; strain and stress diagrams (balanced failure): (a) cross section; (b) balanced strain state; (c) stresses.

\[
\text{required } M_r = \frac{3,992,938}{0.65} = 6,016,028 \text{ in.-lb}
\]

\[
\text{eccentricity } e = \frac{4,916,028}{877,731} = 5.60 \text{ in.} < \text{Limit } e = 12.11 \text{ in.}
\]

Hence failure will be in compression, and \( e = 0.65 \) as assumed.

Conditions for case 2 do not control since failure is still in compression and the required \( P_r \) is less than that for case 1. Adopt the same section 21 in. x 21 in. with five No. 9 reinforcing bars on each of the two faces parallel to the neutral axis.

**Case 2, stability index \( Q \) check**

\[
\begin{align*}
\sum F_x &= 15.5 \times 10^3 \text{ lb} \\
V_y &= 1.45 \times 10^3 \text{ lb} \\
I &= 18.0 + 1.9 = 19.9 = 228 \text{ in.} \\
A_o &= 1.4 \text{ in.}
\end{align*}
\]

From Equation 9.43(a),

\[
Q = \frac{\sum F_x A_o}{V_y I} = \frac{15.5 \times 10^3 \times 1.4}{1.45 \times 10^3 \times 228} = 0.88 > 0.05.
\]

Since, one cannot consider this story column as a moment member.

From Equation 9.43(b),

\[
\delta = \frac{1}{1 - Q} = \frac{1}{1 - 0.88} = 2.94 \ll \infty
\]

Consequently a second-order analysis has to be performed unless one can otherwise, and with lengthy computational work, find relatively accurate values of sway magnitude \( \delta \). An approximate alternative is to proceed using the method computed in the previous section.

### 9.16 COMPRESSION MEMBERS IN BIAXIAL BENDING

**9.16.1 Exact Method of Analysis**

Columns in corners of buildings are compression members subjected to biaxial bending about both the \( x \) and \( y \) axes, as shown in Figure 9.36. Also, biaxial bending occurs due to imbalance of loads in adjacent spans and almost always in bridge piers. Such columns are subjected to moments \( M_x \) about the \( x \) axis, creating a load eccentricity \( e_x \) and a moment \( M_y \) about the \( y \) axis, creating a load eccentricity \( e_y \). Thus the neutral axis is inclined at an angle \( \theta \) to the horizontal.
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(a)

(b)

Figure 9.36  Corner column subjected to axial load: (a) biaxially stressed column cross section; (b) vector moments $M_x$ and $M_y$ in column plane.

The angle $\theta$ depends on the interaction of the bending moments about both axes and the magnitude of the load $P$. The compressive area in the column section can have one of the alternative shapes shown in Figure 9.36(a). Since such a column has to be designed from first principles, the trial-and-adjustment procedure has to be followed where compatibility of strain has to be maintained at all levels of the reinforcing bars. The process is similar to the one briefly outlined in Section 9.5.8 for columns with reinforcing bars on all faces. Additional computational effort is needed because of the position of the inclined n.e. cul-axis plane and the four different possible forms of the concrete compression area.

Figure 9.37 shows the strain distribution and forces on a biaxially loaded rectangular column cross section. $G_x$ is the center of gravity of the concrete compression area having coordinates $x_c$ and $y_c$, from the neutral axis in the directions $x$ and $y$, respectively. $G_y$ is the resultant position of steel forces in the compression area having coordinates $x_s$ and $y_s$, from the neutral axis in the directions $x$ and $y$, respectively. From the equilibrium of internal and external forces,

\[ M_x = P_x + \sqrt{M_{xu}^2 + M_{yu}^2} \]
Figure 9.37 Strain-compatibility and forces in biaxially loaded rectangular columns. (a) cross section; (b) strain; (c) forces.

\[ P_e = 0.85\sigma_f A_e + F_e - F_u \] (9.46)

where \( A_e \) = area of the compression zone covered by the rectangular stress block

\( F_e \) = resultant steel compressive force (2 \( A_e f_c \))

\( F_u \) = resultant steel tensile force (\( \Sigma A_{ue} f_{ue} \))

From equilibrium of internal and external moments,

\[ P_e x_1 = 0.85\sigma_f A_e x_1 + F_e x_1 + F_u x_2 \] (9.47a)

\[ P_e y_2 = 0.85\sigma_f A_e y_2 + F_e y_2 + F_u y_2 \] (9.47b)

The position of the neutral axis has to be assumed in each trial and the stress calculated in each bar using

\[ f_e = E e_u = E e_u \frac{L}{c} < f_e \] (9.48)

9.16.2 Load Contour Method

One method that gives a rapid solution is to design the column for the vector sum of \( M_{ex} \) and \( M_{ey} \), and use a circular reinforcing cage in a square section for the corner columns. However, such a procedure cannot be economically justified in most cases. Another design approach widely prevalent by experimental verification is to transform the biaxial moments into an equivalent uniaxial moment and an equivalent uniaxial eccentricity. The section can then be designed for uniaxial bending, as previously discussed in this chapter, to resist the actual factored biaxial bending moments.

Such a method considers a failure surface instead of failure planes and is generally termed the "Bresler-Pierce contour method" (Ref. 9.14, 9.15). This method involves cutting the three-dimensional failure surfaces in Figure 9.38 at a constant value \( P_e \) to give an interaction plane relating \( M_{ex} \) and \( M_{ey} \). In other words, the contour surface \( S \) can be viewed as a curved surface that includes a family of curves, termed the load contours.

The general non-dimensional equation for the load contour at a constant load \( P_e \) may be expressed as follows:
Figure 9.38  Failure interaction surface for biaxial column bending.

\[
\left( \frac{M_x}{M_y} \right)^2 + \left( \frac{M_y}{M_x} \right)^2 = 1.0
\]  
(9.49)

where \( M_x = P_e \) and \( M_y = P_e \).

\( M_x = M_x \) at such an axial load \( P \), where \( M_x \) or \( e_x = 0 \).

\( M_y = M_y \) at such an axial load \( P \), where \( M_y \) or \( e_y = 0 \).

Figure 9.39 Modified interaction contour plot of constant \( P \) for biaxially loaded column.
The moments \( M_{x0} \) and \( M_{y0} \) are the required equivalent resisting moment strengths about the \( x \) and \( y \) axes, respectively.

\( \alpha_x, \alpha_y \) = exponents depending on the cross-section geometry, steel percentage, and its location and material stresses \( f_x \) and \( f_y \).

Equation 9.49 can be simplified using a common exponent and introducing a factor \( \beta \) for one particular load value \( P_0 \) such that the \( M_{x0}/M_{y0} \) ratio would have the same value as the \( M_{x0}/M_{y0} \) as detailed by Parme and associates. Such simplification leads to

\[
\frac{M_{x0}}{M_{y0}} = \frac{M_{y0}}{M_{x0}} = 1.0
\]  
(9.50)

where \( \alpha \) would have a value of \((\log 0.5/\log \beta)\). Figure 9.39 gives a contour plot ABC from Eq. 9.50.

For design purposes, the contour is approximated by two straight lines BA and BC, and Eq. 9.50 can be simplified to two conditions:

1. For \( AB \) when \( M_{x0}/M_{y0} < M_{y0}/M_{x0} \),

\[
\frac{M_{x0}}{M_{y0}} = \frac{M_{y0}}{M_{x0}} = 1.0
\]  
(9.51a)

2. For \( BC \) when \( M_{x0}/M_{y0} > M_{y0}/M_{x0} \),

\[
\frac{M_{x0}}{M_{y0}} = \frac{M_{y0}}{M_{x0}} = 1.0
\]  
(9.51b)

In both Eqs. 9.51 a and b, the actual controlling equivalent uniaxial moment strength \( M_{x0} \) or \( M_{y0} \) should be at least equivalent to the required controlling moment strength \( M_{x0} \) or \( M_{y0} \) of the chosen column section.

For rectangular sections where the reinforcement is evenly distributed along all the column faces, the ratio \( M_{x0}/M_{y0} \) can be approximately taken as equal to \( b/h \). Hence Figs. 9.36a and b can be modified as follows:

1. For \( M_{x0}/M_{y0} > b/h \),

\[
M_{x0} = M_{y0} \frac{1 - \beta}{\beta} = M_{y0}
\]  
(9.52a)

2. For \( M_{x0}/M_{y0} < b/h \),

\[
M_{x0} = M_{y0} \frac{1 - \beta}{\beta} = M_{y0}
\]  
(9.52b)

The controlling required moment strength \( M_{x0} \) or \( M_{y0} \) for designing the section is the larger of the two values as determined from Eq. 9.57(a) and (b).

Plots of Figure 9.36 are used in the selection of \( \beta \) in the analysis and design of such columns. In effect, the modified load-contour method can be summarized in Eq. 9.55 as a method for finding an equivalent required moment strength \( M_{x0} \) and \( M_{y0} \) for designing the columns as if they were uniaxially loaded.

### 9.16.3 Step-by-Step Operational Procedure for the Design of Biaxially Loaded Columns by the Load Contour Method

The following steps can be used as a guideline for the design of columns subjected to bending in both the \( x \) and \( y \) directions. The procedure assumes an equal area of reinforcement on all four faces.
Figure 9.40 Contour β-factor chart for rectangular columns in biaxial bending.

1. Calculate the uniaxial bending moments assuming an equal number of bars on each column face. Assume a value of an interaction contour β factor between 0.80 and 0.70. Assume a ratio of h/b. This ratio can be approximated to $M_{x}/M_{y}$. Using Eqns. 52(a) and 52(b), determine the equivalent required uniaxial moment $M_{x}$ or $M_{y}$. If $M_{x}$ is larger than $M_{y}$, use $M_{x}$ for the design, and vice versa.

2. Assume a cross-section for the column and a reinforcement ratio $\rho = \rho' = 0.01$ to 0.02 for each of the two faces parallel to the axis of bending of the larger-equivalent moment. Make a preliminary selection of the steel bars. Verify the capacity $P_{y}$ of the assumed column cross-section. In the completed design, the same amount of longitudinal steel should be used on all four faces.

3. Calculate the actual nominal moment strength $M_{x,y}$ for equivalent uniaxial bending about the $x$ and $y$ axes when $M_{x} = 0$. Its value has to be at least equivalent to the required moment strength $M_{r}$.

4. Calculate the actual nominal moment strength $M_{x,y}$ for the equivalent uniaxial bending moment about the $x$ and $y$ axes when $M_{y} = 0$.

5. Find $M_{x,y}$ by entering $M_{x,y}/M_{r}$ and the trial $\beta$ value into the β factor contour plots of Figure 9.40.

6. Make a second trial and adjustment, increasing the assumed $\beta$ value if the $M_{x,y}$ value obtained from entering the chart is less than the required $M_{r}$. Repeat this step until the two values of $M_{x,y}$ converge either through changing $\beta$ or changing the section.

7. Design the lateral ties and detail the section.
9.15.4 Example 9.16: Design of a Biaxially Loaded Column by the Load Contour Method

A nonmember corner column is subjected to a factored compressive axial load \( P = 190,000 \) lb (853 kN), a factored bending moment \( M_{x} = 1,560,000 \) in.-lb (176 kN-m) about the \( x \) axis, and a factored bending moment \( M_{y} = 910,000 \) in.-lb (103 kN-m) about the \( y \) axis, as shown in Figure 9.41. Given:

\[
\begin{align*}
& f'_{c} = 4000 \text{ psi (27.6 MPa)}, \text{ normal-weight concrete} \\
& f'_{c} = 6000 \text{ psi (41.4 MPa)},
\end{align*}
\]

Design a rectangular and column section to resist the biaxial bending moments resulting from the given eccentric compressive load.

**Solution:** Calculate the equivalent uniaxial bending moments assuming equal numbers of bars on all faces (step 1)

Assume that \( d = 0.65 \) for tied columns.

- Required nominal \( P_{n} = \frac{195,000}{9.45} = 20,000 \) lb (87,868 kN)
- Required nominal \( M_{x,n} = \frac{1,560,000}{0.65} = 2,400,000 \) in.-lb (271 kN-m)
- Required nominal \( M_{y,n} = \frac{910,000}{0.65} = 1,400,000 \) in.-lb (164 kN-m)

\( \epsilon_{x} = 2,400,000 \)
\( \epsilon_{y} = 1,400,000 \)
\( r_{x} = \frac{2,400,000}{500,000} = 4.8 \) in.
\( r_{y} = \frac{1,400,000}{500,000} = 2.8 \) in.

Analyze for equivalent moment and equivalent eccentricity about the \( x \) axis since the larger of the two biaxial moments is \( M_{y} = 2,400,000 \) in.-lb about the \( x \) axis.

\[
\frac{M}{M_{y}} = \frac{2,400,000}{1,400,000} = 1.71
\]

![Figure 9.41: Compression and biaxial bending on corner column in Ex. 9.16.](image)
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Assume the section slope \( h = 30 \) in. (Figure 9.42).

Since the column dimensions are proportional to the applied moments, assume that \( \frac{a}{b} = 1.71 \) or \( a = 12 \) in. and \( b = 20 \) in. to give \( \frac{b}{h} = 1.67 \). Assume that the interaction contour factor \( \beta = 0.61 \).

Equivalent moment:

\[
M_{eq} = M_{eq} = M_e + M_b \frac{b - h}{\beta} = 3.989 \text{ kN-m} (3,989 \text{ kN-m})
\]

Verify capacity \( M_{pu} \) of the assumed column section (step 2).

From the previous section, \( b = 20 \) in. (508 mm) and \( h = 30 \) in. (762 mm). Assume that the steel ratio \( \rho = 0.012 \) and \( d' = 2.5 \) in. (64 mm), \( d = 12/2 = 6 \) in. (152 mm).

\[
A_c = A_e = 0.012 \times 12 \times 2.0 = 2.82 \text{ in.}^2
\]

Try 3 no. 8 bars, \( A_e = 5.37 \text{ in.}^2 (567 \text{ mm}^2) \) on each of the two 12-in. faces parallel to the \( x \) axis of bending in Figure 9.13.

Another way of trying a section is to select dimensions in the first trial using an approximation for equivalent transverse moment:

\[
M_{pu} = 1.1 \sqrt{(M_{eq})} - (M_{pu}/h)b
\]

Limit compression-controlled state (Step 3):

\[
\frac{f_c}{f_i} = 0.60, \quad d_i = 17.5 \text{ in.}
\]

\[
c = 0.60 \times 17.5 = 10.5 \text{ in.}
\]

\[
a = 0.65 \times 10.5 = 8.925 \text{ in.}
\]

From Equation 9.9(b),

\[
f_i' = 87,000 \left[ 1 - \frac{d}{c} \right] = 87,000 \left[ 1 - \frac{2.5}{10.5} \right] = 66,258 \text{ in.} > f_i
\]

![Figure 9.42](image.png)  
Figure 9.42  Equivalent column geometry: strain and stress diagrams (balanced failure): (a) cross section; (b) strains; (c) stresses.
9.16 Compression Members in Axial Bending

Use $f_c^* = 60,000$ psi
\[ f_e = f_c = 60,000 \text{ psi} \]
\[ P_e = C_e - C_t \]
\[ C_e = A_e f_e = 0.85 f_e \] b.a. \[ = 0.85 \times 60,000 \times 12 \times 0.935 = 364,140 \text{ lb} \]
\[ C_t = A_t f_t = 2.37 \times 60,000 = 142,200 \text{ lb} \]
\[ T_e = A_e f_e = 0.85 \times 60,000 = 142,200 \text{ lb} \]

Limit $P_{ue}$: $C_e - C_t = 364,140 + 142,200 = 506,340 \text{ lb}$

Limit $M_{ue}$: $C_e - C_t = 364,140 + 142,200 = 506,340 \text{ lb}$

\[ M_{ue} = C_e - C_t = 364,140 + 142,200 \left( \frac{20}{2} - 2.5 \right) + 142,200 \left( \frac{17.5 - 20}{2} \right) = 4,149,425 \text{ in.-lb} \]

Limit $e_{ue} = 4,149,425 = 11.4$ in.

**Trial and adjustment analysis of section for bending about the x-axis (Step 4)**

Since actual $e = 8.0$ in. \(< limit e_r = 11.4$ in., try a neutral axis depth, $t$ larger than limit-state neutral axis depth for this compression-controlled state.

Assume $e = 12.1$ in.

\[ a = 8 \times 12.1 = 96.8 \text{ in.} \]

\[ f_c^* = 60,000 \left( 1 - \frac{20}{12.1} \right) = 50,025 \text{ psi} > f_e, \text{ use } f_e = 60,000 \text{ psi} \]

\[ f_e = 60,000 \left( \frac{17.5 - 1}{12.1} - 1 \right) = 38,826 \text{ psi} \]

\[ P_e = C_e - C_t \]
\[ C_e = 0.85 f_e \] b.a. \[ = 0.85 \times 60,000 \times 12 \times 0.935 = 346,568 \text{ lb} \]
\[ C_t = A_t f_t = 2.37 \times 60,000 = 142,200 \text{ lb} \]
\[ T_e = A_e f_e = 0.85 \times 60,000 = 142,200 \text{ lb} \]
\[ P_e = 346,568 + 142,200 - 92,018 = 456,750 \text{ lb} \]
\[ M_{ue} = 456,750 \left( 10 - \frac{20}{2} \right) + 142,200 \left( 10 - 2.5 \right) + 92,018 \left( 17.5 - 10 \right) = 3,795,735 \text{ in.-lb} \]

\[ e_r = \frac{M_{ue}}{P_e} = \frac{3,795,735}{456,750} = 8.2 \text{ in. = actual } e = 8.0 \text{ in.} \]

Therefore, compatibility analysis is satisfied in bending about the x-axis.

**Calculate the actual nominal resisting moment $M_{ue}$ for equivalent uniaxial bending about the x-axis when $M_{ue} = 0$ (Step 3).**

Required $P_e = 300,000$ lb. Assuming that the compression steel has yielded (to be vertical later), $f_e = 60,000$ and $A_t f_t = A_e f_e = 0$. Hence $P_e = 0.85 f_e$

\[ \sigma = \frac{P_e}{A_e} = \frac{300,000}{0.85 \times 4000 \times 12} = 7.35 \text{ in.} \]
\[ e = \frac{d}{2} \cdot \frac{7.95}{0.85} = 8.65 \text{ in.} \]

\[ f_y = 87,000 \left( \frac{8.65 - 2.5}{8.65} \right) = 61,923 \text{ psi} > 60,000 \text{ O.K.} \]

\[ f_y = 87,000 \left( \frac{17.5 - 8.65}{17.5} \right) = 49,765 \text{ psi} > f_y = 60,000 \text{ psi.} \]

\[ \text{hence } f_y = 60,000 \text{ psi} \]

\[ M_{uw} = P_e e = 0.857h \left( \frac{d}{2} - e \right) + A_f f_y (d - e) \]

or

\[ M_{uw} = 0.85 \times 4000 \times 12 \times 12 \times \frac{20}{2} \cdot \frac{7.25}{2} + 2.37 \times 6000 \left( \frac{20}{2} - 2.5 \right) \]

\[ = 3.37 \times 6090 \left( \frac{17.5}{2} - \frac{20}{2} \right) \]

\[ = 4,029,741 \text{ in.-lb} (553 \text{ kN-m}) > M_{uw}(3,894,787 \text{ in.-lb}) \text{ O.K.} \]

If this calculation showed that \( M_{uw} < M_u \), obtain \( M_u \) in step 1, revise the assumed cross section by increasing the flange area or enlarging the section, or both.

**Calculate the actual nominal resisting moment \( M_{uw} \) for the equivalent uniaxial bending about the y axis when \( M_u = 0 \) (step 2).**

In this condition, \( b = 20 \text{ in.}, c = 12 \text{ in.}, d = 9.5 \text{ in.}, \) and \( A_t = 12 \text{ in.}^2 \). By trial and adjustment, choose compressive stress \( a \) such that the calculated \( P_e \) approximates the required \( P_u \).

At the third trial, \( a = 4.8 \text{ in.} \) and \( e = 8.59 \text{ in.} = 5.65 \text{ in.} \)

\[ P_e = 87,000 \left( \frac{7.65 - 2.5}{3.65} \right) = 48,904 \text{ psi} \]

\[ P_e = 87,000 \left( \frac{d - e}{e} \right) = 87,000 \left( \frac{9.5 - 5.65}{5.65} \right) = 59,283 \text{ psi} \]

\[ P_e = 0.85 \times 4000 \times 20 \times 4.8 = 2.37 \times 48,904 = 2.37 \times 59,283 \]

\[ = 300,894 = \text{required } P_u = 300,000 \text{ O.K.} \]

Hence use \( a = 4.8 \text{ in.} \) for calculating \( M_{uw} \).

\[ M_{uw} = 3.85 \times 4000 \times 20 \times 4.8 \left( \frac{12}{2} - \frac{8.8}{2} \right) + 2.37 \times 48,904 \left( \frac{12}{2} - 2.5 \right) \]

\[ + 2.37 \times 59,283 \left( \frac{5}{2} - \frac{12}{2} \right) = 2,069,123 \text{ in.-lb} \]

**Find \( M_u \) by entering \( M_{uw} / M_u \) and trial \( \beta \) value into the \( \mu \)-factor contour plots in Fig. 9.40 (step 3).**

First trial \( \beta = 0.5 \text{ in.} \); from step 2, \( M_{uw} = 4,029,741 \text{ in.-lb} \)

\[ \frac{M_{uw}}{M_u} = 4,029,741 = 0.996 \]

Enter into Figure 9.40 the values of \( \beta = 0.5 \text{ in.} \) and \( M_{uw} = 0.996 \) to get

\[ \frac{M_{uw}}{M_u} = 0.62 \]
But \( M_{a,\text{req}} \) from step 4 is 2,069.123 in.-lb or

\[
\frac{M_{a,\text{req}}}{2,069.123} = 0.62
\]

Hence

\[M_a = 0.62 \times 3,053.133 = 1,882.867 \text{ in.-lb}\]

< required \( M_a = 1,400,000 \text{ in.-lb} \)

Revising the solution assuming a higher \( \beta \) value. If adjusting \( \beta \) does not give the actual \( M_a \) at least equal to the required \( M_{a,\text{req}} \), increase the reinforcement area or enlarge the section.

Second trial and adjustment (step 6)

Assume the same section but assume that \( \beta = 0.63 \)

\[M_a = 2,400,000 \times 1,400,000 \times 1.87 \times 1 - 0.63 = 3,773.111 \text{ in.-lb}\]

\[M_a = 3,773.111 = M_{a,\text{req}} = 3,795.726 \text{ in.-lb}, \text{ hence O.K.}\]

Trial and adjustment analysis of section for bending about the y-axis (Step 6)

Actual \( \varepsilon = 4.67 \text{ in.}\)

Assume \( \varepsilon = 13.6 \text{ in.} \)

\[a = \beta \varepsilon = 0.63 \times 13.6 = 8.52 \text{ in.}\]

\[f_e = \frac{87,000 \times (1 - 25 \times 0.126)}{12} = 73,057 \text{ psi} \text{ > } f_s = 60,000 \text{ psi}\]

\[f_a = \frac{87,000 \times (17.5 - 1)}{15.6} = 10,596 \text{ psi}\]

\[C_a = 0.85f_a; a = 0.85 \times 4,000 \times 12 \times 13.26 = 541,000 \text{ lb}\]

\[C_s = A_f f_s = 237 \times 60,000 = 14,200 \text{ lb}\]

\[T_a = A_f f_a = 237 \times 10,596 = 25,113 \text{ lb}\]

\[P_{e,\text{a}} = C_s + C_a - T_a = 541,008 + 14,200 - 25,113 = 540,095 \text{ lb}\]

\[M_{a,\text{a}} = 540,095 \times \frac{10 - 12.25}{2} = 142,200 \times (10 - 2.5) = 23,113 (17.5 - 10) = 3,078,044 \text{ in.-lb}\]

\[\varepsilon = \frac{M_{a,\text{a}}}{P_{e,\text{a}}} = \frac{3,078,044}{540,095} = 4.68 \text{ in., actual } \varepsilon = 4.67 \text{ in.}; \text{ O.K.}\]

Therefore, compatibility analysis for bending about the y-axis is satisfied.

From step 3, \( M_{a,\text{a}} = 4,029,741 \text{ in.-lb} \) since this value does not change as long as the section and its reinforcement remain the same. \( M_{a,\text{a}}/M_{a,\text{req}} = 0.596 \) from before and \( \beta = 0.63 \).

From contour plots in Fig. 9.40, \( M_a/M_{a,\text{a}} = 0.63 \). From step 4, \( M_{a,\text{a}} = 2,069,123 \).

\[M_a = 0.63 \times 2,069,123 = 1,407,010 \text{ in.-lb} > \text{ required } M_a = 1,400,000 \text{ in.-lb}\]

Adopt the design.

The use of hand-held or desk computers, or charts in steps 2 to 6 reduces the calculation effort for biaxially loaded columns almost to that for the design of uniaxially loaded columns.
Select the longitudinal and lateral reinforcement (step 7).

**Longitudinal bars:** Provide three No. 8 bars (25.4-mm diameter) on each of the two 12-in. wide faces. Provide one No. 6 bar at the center of the 20-in. wide face so that each face of this column would have an equal number of reinforcing bars.

**Lateral ties:** Try No. 3 bar lateral ties. The spacing should be the minimum of

\[ 16 \times \text{longitudinal bar diameter} = 16 \times \frac{25.4}{2} = 16 \text{ in.} \]

\[ 48 \times \text{lateral tie diameter} = 48 \times \frac{3}{8} = 18 \text{ in.} \]

The minimum lateral dimension is 12 in. Therefore, use No. 3 (9.53-mm diameter) lateral ties at 12 in. (305 mm) center to center. Reinforcing details are shown in Figure 9.43.

### 9.16.5 Reciprocal load Method

This method developed by Bresler relates the desired axial force \( P_a \) value to three other values on a reciprocal of the failure surface (Ref. 9.11). Assume \( S_1 \) denotes the coordinates on the failure surface in Figure 9.38 such that the values of the load and eccentricities as \( P_{ax}, \epsilon_a, \) and \( \epsilon_1. \) If \( S_1 \) is a point on the compatible reciprocal surface to that in Figure 9.38, then \( P_a \) would satisfy the coordinates of that point as \( P_{ax}^2 \epsilon_a^2 + \epsilon_1 \), where \( P_a = 4P_{ax} \) which is the factored (design) load.

If the desired axial load \( P_a \) under biaxial loading about the x and y axes is related to the \( P_a \) values denoted by \( P_{ax}, P_{ay}, \) and \( P_{xy} \), then

\[
\frac{1}{P_a} = \frac{1}{P_{ax}} + \frac{1}{P_{ay}} + \frac{1}{P_{xy}}
\]

or

\[
\frac{P_a}{P_{ax}} = \frac{P_a}{P_{ay}} = \frac{P_a}{P_{xy}} = 1
\]

- \( P_{ax} = \) nominal axial load at eccentricity \( \epsilon_a \) along the x-axis, \( \epsilon_a = 0 \)
- \( P_{ay} = \) nominal axial load at eccentricity \( \epsilon_y \) along the y-axis, \( \epsilon_y = 0 \)
- \( P_a = \) nominal axial load, namely \( \epsilon_a = \epsilon_y = 0 \)
- \( M_{ax} = \) moment about the x-axis = \( P_a \epsilon_a \)
- \( M_{xy} = \) moment about the y-axis = \( P_a \epsilon_y \)

![Figure 9.43 Biaxially loaded column section.](image)
Figure 9.44 Biaxial-bending reciprocal load solution (Ref. 9.8).

- $e_x = \text{eccentricity measured parallel to the x-axis as in Figure 9.7a$, namely, } e_x = M_x/P_v$
- $e_y = \text{eccentricity measured parallel to the y-axis as in Figure 9.7a$, namely, } e_y = M_y/P_v$
- $x = \text{column cross-section dimension parallel to the x-axis}$
- $y = \text{column cross-section dimension parallel to the y-axis}$

The step-by-step operational procedure essentially follows the logic in the steps presented in Sec. 9.15.3.

The set of plots in Figure 9.44 enables obtaining a rapid solution for the axial load $P_v$ of a biaxially loaded compression member.

9.16.6 Modified Load Contour Method

In lieu of Equation 9.54, Hsu in Ref. 9.16 proposed a modified expression which can represent both the strength interaction diagram and the failure surface of a reinforced concrete biaxially loaded column as in Figure 9.45 modifying the approach presented in Sec. 9.16.2. This method as well as the reciprocal load method seems to demand less computational rigor as can be seen from the two design examples to follow.
Figure 9.45 Failure Surface Interaction Diagram (Ref. 9.13) (a) axial bending on compression, (b) bi-axial bending and tension.

The interaction expression for the load and bending moments about the two axes is

\[
\left( \frac{P_e - P_{e,0}}{P_{e,0}} \right) + \left( \frac{M_{e,0}}{M_{e,0}} \right)^\alpha + \left( \frac{M_{e,0}}{M_{e,0}} \right)^\beta = 1.0
\]

(9.53)

where

- \( P_e \) = nominal axial compression (positive), or tension (negative)
- \( M_{e,0} \), \( M_{e,0} \) = nominal bending moments about the \( x \)- and \( y \)-axes respectively
- \( P_{e,0} \) = minimum nominal axial compression (positive) or axial tension (negative) \( = 0.85 f_y [A_y - A_x] + f_x A_x \)
- \( P_{e,0} \) = nominal axial compression at the limit strain ratio \( (\varepsilon_y = 0.002) \)
- \( M_{e,0} \), \( M_{e,0} \) = minimum bending moments about the \( x \)- and \( y \)-axes respectively, at the limit strain state \( (\varepsilon_y = 0.002) \).
The value of \( P_m \) and \( M_{ef} \) can be obtained from:

\[
P_m = 0.85f'_c/b_c/c_b + A_{fl}' - A_f
\]

(9.56a)

and

\[
M_{ef} = P_m c_e = C_e \left( \frac{d - \frac{b}{2} - d''}{2} \right) + C_e (d - d' - d'') + Td'
\]

(9.56b)

where
- \( a \) = depth of the equivalent block = \( b/c_b \)
- \( c_e = \left( \frac{0.003}{\sqrt{f'_c + 0.003}} \right) d = \left( \frac{87,000}{87,000 + f'_c} \right) d \)
- \( f'_c \) = stress in the compressive reinforcement closest to the load
- \( f'_c \), if \( f'_c \neq f'_t \)
- \( T \) = Force in the tensile side reinforcement.

The step-by-step operational procedure for the design of biaxially loaded columns essentially follows the procedure of Sec. 9.16.3.

9.16.7 Example 9.17: Design of a Biaxially Loaded Column by the Reciprocal Load Method

Design the reinforced concrete rectangular nonflanged column of Example 9.16 by the reciprocal load method.

Solution:

- \( P_c = 195,000 \text{ lb (866 kN)} \)
- \( M_{ce} = P_c r = 1,560,000 \text{ in.-lb (176 kN-m)} \) about the \( x \)-axis.
- \( M_{de} = P_d r = 910,000 \text{ in.-lb (105 kN-m)} \) about the \( y \)-axis.
- \( f'_c = 4000 \text{ psi (28 MPa)} \) normal weight.
- \( f'_t = 60,000 \text{ psi (414 MPa)} \).

Hence:

\[
r_e = \frac{M_{ce}}{P_c} = \frac{1,560,000}{195,000} = 8.0 \text{ in.}
\]

\[
r_e = \frac{M_{de}}{P_d} = \frac{910,000}{195,000} = 4.7 \text{ in.}
\]

\( x \)-axis parallel to the shorter side \( b \).
\( y \)-axis parallel to the longer side \( h \).

(a) Preliminary choice of column section

Assume a total reinforcement percentage of 2.5%. Use a section with reinforcing bars on all faces.

\[
P_e = \frac{P_c}{\phi} = \frac{195,000}{0.65} = 300,000 \text{ lb.}
\]

From Equation 9.3(a), the estimated preliminary gross area of the column section if axially loaded is
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\[ A_d > \frac{P_c}{0.45f_c' + f_e}(\frac{300.000}{0.45(4000 + 60,000 \times 0.025)}) > 121.2 \text{ in}^2 \]

Since the column is initially loaded with eccentricities \( e_e = 8.0 \text{ in.} \) and \( e_p = 4.67 \text{ in.} \), assume that \( A_d = \text{net area obtained from Eq. 9.3(a)} \) for axial loading with zero eccentricity.

For a first trial assume a section \( = (12.12) \times 242 \text{ in}^2 \)

\[ \begin{align*}
\text{Required} & \quad \frac{M_{cr}}{M_{ep}} = \frac{90000}{1560000} = 0.58 = \frac{b}{h} \\
A_d & = bh = 0.58(242) = 140 \text{ in.}^2 \\
b & = 20.4 \text{ in.} \quad h = 11.9 \text{ in.}
\end{align*} \]

Try a section \( 12 \text{ in.} \times 26 \text{ in.}, A_d = 240 \text{ in.}^2 \)

(b) Bending about the x-axis

\[ A_d = 12 \times 20 = 240 \text{ in.}^2 \]

\[ \begin{align*}
P_{cr} & = 466750 \text{ lb (from Step 4, Ex. 9.16)} \\
P_e & = 658095 \text{ lb (from Step 6, Ex. 9.16)} \\
A_e & = 8 \text{ No. 8 bars } = 8 \times 0.73 = 5.84 \text{ in.}^2 \\
f_e & = 6.32 \frac{12}{20} = 0.056 \\
P_{nc} & = 0.85 f_y (A_d - A_e) + A_e f_e \\
& = 0.85 \times 4000 (240 - 6.32) + 6.32 \times 60000 = 1,173,712 \text{ lb} \\
\frac{1}{P_e} & = \frac{1}{P_{nc}} + \frac{1}{P_{cr}} \\
& = \frac{1}{466,750} \quad \frac{1}{658,095} \\
& = 1.173,712 \times 2.81 \times 10^{-4}
\end{align*} \]

\[ P_e = \frac{1}{2.81 \times 10^{-4}} = 355,872 \text{ lb} > 300,000 \text{ lb}, \quad \text{O.K.} \]

Note that the reciprocal method tends to be usually conservative.

Adopt the section and the reinforcement as shown in Fig. 9.43 of Ex. 9.16

Alternate Graphical Solution Using the Chart in Fig. 9.44

\[ \begin{align*}
f_e & = 6.32 \frac{12}{20} = 0.056 \\
P_{nc} & = 0.85 f_y (A_d - A_e) + A_e f_e \\
& = 0.85 \times 4000 (240 - 6.32) + 6.32 \times 60000 = 1,173,712 \text{ lb} \\
P_{cr} & = 466,750 \text{ lb (from Step 4, Ex. 9.16)} \\
P_e & = 658,095 \text{ lb (from Step 6, Ex. 9.16)} \\
P_{nc} & = 466,750 \\
P_e & = 1,173,712 \\
P_e & = 658,095 \\
P_e & = 1,173,712 \\
P_e & = 658,095 \\
P_e & = 0.36
\end{align*} \]
From the chart in Fig. 4.44, \( \phi_P = 0.3 \)
Hence, \( P_{cr} = 0.3 \times 1,173,712 = 352,114 \text{ lb} > \text{Required } P_{cr} = 300,000 \text{ lb, O.K.} \)

9.16.8 Example 9.18: Design of a Biaxially Loaded Column by the Modified Load Contour Method

Design the reinforced concrete rectangular non-sleender column of Example 9.17 by the modified load contour method.

Solution:
\[
\begin{align*}
P_c &= 195,000 \text{ lb (878 kN)} \\
M_{xx} &= P_c e_x = 1,560,000 \text{ in.-lb (1,760 kN-m)} \text{ about the } x \text{-axis.} \\
M_{yy} &= P_c e_y = 910,000 \text{ in.-lb (103 kN-m)} \text{ about the } y \text{-axis.} \\
f_y &= 4000 \text{ psi (27.6 MPa), normal weight} \\
f_f &= 60,000 \text{ psi (414 MPa)}
\end{align*}
\]

Hence:
\[
\begin{align*}
\varepsilon_x &= \frac{M_{xx}}{P_c} = \frac{1,560,000}{195,000} = 8.0 \text{ in.} \\
\varepsilon_y &= \frac{M_{yy}}{P_c} = \frac{910,000}{195,000} = 4.7 \text{ in.}
\end{align*}
\]

Axial parallel to the shorter side b.

y-axis parallel to the longer side A.

The expression for interaction of biaxially loaded columns from equation 9.59 is:
\[
\left( \frac{P_e - P_{cr}}{P_{cr}} \right) + \left( \frac{M_{xx}}{M_{cr}} \right)^{3/2} + \left( \frac{M_{yy}}{M_{cr}} \right)^{3/2} = 1.0
\]

Assume a column section based on the same assumptions as in part (a) of Ex. 9.17.

\( b = 12 \text{ in.} \)
\( h = 20 \text{ in.} \)
\( d = 2.5 \text{ in.} \)
\( d_t = \text{ (eight No. 8 bars) } = 6.32 \text{ in.} \)

From example 9.16, \( d = 0.65 \), Limit \( a = 8.925 \text{ in.} (227 \text{ mm})\)
\( f_y = 66,286 > f_y \), hence compression steel yielded
\( P_{cr} = 195,000 \text{ lb} \)
\( f_y = 0.65 = 300,000 \text{ lb} \)

Limit \( P_{cr} \) = 368,800 lb, from Ex. 9.16.

Since three No. 8 bars are at each of the two faces and two bars at the neutral axis, use equation 9.13 in lieu of Eq. 9.69.

From Eq. 9.16, Limit \( M_{cr} = 4,113,875 \text{ in.-lb} \)
\[
\begin{align*}
e_0 &= \frac{M_{cr}}{P_{cr}} = \frac{4,113,875}{368,800} = 11.5 \text{ in.} \\
e_0 &= \varepsilon_0 \geq 8.0 \text{ in.}, \text{ hence compression failure and } \phi = 0.65
\end{align*}
\]
\[
\begin{align*}
e_{cr} &= B \frac{87,000}{87,000 + f_y} d \\
e_{cr} &= 0.85 \left( \frac{87,000}{87,000 + 60,000} \right) (12 - 2.5) = 4.78 \text{ in.} (122 \text{ mm}).
\end{align*}
\]

Limit \( P_{cr} = 0.85 f_y d_0 = 0.85 \times 4000 \times 20 \times 4.78 \)
\[ M_{nc} = 0.85 \times 4000 \times 20 \times 4.76 \left( \frac{12}{2} - \frac{12}{2} \right) + 3 \times 0.79 \times 60,000 \left( 0.5 - \frac{12}{2} \right) \\
= 2,006,270 \text{ in.-lb} (227 \text{ kN-m}) \]

\[ P_{nc} = 0.85 f_c (A_s - A_d) + A_d f_t \\
= 0.85 \times 4000 \times (240 - 6.32) + 6.32 \times 60,000 \\
= 784,500 + 379,200 = 1,173,700 \text{ lb} \]

\[ M_{nc} = 1,560,000 \text{ in.-lb} (217 \text{ kN-m}) \]

\[ b = 12 \text{ in., } h = 20 \text{ in., } d = 17.5 \text{ in.} \]

Using the interaction surface expression for flexural bending in Equation 9.55.

\[ \left( \frac{P_n - P_{nc}}{P_{nc}} \right)^{0.35} + \left( \frac{M_n - M_{nc}}{M_{nc}} \right)^{0.35} + \left( \frac{M_{nc}}{M_{nc}} \right)^{0.35} = \frac{300,000 - 325,000}{1,173,700 - 325,000} + \left( \frac{2,490,000}{2,006,270} \right)^{0.35} = 0.039 + 0.446 + 0.583 = 0.956 \approx 1.0 \]

Hence, accept the design. Namely:

- \( b = 12 \text{ in.} \)
- \( h = 20 \text{ in.} \)
- \( d = 17.5 \text{ in.} \)

9.17 SI EXPRESSIONS AND EXAMPLE FOR THE DESIGN OF COMPRESSION MEMBERS

1. Limit \( c = \frac{600}{f_y} \)

2. \( f_s = 0.003 E_s \frac{c}{d} = f_y \)

3. \( E_s = 10^6, 0.043 \sqrt{f_y} \text{ MPa where } \psi = 1500 \text{ to } 2500 \text{ kg/m}^3 (90 \text{ to } 1554 \text{ lb/ft}^3). \text{ For standard normal-weight concrete, } \psi = 2600 \text{ kg/m}^3 \text{ to give } E_s = 29,700 \text{ MPa.} \)

4. Modulus of rupture \( f_y = 0.7 \sqrt{f_y} \)

5. \( E_s = 200,000 \text{ MPa.} \)

6. \( \phi P_{con} = 0.80 \phi f_t (A_s - A_d) + f_t A_d \) where \( \phi = 0.8 \text{ for tied columns and } 0.85 \text{ for spirally reinforced columns.} \)

7. Ratio of spiral reinforcement should not be less than the value

\[ \rho_s = 0.45 \left( \frac{A_s}{A_d} \right) \frac{f_t}{f_y} \] where \( f_y \) is the specified yield strength of the spirals, but not to exceed 400 MPa.

8. \[ k \frac{f_t}{f_y} \leq \left( \frac{34 - 12 M_{nc}}{M_{nc}} \right) \]

9. \( EI = \frac{0.2 E_s I_s + f_t I_s}{1 + \beta_d} \text{ or } EI = \frac{0.4 E_s I_s}{1 + \beta_d} \)
10. For compression members without transverse loads
   \[ c_n = 0.6 + 0.4 \frac{M_L}{M_t} \geq 0.4 \]
   where \( M_t/M_L \) is positive for single curvature. For members with transverse loads between supports, \( c_n = 1.0 \).

11. \( M_{L,max} = P(1.5 + 0.03h) \) at each axis separately, where 1.5 and h are in millimeters.

12. \( M_L = M_{L,max} + \delta M_L \) and \( M_t = M_{t,max} + \delta M_t \).

13. \( \delta M_L = \frac{M_L}{1 - \sum P_i(0.75 \sum P_i)} \geq M_L, \) where \( \Sigma P_i = \sum \frac{P_i \pi E}{L_i} \) as the Euler buckling load.

9.17.1 SI Example on Column Design

Solve Ex. 9.5 using SI units.

\[
\begin{align*}
\sigma_s &= 276 \text{ MPa} \quad \text{(MPa = N/mm}^2) \\
\sigma_p &= 414 \text{ MPa} \\
\rho &= 3.5 \text{ mm} \\
\beta &= 305 \text{ mm} \\
\h &= 381 \text{ mm} \\
A &= A' = \text{three No. 9 bars (28.7-mm diameter) } = 1936 \text{ mm}^2
\end{align*}
\]

Use in this solution: \( A = A' = \) four: \( \text{No. 25 M = 2000 mm}^2 \)

Assume \( \rho = 66 \text{ mm to give } \rho = 381 - 66 = 315 \text{ mm} \)

Solution:

**Test and adjustment procedure**

Assume that \( c = 180 \text{ mm to give } a = 133 \text{ mm} \). From similar triangles,

\[
\begin{align*}
\sigma_s &= 0.003 \times \left( \frac{122}{100} \right) = 0.03 \times 122 \times 100 \times \left( \frac{180 - 66}{180} \right) \\
&= 380 \text{ MPa} \\
P_x &= 0.85 \times 276 \times 305 \times 158 + 2000 \times 387 - 2000 \times 414 \\
&= 13.9 \times 10^6 \text{ N} \\
M_L &= 0.85 \times 276 \times 305 \times 158 \left( \frac{190.5 - 133}{2} \right) + 2000 \times 387(190.5 - 66) \\
&+ 2000 \times 414(133 - 90.5) + 32 \times 10^6 \text{ N-mm} \\
&= 322 \times 10^6 \text{ N-mm} > 312 \text{ mm > actual } \rho = 305 \text{ mm}
\end{align*}
\]

Proceed to second cycle: assume that \( c = 186 \text{ mm to give } a = 158 \text{ mm} \).

\[
\begin{align*}
\sigma_s &= 0.003 \times 200 \times \left( \frac{186 - 66}{180} \right) = 387 \text{ MPa} \\
P_x &= 0.85 \times 276 \times 305 \times 158 + 2000 \times 387 - 2000 \times 414 \\
&= 13.9 \times 10^6 \text{ N}
\end{align*}
\]
Chapter 9  Combined Compression and Bending: Columns

\[ P = 1.0 \times 10^6 \text{ N} \]
\[ M_c = 0.85 \times 27.6 \times 315 \times 150 \left(190.5 - \frac{158}{2}\right) + 2000 \times 315(190.5 - 60) \]
\[ + 200 \times 4(315 - 190.5) = 326 \times 10^6 \text{ N-mm} \]
\[ c = \frac{326 \times 10^6}{1.07 \times 10^9} = 304 \text{ mm} = \text{actual } c = 305 \text{ mm} \]

Hence, column capacity \( P_c = 1070 \text{ kN} \).

**Check Limit States of Neutral Axis Depth \( c \)**

From Equation 9.12 (b),

- Limit tension-controlled state neutral axis ratio \( c/d_d = 0.375 \)
  \[ \text{limit } c = 118 \text{ mm} < \text{actual } c = 180 \text{ mm}. \]
- Limit compression-controlled state \( c/d_c = 0.60 \)
  \[ \text{limit } c = 0.60 \times 315 = 189 \text{ mm} = \text{actual } c = 180 \text{ mm}. \]

Adopt the solution giving \( P_c = 1070 \text{ kN} \).

**SELECTED REFERENCES**


### Problems for Solution

Figure 9.46 Column sections.


### PROBLEMS FOR SOLUTION

9.1. Calculate the axial load strength $P_r$ for columns having the cross-sections shown in Figure 9.41. Assume zero eccentricity for all cases. Cases (a), (b), (c), and (d) are tied columns; case (c) is spirally reinforced.
9.2. Calculate $P_1$ and $e$ in Figure 9.46a and $e$ of Problem 9.1. Assume that the stresses in the tension steel are zero.

9.3. For the cross section shown in Figure 9.46a of Problem 9.1, determine the safe eccentricity $e$ if $P_1 = 200,000$ lb and the safe $P_1$ if $e = 15$ in., satisfying compatibility of strains.

9.4. For the cross section shown in Figure 9.46a of Problem 9.1, determine the safe eccentricity $e$ if $P_1 = 500,000$ lb. Use the trial-and-adjustment method satisfying the compatibility of strains.

9.5. Repeat Problem 9.4 using Whitney's approximate procedure. Compare the results.

9.6. Construct the load-moment interaction diagram for the cross-sections shown in Figure 9.46a and $e$ of Problem 9.1.

9.7. For the cross section shown in Figure 9.46a of Problem 9.1, calculate the design load $P_1$ if $e = 6$ in. Repeat the calculation for $e = 20$ in. Use the strain-compatibility trial and adjustment procedure.

9.8. Design the reinforcement for a noncylinder 15 in. × 70 in. column to carry the following loading. The factored ultimate axial force $P_1 = 300,000$ lb. The eccentricity $e$ to geometric centroid is 6 in. Given:

$$f'_c = 6000 \text{ psi}$$

$$f'_e = 60,000 \text{ psi}$$

9.9. Design a noncylinder column to support the following service loads and moments: $P_1 = 100$ kips, $P_2 = 50$ kips, $M_1 = 2500$ in.-kips, and $M_2 = 1000$ in.-kips. Given:

$$f'_c = 5000 \text{ psi}$$

$$f'_e = 60,000 \text{ psi}$$

9.10. Design a noncylinder circular column to support a factored ultimate load $P_1 = 250,000$ lb and a factored moment $M_1 = 5 \times 10^6$ in.-lb. Given:

$$f'_c = 6000 \text{ psi, normal-weight concrete}$$

$$f'_e = 60,000 \text{ psi}$$

$$d' = 2.50 \text{ in.}$$

9.11. Design the reinforcement for a 16 in. × 22 in. braced rectangular reinforced concrete column that can support a factored axial load $P_1 = 500,000$ lb and a factored moment $M_1 = 3,500,000$ in.-lb. The unsupported length, $l_0$, of the column is 10 ft. Assume that the end moments $M_1$ and $M_2$ are equal. Given:

$$f'_c = 4000 \text{ psi, sand-lightweight concrete}$$

$$f'_e = 60,000 \text{ psi}$$

$$d' = 2.5 \text{ in.}$$

9.12. A rectangular braced column of a multi-story frame building has a floor height $l_0 = 25$ ft. It is subjected to service-dead loads moments $M_1 = 3,500,000$ in.-lb and $M_2 = 2,500,000$ in.-lb at the bottom. The service live-load moments are 50% of the dead-load moments. The column carries a service axial dead load $P_2 = 200,000$ lb and a service live load $P_1 = 350,000$. Design the cross-section size and reinforcement for this column. Given:

$$f'_c = 7000 \text{ psi}$$

$$f'_e = 60,000 \text{ psi}$$

$$\Phi_1 = 1.3, \quad \Phi_2 = 0.9$$

$$d' = 2.5 \text{ in.}$$

9.13. A rectangular braced exterior column of a multi-story, multi-storey frame system is subjected to $P_1 = 300,000$ lb, factored end moments $M_1 = 2,500,000$ in.-lb and $M_2 = 3,500,000$ in.-lb. The braced length $l_o$ of the column is 18 ft. Design the column if

(a) it is subjected to gravity loads with sideways considered as negligible.

(b) it is subjected to wind load resulting in a sideways factored moment $M_2 = 2,000,000$ in.-lb. Assume the total loading of all interior and exterior columns in a single floor is $P_1 = 20 \times 10^6$ lb and $P_2 = 44 \times 10^6$ lb. Given:

9.15. The columns of the first floor in a nine-story 7 x 7 bay office building have a clear height of 18 ft (5.49 m). They are not braced against sway and the clear height above the first floor is 11 ft (3.32 m). Assume in your solution that the exterior columns have the same section as the interior columns. Design a typical interior column in this floor. Given:

\[ \Sigma P_i = 38 \times 10^6 \text{ lb}, \quad \Sigma M_i = 16 \times 10^6 \text{ lb} \]

For the connecting beams = 454 \times 10^6 \text{ in.-lb/ft (50,880 KN-m)}

Service loads for interior columns (lb) are

\[ D = 360,000, \quad L = 130,000, \quad W = 3000 \]

Service loads for exterior columns (lb) are

\[ D = 40,000, \quad L = 65,000, \quad W = 5000 \]

Service moments for the interior columns (in.-lb) are

Top: \[ D = 200,000, \quad L = 140,000, \quad W = 600000 \]
Bottom: \[ D = 300,000, \quad L = 400,000, \quad W = 600000 \]

Service moments for exterior columns (in.-lb) are

Top: \[ D = 400,000, \quad L = 240,000, \quad W = 300000 \]
Bottom: \[ D = 500,000, \quad L = 360,000, \quad W = 300000 \]

Check also by the Q index method. Assume \( V_s = 150,000 \text{ lb/ft} \)

9.16. A non slender square corner column is subjected to biaxial bending about the x and y axes. It supports a factored load \( P_1 = 200,000 \text{ lb} \) acting at eccentricities \( e_x = 7 \text{ in.} \) and \( e_y = 9 \text{ in.} \). Design the column size and reinforcement needed to resist the applied stresses. Given:

\[ f'_c = 5000 \text{ psi} \]
\[ f'_t = 60,000 \text{ psi} \]

given reinforcement percentage \( p_s = 0.03 \)

\[ d'' = 2.5 \text{ in.} \]

Solve by using all the above methods.

9.17. Design a non slender rectangular end column to support a factored load \( P_2 = 200,000 \text{ lb} \) acting at eccentricities \( e_x = 9 \text{ in.} \) and \( e_y = 6 \text{ in.} \). Try a section with a total gross reinforcement percentage not less than \( p_s = 0.025 \). Given:

\[ f'_c = 5000 \text{ psi}, \text{ normal-weight concrete} \]
\[ f'_t = 40,000 \text{ psi} \]
\[ d'' = 2.5 \text{ in.} \]

Solve by using all the three methods.
10.1 INTRODUCTION

Reinforcement for concrete to develop the strength of a section in tension depends on the compatibility of the two materials to act together in resisting the external load. The reinforcing element, such as a reinforcing bar, has to undergo the same strain or deformation as the surrounding concrete in order to prevent the discontinuity or separation of the two materials under load. The modulus of elasticity, the ductility, and the yield or rupture strength of the reinforcement must also be considerably higher than those of the concrete in order to raise the capacity of the reinforced concrete sections to a meaningful level. Consequently, materials such as brass, aluminum, rubber, or bamboo are not suitable for developing the bond or adhesion necessary between the reinforcement and the concrete. Steel and fiberglass do possess the principal factors necessary: yield strength, ductility, and bond value.

Bond strength results from a combination of several parameters, such as the mutual adhesion between the concrete and steel interfaces and the pressure of the hardened concrete against the steel bar or wire due to the drying shrinkage of the concrete. Additionally, friction interlock between the bar surface deformations or projections and the

Photo 10.1 Ladd Canyon overpass, Oregon. (Courtesy of Portland Cement Association.)
concrete caused by the micro movements of the tensioned bar results in increased resistance to slippage. The total effect of this is known as bond. In summary, bond strength is controlled by the following major factors:

1. Adhesion between the concrete and the reinforcing elements
2. Gripping effect resulting from the drying shrinkage of the surrounding concrete and the shear interlock between the bar deformations and the surrounding concrete
3. Frictional resistance to sliding and interlock as the reinforcing element is subjected to tensile stress
4. Effect of concrete quality and strength in tension and compression
5. Mechanical anchorage effect of the ends of bars through development length, splicing, hooks, and crossbars
6. Diameter, shape, and spacing of reinforcement as they affect crack development

The individual contributions of these factors are difficult to separate or quantify. Shear interlock, shrinking confining effect, and the quality of the concrete can be considered as major factors.

10.2 BOND STRESS DEVELOPMENT

Bond stress is primarily the result of the shear interlock between the reinforcing element and the enveloping concrete caused by the various factors previously enumerated. It can be described as a local shearing stress per unit area of the bar surface. This direct stress is
transferred from the concrete to the bar interface so as to change the tensile stress in the reinforcing bar along its length.

Three types of tests can determine the bond quality of the reinforcing element: the pull-out test, the embedded-rod test, and the beam test. Figure 10.1 shows the first two types of test. The pull-out test can give a good comparison of the bond efficiency of the various types of bar surfaces and the corresponding embedment lengths. It does not, however, truly represent the bond stress development in a structural beam. The concrete is subjected to compression and the reinforcing bar acts in tension in this test, whereas both the bar and the surrounding concrete in the beam are subjected to the same stress.

In the embedded-rod test (Figure 10.1b), the number of cracks, their widths, and their spacing at the various loading levels are a measure of the bond stress development and bond strength. The process resembles more closely the behavior in beams as the progressive increase in crack widths ultimately leads to bar slippage and beam failure.

The progressive slippage of the reinforcing bar in a beam and the redistribution of stresses is represented schematically in Figure 10.2. As the resistance to slippage over length l becomes larger than the tensile strength of concrete, a new crack forms in that area and a new stress distribution develops around the newly formed crack. The bond stress peak in Figure 10.2a continues to progress to the right from position A to position B. passing the center line between the two potential cracks until a second crack forms at a distance x, from crack 1.

![Diagram](image_url)
Figure 10.2 Stress redistribution with reinforcement slipage: (a) bond stress propagation; (b) reinforcement force or stress; (c) bending stress distribution.

Consequently, it is important to choose the appropriate length of the reinforcing bars that can minimize cracking and bond slipage. As a result, the reinforcement can attain its full strength in tension, that is, its yield strength within the structural element without bond failure.

10.2.1 Anchorage Bond

Assume $l_e$ in Figure 10.3a to be the length of the bar embedded in the concrete subjected to a net pulling force $dT$. $d_e$ is the diameter of the bar, $\mu$ is the average bond stress, and $f_t$ is the stress in the reinforcing bar due to direct pull or bending stresses in a beam, the anchorage pulling force $dT$ would be $\mu \pi d_e l_e$ and equal to the tensile force $dT$ on the bar cross section, that is:

$$d_T = \frac{\mu \pi d_e l_e}{2} f_t$$

Hence

$$\mu \pi d_e l_e = \frac{2}{d_e} f_t$$
from which the average bond stress

\[ \mu = \frac{f_d d_b}{4d_b} \]  

(10.1a)

and the development length

\[ l_d = \frac{f_d}{4\mu} d_b \]  

(10.1b)

10.2.2 Flexural Bond

The change in stress along the length of a bar in a beam due to the variation of moment along the span is shown schematically in Figure 10.3b. If \( f \) is the lever arm of the couple \( T \) due to moment \( M \), and \( T = M/df \). In terms of the moment difference between cracked sections 1 and 2,

\[ dT = \frac{dM}{fd} \]  

(10.2a)

Also,

\[ dT = \mu f C \Sigma \sigma \]  

(10.2b)
where $\Sigma o$ is the total circumference of all the reinforcement subjected to the bond stress pull, to get $dM/dx = \mu \Sigma o$; since $dM/dx = \text{shear} V$,

$$\mu = \frac{V}{\Sigma o}$$

Equation 10.2c is primarily of academic importance since it is indirectly accounted for in the development length approach given in Eq. 10.1b and the expressions to follow.

### 10.3 Basic Development Length

From the discussion in the preceding section, it can be concluded that the development length $l_d$ as a function of the size and yield strength of the reinforcement determines the resistance of the bars to slippage and hence the magnitude of the beam's failure capacity. It has been verified by tests that the bond strength $\mu$ is a function of the compressive strength of concrete such that

$$\mu = k \sqrt{f_c}$$

(10.3a)

where $k$ is a constant. If the bond strength equals or exceeds the yield strength of a bar of cross-sectional area $A_y = \pi d^2 / 4$, then

$$2d\mu \geq A_y f_y$$

(10.3b)

From Eqs. 10.1b, 10.3a, and 10.3b and considering $l_o$ as the basic development length,

$$l_o = k_1 \frac{A_y f_y}{\sqrt{f_c}}$$

(10.4)

or

$$\frac{d_o}{k_2} = k_3 \frac{d}{\sqrt{f_c}}$$

(10.5)

where $k_1$ is a function of the geometrical property of the reinforcing element and the relationship between bond strength and compressive strength of concrete.

Equation 10.5 is consequently the basic model for defining the minimum development length of bars in structural elements, with the factor $k_1$ being the experimental con-
Chapter 10  Bond Development of Reinforcing Bars

staat that covers the various factors affecting the development length. These factors include bar size, bar spacing, concrete cover, type of concrete, spacing and amount of transverse reinforcement, effect of use of excess main reinforcement, whether bars are coated, and the effect of bar splicing. These factors have been extensively investigated over the past 30 years, particularly by the group at the University of Texas at Austin.

10.3.1 Development of Deformed Bars in Tension

Equation 10.5 is transformed in the ACI 318 Code by replacing the coefficient $k$ by multipliers that reflect the effects of spacing of bars, cover, confinement by transverse reinforcement, type of concrete, and whether the reinforcement is coated.

The full development length $\bar{L}_d$ for deformed bars or wires obtained by applying these multipliers to the basic development length $L_d$ in Eq. 10.5 in terms of the bar diameter $d_b$ is

$$\bar{L}_d = \frac{3}{40} \sqrt{\frac{f_t}{f_{ct}}} \psi \alpha \lambda \frac{d_b}{d_b}$$  \hspace{1cm} (10.6)
where the term \( f + K \sigma_d \) should not exceed a value of 2.5, but not less than 1.5 for usual structures and \( \sqrt{f} \) shall not exceed 100 psi (690 Mpa).

### 10.3.2 Modifying Multipliers of Development Length for Bars in Tension

\( \psi_i = \text{bar location factor} \)

- For horizontal reinforcement so placed that more than 12 in. of fresh concrete is below the development length or splice (top reinforcement): 1.3
- Other reinforcements: 1.0

\( \psi_c = \text{coating factor} \)

- Epoxy-coated bars or wires with cover less than 3\( d_p \) or clear spacing less than 6\( d_p \): 1.5
- All other epoxy-coated bars or wires: 1.2
- Uncoated reinforcement: 1.0

However, the product of \( \psi_i \psi_c \) should not exceed 1.7.

\( \psi_b = \text{bar size factor} \)

- No 6 and smaller bars and deformed wires (No. 20 and smaller, SI): 0.8
- No. 7 and larger bars (No. 25 and larger, SI): 1.0
- \( \lambda = \text{lightweight concrete aggregate factor} \): 1.3

\( c = \text{spacing or cover dimension, in.} \)

Use the smaller of either the distance from the center of the bar to the nearest concrete surface or one-half center-to-center spacing of the bars being developed.

\( K_v = \text{transverse reinforcement index} = A_v f_v / 1500 \text{ psi} \), where constant 1500 carries units of psi

\( A_v = \text{total cross-sectional area of all transverse reinforcement that is within the spacing} s \) and that crosses the potential plane of splitting through to the reinforcement being developed, in.\(^2\) (mm\(^2\))

\( f_v = \text{specified yield strength of transverse reinforcement, psi (MPa)} \)

\( s = \text{maximum spacing of transverse reinforcement within} f_v \) center to center, in. (mm)

\( n = \text{number of bars or wires being developed along the plane of splitting} \)

The ACI Code permits using \( K_v = 0 \) as a conservative design simplification even if transverse reinforcement is present.

\( \lambda = \text{lightweight aggregate concrete factor} \)

- When lightweight aggregate concrete is used: \( \lambda = 1.3 \)
- However, when \( f_v \) is specified, use \( \lambda = 6.7(\sqrt{f}) / f_v \), but not less than 1.0.
- When normal-weight concrete is used: \( \lambda = 1.0 \)

The minimum development length in all cases is 12 in.

\( \lambda_e = \text{excess reinforcement factor} \)

The ACI Code permits the reduction of \( \xi \) if the longitudinal flexural reinforcement is in excess of that required by analysis except where anchorage or development for \( f_v \) is specifically required or the reinforcement is designed for seismic effects.
Reduction multiplier $\lambda = \frac{A_i}{A_{ij}}$ required $A_{ij}$ provided and $\lambda_0 = \frac{f_i}{60,000}$ for cases where $f_i > 60,000$ psi. In lieu of using a refined computation of the development length of Eq. 10.6, Table 10.1 can be utilized for typical construction practices using a value of $(c + K_c)h_i = 1.5$ in such cases and $f_i^\prime = 4000$ psi.

Table 10.2 is a general table for usual construction conditions giving the required development length for deformed bars of sizes Nos. 3 to 18.

Table 10.3 gives minimum beam width (inches) to satisfy two bar-diameter clear spacing, while Table 10.4 satisfies one-bar-diameter or 1-in. clear spacing. In these two tables, the following assumptions are made:

Side cover is 1.5 in on each side.
No. 5 stirrups for bars No. 11 or smaller.
No. 4 stirrups for bars No. 14 or No. 18.
Stirrups are bent around four bar diameters. Hence the distance from the centroid of the bar nearest the side face of the beam to the inside face of the No. 3 stirrup is taken as 0.75 in. for bars No. 11 or smaller and equal to the longitudinal bar radius for No. 14 and No. 18 bars.

10.3.3 Development of Deformed Bars in Compression and the Modifying Multipliers

Bars in compression require shorter development length than bars in tension. This is due to the absence of the weakening effect of the tensile cracks. Hence the expression for the basic development length is

$$l_0 = 0.02 \frac{d_i f_i^\prime}{\sqrt{f_i}} \quad (10.7 a)$$

and

$$l_a = 0.00033 d_i f_i \quad (10.7 b)$$

where the constant 0.0003 carries the unit of in.$^2$/lb.

<table>
<thead>
<tr>
<th>Table 10.1 Simplified Development Length $l_0$ Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>No. 6 and Smaller Bars</strong></td>
</tr>
<tr>
<td><strong>and Deformed Wires</strong></td>
</tr>
<tr>
<td>(f)</td>
</tr>
<tr>
<td>Clear spacing of bars being developed or spliced not less than $d_i$, clear cover not less than $d_i$, and stirrups or ties throughout $d_i$ not less than the Code minimum</td>
</tr>
<tr>
<td>$d_i$</td>
</tr>
<tr>
<td>or</td>
</tr>
<tr>
<td>$h_i$, $h_f$, $h_0$, $l_i = 1.0$</td>
</tr>
<tr>
<td>$c_0 = 0.8$</td>
</tr>
<tr>
<td>Clear spacing of bars being developed or spliced not less than $2d_i$ and clear cover not less than $d_i$</td>
</tr>
<tr>
<td>Other cases (1.5 times the above values)</td>
</tr>
<tr>
<td>$d_i$</td>
</tr>
<tr>
<td>$l_0 = 57d_i$</td>
</tr>
</tbody>
</table>
10.3 Basic Development Length

Table 10.2 Tension Reinforcement and Development Length (inches) for $f'_c = 4000$ psi
Normal-weight Concrete, $f_y = 60,000$ psi Steel

<table>
<thead>
<tr>
<th>Bar Size (in.)</th>
<th>Cross-sectional Area (in.$^2$)</th>
<th>Bar Diameter (in.)</th>
<th>Development Length, $L_{dev}$ (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$a \geq 2d_b$ or $d_b^a$ and Clear Cover $\geq d_b$</td>
</tr>
<tr>
<td>3</td>
<td>0.11</td>
<td>0.375</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>0.20</td>
<td>0.500</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>0.31</td>
<td>0.625</td>
<td>24</td>
</tr>
<tr>
<td>6</td>
<td>0.44</td>
<td>0.750</td>
<td>29</td>
</tr>
<tr>
<td>7</td>
<td>0.60</td>
<td>0.875</td>
<td>42</td>
</tr>
<tr>
<td>8</td>
<td>0.79</td>
<td>1.000</td>
<td>48</td>
</tr>
<tr>
<td>9</td>
<td>1.00</td>
<td>1.128</td>
<td>50</td>
</tr>
<tr>
<td>10</td>
<td>1.27</td>
<td>1.270</td>
<td>61</td>
</tr>
<tr>
<td>11</td>
<td>1.56</td>
<td>1.410</td>
<td>68</td>
</tr>
<tr>
<td>14</td>
<td>2.23</td>
<td>1.693</td>
<td>82</td>
</tr>
<tr>
<td>18</td>
<td>4.00</td>
<td>2.257</td>
<td>108</td>
</tr>
</tbody>
</table>

$\psi, \psi_c = 1.0; \psi_v = 0.6$ for No. 6 bars or smaller and = 1.0 for No. 7 bars and larger.

*For $f'_c$ values different from 4000 psi, multiply table values by $\sqrt{f'_c/4000}$. For $f'_c = 4000$ psi, multiply by $\psi_v \sqrt{f'_c}$ should not exceed 100.

*Continued by symbols.

For compression development length, $L_{dev} = multiplier \times L_{bar}$, where $L_{bar} = \psi_v f'_c / \psi_\lambda \geq 0.00056 f'_c$.

Multiply table values by $\lambda = 1.3$ for top reinforcement; $\lambda = 1.3$ for high-strength concrete; $\lambda = 1.5$ for epoxy-coated bars with cover less than $d_b$, or clear spacing less than $d_b$, and $\lambda = 1.2$ for other epoxy-coated bars.

Minimum $f'_c$ for all cases = 12 in.

with the modifying multiplier for

1. Excess reinforcement: $\lambda = required A_b / provided A_b$.
2. Spirally enclosed reinforcement: $\lambda = 2.75$.

10.3.4 Development of Bundled Bars in Tension and Compression

If bundled bars are used in tension or compression, $L_{dev}$ has to be increased by 20% for three-bar bundles and 33% for four-bar bundles. $\sqrt{f'_c}$ should not be taken greater than 100 psi. A unit of bundled bars is treated as a single bar of a diameter derived from the equivalent total area for the purpose of determining the modifying factors. However, although splice and development lengths of bundled bars are based on the diameter of individual bars increased by 20 or 23% as applicable, it is necessary to use an equivalent diameter of the entire bundle derived from the equivalent total area of bars when determining the factors that consider cover and clear spacing and represent the tendency of concrete to split.

10.3.5 Flowchart for Reinforcement Development Length Computation

A flowchart for reinforcement development length computation is shown in Figure 10.4.
Table 10.3  Minimum Beam Width (in.) to Satisfy Two-bar-diameter Clear Spacing

<table>
<thead>
<tr>
<th>Bar Size</th>
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Table 10.4  Minimum Beam Width (in.) to Satisfy the Larger of One-bar-diameter or 1-inch Clear Spacing

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Table 10.5  SI Development Length: Simplified Expressions

\[
\begin{align*}
\text{si} \text{ No. M. 20} & = 22 \sqrt{\frac{f_y}{k_{	ext{M. 20}}} r} \\
\text{si} \text{ No. M. 25} & = 22 \sqrt{\frac{f_y}{k_{	ext{M. 25}}} r} \\
\text{*main case} & \quad \frac{\varepsilon}{d_0} = \frac{5}{2} \frac{f_y}{k_{	ext{M. 20}}} r \\
\quad \frac{\varepsilon}{d_0} = \frac{3}{8} \frac{f_y}{k_{	ext{M. 25}}} r \\
\text{*other case} & \quad \frac{\varepsilon}{d_0} = \frac{15}{4} \frac{f_y}{k_{	ext{M. 20}}} r \\
& \quad \frac{\varepsilon}{d_0} = \frac{15}{16} \frac{f_y}{k_{	ext{M. 25}}} r
\end{align*}
\]

*See Table 10.1
10.3 Basic Development Length

![Flowchart](image)

- \( V_d \) should not be taken greater than \( 1W \).
  - **A. Tension reinforcement**
    - For normal construction practice
      1. For spacing \( s \leq 2d_y \), if \( b_x \times b_y \times a < 20d_y \):
        - No. 8 bars: \( \phi_y = 6000 \) psi
        - No. 10 bars: \( \phi_y = 4000 \) psi
        - No. 12 bars: \( \phi_y = 3000 \) psi
      2. For larger bars, multiply by \( 0.85 \) for the previous over 10;
        \( \phi_y \) in kip for larger bars.
      - **B. Compression reinforcement**
        - \( \phi_p = 0.85 \phi_y \), but not less than
        \( \phi_p = 0.005 \phi_y \).
        - \( \phi_p \) in kip.
      - If the bars are confined, use any factor of \( 0.75 \).

---

**Multiplication Factors for Tension Reinforcement**

- **(1)** Per section factor \( x \):
  - Tension reinforcement: 1.0
  - Other: 0.5

- **(2)** Cover factor \( v_p \):
  - Very small, \( d < 0.02d_y \) or \( a < 0.04d_y \)
  - Other: \( 1.0 \)
  - Unreinforced: \( 1.0 \)
  - \( v_p = \phi_y \times v \times \phi_y \) in kip.

- **(3)** Bar size factor:
  - No. 8 and smaller bars: \( 0.8 \)
  - No. 10 and larger bars: \( 1.0 \)

- **(4)** Lightweight aggregate counter factor \( x \):
  - \( x = 1.0 \) unless \( b_x \) is specified, in which case \( x = 0.7 \phi_y \).
  - but not less than 1.0. When normal weight concrete is used, \( x = 1.0 \).

- **(5)** Excess reinforcement factor \( k_p \):
  - \( k_p = k \times \phi_y \), provided.

- **(6)** Tension reinforcement factor \( k_p \) for compression reinforcement:
  - \( k_p = 0.15 \)

- **(7)** \( A \), \( b_x \), \( b_y \), \( a \), factor \( \phi_y \):
  - \( \phi_y \) in kip, \( b_x \times b_y \times a \) or \( 20d_y \).

---

**Determine the full development length \( L_d \) using the applicable multiplying factor in (3).**

**Check minimum length.**

---

**Figure 10.4 Flowchart for reinforcement development length computation.**
10.3.6 SI Metric Conversion

\[ K_s = \frac{A_{fs}}{250 \text{ m}} \]

where \( f_s \) is in MPa.

Equation 10.6:

\[ \frac{\varepsilon}{d_s} = \frac{15f_{c,b}b}{6 \sqrt{f_s}} \left( \frac{c_s + K_s}{d_s} \right) \]

10.3.7 Example 10.1: Development Length of Deformed Bars

Calculate the required embedment length of the deformed bars in the following four cases:

(a) No. 7 bars (22.2-mm diameter), top reinforcement in single layer in a beam with No. 3 stirrups. Given:

\( f_s = 60,000 \text{ psi} (414 \text{ MPa}) \)

\( f_c = 4000 \text{ psi} (27.6 \text{ MPa}) \), normal weight concrete

clear spacing between bars = \( d_s \)

clear side cover = 1.5 in. on each side

bars not spliced

(b) Same as part (a) except that clear spacing between bars = \( d_s \) or 1-in. minimum. The bars are epoxy coated.

(c) Same as part (a) except that clear spacing between bars = 3\( d_s \), and the bars are not top bars.

(d) Assume that the No. 7 bars in part (a) are in compression and the concrete is lightweight. Also assume that the provided \( A_s \) is 10% higher than the required \( A_s \).

Solution:

(a) Development length from Eq. 10.5a

\[ \ell_s = d_s \left[ \frac{3}{4} \frac{f_s}{\sqrt{f_c}} \left( \frac{c_s + K_s}{d_s} \right) \right] \]

\( \alpha = 1.3 \) (top bar), \( \beta = 1.0 \), \( \gamma = 1.0 \) for No. 7, \( h = 1.0 \), \( d_s = 0.875 \text{ in.} \), and \( c = \) smaller of distance from outer of bar to the nearest concrete surface or one-half center-to-center spacing of bars.

\[ c = \frac{1.5 + 0.875}{2} = 1.94 \text{ in.} \]

or

\[ c = \text{bar spacing} = \frac{1.1 + 0.875}{2} = 0.96 \text{ in.} \]

\( K_s \) can be assumed zero as a design simplification even if transverse reinforcement is present, but the term \( (c + K_s) \) cannot be larger than 2.5 or less than 1.5. = 0.936(0.875 + 1.072) = 1.5; use 1.5

\[ \sqrt{f_s} = \sqrt{60,000} = 64 < 100 \quad \text{O.K.} \]

\[ \ell_s = d_s \left( \frac{3}{4} \times \frac{60,000}{\sqrt{4000} \times 1.5} \right) = 61.7d_s \]

\( = 54 \text{ in.} (1370 \text{ mm}) \)

If \( \ell_s = 48d_s \), from Table 10.1 is used with a value of \( \frac{c_s + K_s}{d_s} = 1.5 \),

\[ \ell_s = 48.3d_s = 54 \text{ in.} \] (also from Table 10.2, 48 x 0.875 x 1.3 = 54 in.)
10.3 Basic Development Length

(b) \( \psi_1 = 1.3 \) (top bar), \( \psi_2 = 1.5 \), \( \psi_3 = 1.0 \), and \( \lambda = 1.0 \). Using Table 10.1, \( \psi_1 \psi_2 = 1.95 > 1.7 \). Use \( m_a = 1.70 \)

\[ c_a = \frac{48k}{a_a} = \frac{48 \times 1.70 \times 0.875}{\sqrt{2}} = 72 \text{ in. (1830 mm)} \]

(c) \( \psi_1 = 1.0 \) (bottom bar), \( \psi_2 = 1.0 \), \( \psi_3 = 1.0 \), and \( \lambda = 1.0 \). From Table 10.1

\[ c_a = \frac{48a}{a_a} = 42 \text{ in. (1070 mm)} \]

(d) \( \lambda = 1.3 \) for lightweight aggregate concrete. For compression steel, from Eq. 10.7a,

\[ c_a = \frac{0.02}{f_y'} = \frac{0.02 \times 0.875 \times 60,000}{\sqrt{4000}} = 16.6 \text{ in. (422 mm)} \]

From Eq. 10.7b,

\[ \psi = 0.0003d / f_y' = 0.0003 \times 0.875 \times 60,000 = 15.3 \text{ in.} \]

\[ c_a = 16.6 \text{ in. controls} \]

\( \lambda = 1.3 \)

\( \lambda \) for excess reinforcement = 1/1.1. Hence \( \epsilon_a = 16.6 \times 1.3 \times 1/1.1 = 20 \text{ in. (508 mm)} \).

10.3.8 SI Example on Development Length Evaluation

Solve Ex. 10.1 using SI units.

Data

\[ f_y = 27.6 \text{ MPa}, \quad d_o = \frac{7}{8} \text{ in.} = 22.2 \text{ mm (soft conversion)} \]

\[ f_y' = 414 \text{ MPa} \]

Solution: Use closest in millimeters to No. 7 bars in column 3 of Table 10.5.

(a)

\[ c_a = d_0 \left( \frac{556.6 \psi_1}{8 \sqrt{f_y'}} \right) \]

\[ f_y' = 414 \text{ MPa}, \quad f_y = 27.6 \text{ MPa} \]

\[ \psi_1 = 1.3 \] (top bar), \( \psi_2 = \psi_3 = 1.0 \)

\[ c_a = 22.3 \left( \frac{556.6 \times 1.3}{8 \sqrt{27.6}} \right) = 1420 \text{ mm (55 in.)} \]

(b) \( \psi_1 = 1.3, \psi_2 = 1.5, \psi_3 = 1.0, \) and \( \lambda = 1.0 \)

\[ c_a = 1420 \times 1.5 = 2130 \text{ mm} \]

(c) \( \psi_1 = \psi_2 = \psi_3 = \lambda = 1.0 \)

\[ c_a = 1420 / 1.3 = 1091 \text{ mm} \]

(d) \( \lambda = 1.3 \) for lightweight aggregate concrete; \( \lambda = 1/1.1 \). From before, \( \epsilon_a = 16.6 \text{ in.} = 422 \text{ mm} \)

\[ c_a = 422 \times 1.3 \times \frac{1}{1.1} = 498 \text{ mm} \]
10.3.9 Mechanical Anchorage and Hooks

Hooks are used when space limitations in a concrete section do not permit the necessary straight embedment length. Hooks in structural members are placed relatively close to the free surface of a concrete element, where splitting forces proportional to the total bar force may determine the hook capacity. The standard hook does not develop the tension yield strength of the bar. If \( l_{dh} \) is the basic development length for the standard hook in tension, it has to be multiplied by the appropriate factors, but not less than \( 6d_h \) or 6 in., whichever is greater. \( l_{dh} \) length is shown in Figure 10.5. The length \( l_{dh} \) varies with the bar size, reinforcement yield strength, and compressive strength of concrete. For \( f_y = 60,000 \) psi steel,

\[
l_{dh} = \frac{0.0260f_y}{\sqrt{f_c}}
\]

where \( d_h \) is the diameter of the hook bar.

Figure 10.6 Standard bar-hook details: (a) 90° hook; (b) 180° hook; (c) hook in small concrete cover.
Modifying multipliers for hooks in tension

1. Yield strength $f_y$, effective for a yield-strength different than 60,000, $\lambda_{y} = f_y/60,000$.

2. Concrete cover effect: for No. 11 bars and smaller, side cover normal to the plane of hook not less than 2\(\frac{1}{4}\) in. and for 90° hook with cover on bar extension beyond the hook not less than 2 in., $\lambda_{c} = 0.7$ (see Figure 10.5c).

3. Ties of stirrups: for No. 11 bars and smaller, hook enclosed vertically or horizontally within ties or stirrup spaced not greater than 3$d_{b}$, where $d_{b}$ is diameter of hook bar, $\lambda_{t} = 0.8$.

4. Excess reinforcement: where anchorage or development for $f_y$ is not specifically required but the reinforcement area $A_{s}$ used is in excess of $A_{d}$ required for analysis, required $A_{d}$

$$\lambda_{s} = \frac{A_{d}}{A_{s}}$$
provided $A_{s}$.

5. Bars developed by standard hooks at discontinuous ends: if the concrete cover is less than 2\(\frac{1}{4}\) in., bars should be enclosed within ties or stirrups along the full development length $l_{d}$ spaced at no greater than 3$d_{b}$, for this case $\lambda_{d} = 0.8$ from item 3 above; modifying multiplier shall not apply.

6. Lightweight concrete: $\lambda = 1.3$. It should be noted that hooks cannot be considered effective in developing bars in compression. The total development or embedment length

$$l_{d} = l_{d0} \times \lambda$$

(10.9)

Figure 10.5a and b shows details of standard 90° and 180° hooks used in axial tension or bending tension, and Figure 10.5c gives details of bar hooks susceptible to concrete splitting when the cover is small, that is, less than 2\(\frac{1}{4}\) in. Confinement is enhanced through the use of closed ties or stirrups. No distinction is made between a top bar and a bottom bar if hooks are used.

10.3.10 Example 10.2: Embedment Length for a Standard 90° Hook

Compute the development length required for the top bars of a lightweight concrete beam extending into the column support shown in Figure 10.6 assuming No. 9 reinforcing bars (38.5-mm diameter) hooked at the end. The concrete cover is 2 in. (50.8 mm). Given:

- $l_{d0} = 25$ in.
- Cover = 2\(\frac{1}{4}\) in.
- 12$d_{b}$ = 14 in.
- $3d_{b}$ = 5.65 in.
- Critical section for bar development at support

Figure 10.6 Hook embedment detail.
Chapter 10  Bond Development of Reinforcing Bars

\[ f_y = 5000 \text{ psi (34.47 MPa)} \]
\[ f_s = 60,000 \text{ psi (413.7 MPa)} \]

**Solution:** Top bars for hooks behave similarly to bottom bars. Hence, no modifier is needed. For No. 9 bars, \( d_b = 1.128 \text{ in. (28.65 mm)} \).

\[
\text{basic development length } l_d = \frac{0.02 \lambda d_b f_s}{\sqrt{f_y}}
\]

or

\[
l_d = 0.02 \times 1.0 \times 1.0 \times 60,000 \sqrt{5000} = 17.0 \text{ in.}
\]

For lightweight concrete, \( \lambda = 1.3 \).

\[
l_d = 1.3 \times 17.0 = 22.1 \text{ in.} > 6d_b \text{ or } 6 \text{ in.} \text{ O.K.}
\]

Use a 90° hook with embedment length \( l_\theta = 24 \text{ in. (610 mm)} \) beyond the critical section (face of support). Figure 10.6 shows the geometrical details of the hook.

**10.3.11 Development of Web Reinforcement**

1. For No. 5 bars and D31 wire and smaller, and for No. 6, 7, and 8 bars with \( f_s = 40,000 \text{ psi or less} \), a standard hook has to be used around the longitudinal reinforcement.

2. For Nos. 6, 7, and 8 stirrups with \( f_s \) greater than 40,000 psi, a standard stirrup hook around a longitudinal bar plus an embedment between midheight of the member and the outside end of the hook has to be used such that the length is equal to or greater than

\[
0.014d_b \frac{f_s}{\sqrt{f_y}}
\]

**10.3.12 Development of Weaved Plain Wire Fabric Reinforcement**

\[
l = 0.25 \frac{A_s}{s} \left( \frac{f_s}{\sqrt{f_y}} \right) \lambda
\]

where \( A_s \) = area of individual bar or wire, in.².

\( s \) = spacing between wires to be developed, in.

\( \lambda \) = lightweight aggregate concrete factor

**10.4 DEVELOPMENT OF FLEXURAL REINFORCEMENT IN CONTINUOUS BEAMS**

As discussed earlier, reinforcing bars should be adequately embedded in order to prevent serious bar slippage resulting in bond pull-out failure. The critical locations for bar discontinuance are points along the structural member where there is a rapid drop in the bending moment or stress, such as the inflection points or bending-moment diagram of a continuous beam.

Tension reinforcement can be developed by bending the lower tension bars at a 45° inclination across the web of the beam and can be anchored or made continuous with the reinforcing bars on the top of the member. To ensure full development reinforcement has to be extended beyond the point at which it is no longer required to resist flexure for a distance equal to the effective depth \( d \) or \( 12d_0 \), whichever is greater, except for supports of simple-span beams or at the free end of a cantilever. Figure 10.7 shows details of flex-
Figure 10.7 Development of reinforcement in continuous beams.

Reinforcement development in typical continuous beams for both the positive and the negative steel reinforcement.

The following are general guidelines for full development of the reinforcement and for ensuring continuity in the case of continuous beams:

1. At least one-third of the positive moment reinforcement in simple beams and one-fourth of the positive moment reinforcement in continuous beams should be extended at least 6 in. into the support without using bent.

2. At simple supports without confinement by the reaction as in Figure 10.8a and at points of inflection as in Figure 10.8b, the positive moment reinforcement should be limited to such a diameter that the development length

$$l_e = \frac{M_d}{V_{cf}} + l_i$$  \hspace{1cm} (10.10)

where

- $M_d$ = nominal moment strength where all the reinforcement is stressed to $f_y$
- $V_{cf}$ = factored shear force at the section under consideration
- $l_i$ = effective depth of 12$d_e$, where $d_e$ is the bar...
Chapter 10 Bond Development of Reinforcing Bars

Figure 10.8 Cutoff points for reinforcement: (a) simply supported beams; (b) continuous beams.

1. Diameter, whichever is greater and beam \( l_e = \) additional embedment length of support

Equation 10.10 imposes a design limitation on the flexural bond stress in areas of large shear and small moment in order to prevent splitting. Such a condition exists in short-span, heavily loaded simple beams. Thus the bar diameter for positive moment is so chosen that, even if length AC to the critical section in Figure 10.8a is larger than length AB, the bar size must be chosen such that \( l_e \leq 1.3 \sqrt{\frac{M}{f_e}} + l_e \)

where for confining reactions such as at simple supports, the value \( M/\sqrt{f_e} \) in Eq.

10.10 is increased by 30%.

3. At least one-third of the total tension reinforcement provided for negative bending moment at the support should extend beyond the inflection point not less than the effective depth \( d \) of the member, 12\( d_e \), or \( 3/4 \) of the clear span, whichever has the larger value.

4. Web stirrups have to be carried as close to the compression and tension surfaces of the member as the minimum concrete cover requirements allow. The ends of the stirrups without hooks should have an embedment of at least \( d/2 \) above or below the compression side of the member for full development length \( l_e \), but not less than 12 in. or 24 \( d_e \). For stirrups with hook end, the total embedment length should equal \( l_e \) plus the standard hook.
A typical detail of cutoff points for continuous one-way beam and joint construction from Ref. 10.6 is given in Figure 10.9. Typical cutoff points for one-way slabs are shown in Figure 10.10 and for beams with diagonal tension stirrups are given in Figure 10.11.

10.5 SPlicing OF REINFORCEMENT

Steel reinforcing bars are produced in standard lengths controlled by transportability and weight considerations. In general, 60- to 65-ft lengths are normally produced. But it is impractical in beams and slabs spanning over several supports to interweave bars of such lengths on site over several spans. Consequently, bars are cut to shorter lengths and lapped at the least critical bending moment locations for bar sizes No. 11 or smaller. A general rule of thumb for maximum bar length is about 40 0. for shipping purposes. The most effective means of continuity in reinforcement is to weld the cut pieces without reducing the mechanical or strength properties of the welded bar at the weld. However, cost considerations require alternatives. There are basically three types of splicing:

1. Lap splicing: depends on full bond development of the two lapping bars at the lap for bars of size not larger than No. 11.
2. Welding by fusion of the two bars at the connection can be economically justifiable for bar sizes larger than No. 11 bars.
3. Mechanical connecting: can be achieved by mechanical sleeves threaded on the ends of the bars to be interconnected. Such connectors should have a yield strength at least 1.25 times the yield strength of the bars they interconnect. They are also more commonly used for large-diameter bars.

10.5.1 Lap Splicing

Figure 10.12a shows a bar lap splice and the force and stress distribution along the splice length \( l_s \). Failure of the concrete at the splice region develops by a typical splitting mechanism as shown in Figure 10.12b. At failure, one bar slips relative to the other. The idealized tensile stress distribution in the bars along the splice length \( l_s \) has a maximum value \( f_y \) at the splice end and \( f_y / 2 \) at \( l_s / 2 \). At failure, the expected magnitude of slip is approximately \((0.5f_y/E_y) \times \text{half splice length} \) in Figure 10.12a.

10.5.2 Splices of Deformed Bars and Deformed Bars or Wires in Tension

Two classes of lap splices are specified by the ACI Code. The minimum length \( l_s \), but not less than 12 in., is

- **Class A:** \( 1.0l_s \)
- **Class B:** \( 1.3l_s \)

Table 10.6 gives the maximum percentage of tensile steel area \( A_s \) to be spliced. Splicing should be avoided at maximum tensile stress if at all possible; splicing may be by simple lapping of bars either in contact or separated by concrete. However, every effort should be made to stagger the splice, rather than having all the bars spliced within the required lap length.

10.5.3 Splices of Deformed Bars in Compression

The lap length \( l_s \) should be equal to at least the development length in compression as given in Section 13.3 and Eqs. 10.7a and 10.7b and the modifiers, \( t_s \), should also satisfy the following, but not be less than 12 in.
Figure 10.9 Cutoff point for one-way joint construction, Ref. 10.6. (Note: Continuing reinforcement shall have an embedment length not less than the required development length, but beyond the point where bars or terminated tension reinforcement is no longer required to resist tension.)
Figure 10.10  Cutoff points for one-way slabs, Ref. 10.3. (Note: Continuous reinforcement shall have an embedment length no less than the required development length. Beyond the point where bars or terminated tension reinforcement is no longer required to resist flexure.)
Figure 10.11 Reinforcing details for continuous beams with diagonal tension steel. Ref. 10.8. (Note: Continuing reinforcement shall have an embedment length not less than the required development length \( L \) beyond the point where bent or terminated tension reinforcement is no longer required to resist flexure.)
Figure 10.12: Reinforcing bar splicing: (a) lap splice idealized stress distribution; (b) splice-splitting failure.

\[ f_s \leq 60,000 \text{ psi} \quad t_s \geq 0.0085f_yd \]  
\[ f_s > 60,000 \text{ psi} \quad t_s \geq (0.0005f_y - 24)d \]  \hspace{1cm} (10.11a) \hspace{1cm} (10.11b)

If the compressive strength \(f_c\) of the concrete is less than 3000 psi, such as might occur in foundations, the splice length \(t_s\) has to be increased by one-third.

Modifying multipliers with values less than 1.0 are used in heavily reinforced tied compression members (0.65) and in spirally reinforced columns (0.75), but the lap length should not be less than 12 in.

<table>
<thead>
<tr>
<th>Table 10.6: Tension Lap Splice</th>
<th>Maximum Percent of</th>
<th>Required Lap Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_s ) Provided (^a)</td>
<td>( A_s ) Required</td>
<td>0.00 100</td>
</tr>
<tr>
<td>Equal to or greater than 2</td>
<td>Class A</td>
<td>Class B</td>
</tr>
<tr>
<td>Less than 2</td>
<td>Class B</td>
<td>Class B</td>
</tr>
</tbody>
</table>

\( ^a \) Ratio of area of reinforcement provided to area of reinforcement required by analysis in splice vicinity.
10.5.4 Development of Welded Deformed Wire Fabric in Tension

The development length, $l_d$, for deformed welded wire fabric should be taken as the $f_y$ value obtained from Eq. 10.6 or Table 10.1 multiplied by a fabric factor. The fabric factor, with at least one cross wire within the development length and not less than 2 in. from the point of the critical section, should be taken as the greater of the following two expressions:

$$\frac{f_y - 35,000}{f_y} \quad (10.12a)$$

or

$$\frac{5d_s}{x} \quad (10.12b)$$

but should not be taken greater than 1.0, where $x$ = spacing of wire to be developed or spliced ($i_0$).

For plain welded wire fabric,

$$l_d = 0.27 \frac{A_s}{s} \left( \frac{f_y}{\sqrt{f_{yd}}} \right)$$  \hspace{1cm} (10.12c)

where $A_s$ = cross-sectional area of one wire to be developed.

10.5.5 Splices in Deformed Welded Wire Fabric

The minimum lap length, $l$, measured between the ends of the two looped fabric sheets of welded deformed wire has to be 21.3 in. or 8 in. (204 mm), whichever is greater. Additionally, the overlap measured between the outermost cross wires of each fabric sheet should not be less than 2 in. (51 mm).

10.6 EXAMPLES OF EMBEDMENT LENGTH AND SPLICE DESIGN FOR BEAM REINFORCEMENT

10.6.1 Example 10.3: Embedment Length at Support of a Simply Supported Beam

Calculate the maximum development length that can be used for bars $a$ at the support of the simply supported superstructure beam in Figure 10.13 if the distance $AC$ from the theoretical cutoff point of bars $b$ is 48 in. (1,220 mm). The beam is integral with the support (nonconfin-
10.6 Examples of Embedment Length and Splice Design for Beam Reinforcement

(a) No. 7 deformed (22.2 mm) and (b) No. 14 deformed (43.0 mm), if the beam was a full section concrete beam (a maximum No. 11 bar is usually used in superstructure normal-size beams). Given:

- \( s = \) clear spacing between bars = 3\( d_b \)
- \( V_c = 100,000 \text{ lb} (444.8 \text{ kN}) \)
- \( M_c = 2,353,000 \text{ in} \cdot \text{lb} (225 \text{ kN} \cdot \text{m}) \)
- \( f'_c = 4,000 \text{ psi} (27.6 \text{ MPa}), \) normal-weight concrete
- \( f = 50,000 \text{ psi} (343.7 \text{ MPa}) \)
- \( l_e = 12 \text{ in.} (305 \text{ mm}) \)

Solution: \( \phi_e = \phi = \phi_b = 1. \)

(a) No. 7 bars: \( d_b = 1.0 \text{ in.} \)

\[ \sqrt{\frac{V_c}{f'}} = \sqrt{\frac{100,000}{4,000}} = 33.3 < 100 \text{ O.K.} \]

From column 2 of Table 10.1, \( d_b = 48 \text{ in.} \)

From Eq. 10.10, \( l_e \geq 1.3 M_c V_c + l_b, \) where \( l_b = \) effective depth or 12 \( d_b \), whichever is greater.

\[ l_e = \text{embedment length beyond support center} = 12 \text{ in.} \]

 maximum \( l_e = 1.3 \times 23.3 \times 12 = 42.59 \text{ in.} = 42 \text{ in.} \text{ O.K.} \)

Hence, bar size is O.K. and max. \( l_e = 42 \text{ in.} \)

(b) No. 14 bars:

\( \phi_e = \phi = \phi_b = 1.0 \text{ in.} = 1.593 \text{ in.} \)

From column 2 of Table 9.1.

\[ d_b = 48, d_b = 48 \times 1.651 = 82 \text{ in.} \]

\[ > 1.3 M_c V_c + l_b \]

Hence, reduce bar size to No. 7 bars using a larger number of bars since the maximum \( l_e = 42.59 \text{ in.} \)

10.6.2 Example 10.4: Embedment Length at Support of a Continuous Beam

A continuous reinforced concrete beam has clear spans \( L_w = 30 \text{ ft} (9.14 \text{ m}) \) and \( L_s = 22 \text{ ft} (6.7 \text{ m}) \) and the bending moment diagram segment at an interior support is shown in Figure 10.14. Calculate the cutoff lengths of the negative moment top reinforcement bars to satisfy the development length requirements at the cutoff point. The beam is simply supported and has the dimensions \( b = 27 \text{ in.} (686 \text{ mm}) \), \( d = 23.5 \text{ in.} (597 \text{ mm}) \), and \( h = 15 \text{ in.} (381 \text{ mm}) \). It is subjected to a factored negative bending at the crown of the intermediate support.

\[-M_s = 6,122,000 \text{ in} \cdot \text{lb} (692.4 \text{ kN} \cdot \text{m}) \]

Given:

- \( s = \) clear spacing between bars = 3\( d_b \)
- \( f'_c = 4,000 \text{ psi} (27.6 \text{ MPa}), \) normal-weight concrete
- \( f = 50,000 \text{ psi} (343.7 \text{ MPa}) \)

Required \( A \) = 562 in.²

Provided \( A \) = 6.00 in.² (six No. 9 bars)
Solution:
\[ \sqrt{\frac{d}{A}} = \sqrt{\frac{300}{6}} = 63.2 < 100 \quad \text{O.K.} \]

\[ A_p = 1.0 \quad \psi \text{ for top bars} = 1.3; \quad \psi_1 = \psi_2 = 1.0 \]

From Table 10.1, column 2,
\[ l_2 = 4.48d_1 = 1.3 \times 48 \times 1.128 = 70 \text{ in.} \]
\[ l_2 = \text{required} A_p \times 70 = \frac{5.02}{0.60} \times 70 = 66 \text{ in.} \]

Use \( l_2 = 66 \text{ in.} \) (1630 mm) for the six No. 9 bars.

**Cutoff points:**
At least one-third of the bars have to extend beyond the point of inflection by the largest of \( \frac{d}{L} \) (span \( L_2 \) or \( 12d_1 \)).

1. Right span \( L_2 = 36 \text{ ft.} \)
\[ \frac{1}{3} A_p = \text{two No. 9 bars} \]
\[ 12d_1 = 12 \times \frac{1}{8} = 15 \text{ in.} \]

2. Left span \( L_2 = 22 \text{ ft.} \)
\[ \frac{1}{18} L_2 = \frac{22}{18} = 12 = 15 \text{ in.} \]
\[ d = 23.5 \text{ in.} \]

As given in Figure 10.7, details of the development length dimensions at all cutoff points for this continuous beam are shown in Figure 10.15.

10.6.3 Example 10.5: Splice Design for Tension Reinforcement

Calculate the lap splice length for No. 7 tension bottom bars (22.2-mm diameter) spaced at \( 3d_p \), minimum spacing. The ratio of the provided \( A_p \) to the required \( A_p \) is \( (A_p) \) \( > 2.0 \) \((A_p) \leq 2.0 \).

Use \( 75\% \) as percentage of \( L_2 \), splice within this section is 75\%. Given:
Section 10.6: Examples of Embedment Length and Splice Design for Beam Reinforcement

Figure 10.15 Bar development length of cutoff points.

\[
f' = 5000 \text{ psi (34.5 MPa)}
\]

\[
f = 60,000 \text{ psi (413.7 MPa)}
\]

\[
s = \text{clear spacing between bars} = \frac{2}{3}d_p
\]

Solution:

\[
d_s = 0.875 \text{ in. for No. 7 bar}
\]

\[
\phi_1 = \phi_2 = \phi_3 = \lambda = 1.0
\]

Check \( \sqrt{5000} = 70.7 < 100 \) O.K.

From Table 10.1, column 2, \( l_i = 48 \), \( d_p = 0.875 \times 0.075 = 0.42 \) in.

(a) For provided \( A_r \), required \( A_r > 2.0 \), class A splice: \( l_i = 1 \times l_p = 42 \) in.

(b) For provided \( A_r \), required \( A_r < 2.0 \), class B splice:

lap splice length \( l_i = 1.3 \lambda_i = 1.3 \times 42 = 55 \) (1400 mm)

10.6.4 Example 10.6: Splice Design for Deformed Compression Reinforcement

Calculate the lap splice length for No. 9 compression deformed bars (28.7-mm diameter) in a normal-weight concrete beam at clear spacing \( d_p \). Given:

\[
f' = 7000 \text{ psi (48.3 MPa)}
\]

\[
f = 80,000 \text{ psi (551.6 MPa)}
\]
Solution:

\[ \phi = 1.128 \text{ in. for No. 9 bar} \]
\[ \sqrt{7000} = 83.7 < 100 \quad \text{O.K.} \]
\[ \phi_k = \phi \cdot 0.6 = 1.0 \]

Splice length \( l_s \): From Eq. 10.11b, for \( f_s > 40,000 \) psi,

\[ l_s = \frac{0.0009 f_s - 24 \phi_k}{0.0009 \times 40,000 - 24(1.128)} = 54.14 \text{ in.} \]

Use lap splice length \( l_s = 55 \text{ in.} (1400 \text{ mm}) \).

10.7 TYPICAL DETAILING OF REINFORCEMENT AND BAR SCHEDULING

The design examples for bond development length, lap splicing, and spacing reinforcement are applied in Figures 10.16 to 10.20. Additional examples from the author's parking-garage working drawing details are given in Figures 10.21 to 10.25. These representative examples can serve as a good guideline for producing correct engineering working drawings. It should be recognized that successful execution of a designed system is directly dependent on the availability of clear and correct detailing and the avoidance of any congestion of the reinforcement. Such congestion can only lead to honeycombing in

Figure 10.16 Column ties for preassembled lap-spliced cages. (From Ref. 10.6.)
## Figure 10.17

Column ties for standard columns (from Ref. 10.8).
Figure 10.18  Ties for large and special columns (from Ref. 10.6.).
Figure 10.19 Corner and joint connection details: (a) retaining walls; (b) T-joints; (c) corners (from Ref. 10.8.).
Figure 10.21  Typical beam and slab reinforcing working drawing. (Design by E. G. Newy.)
Figure 10.22 Raft foundation details. (Design by E. G. Navy.)
Figure 10.26 Typical reinforcement bar bending schedule. (Design by E. G. Navy.)
the concrete, resulting in possible cracking, capacity reduction, and even failure. Consequently, equal attention has to be given to detailing as to design if a constructed system is to perform the structural functions for which it is intended.

SELECTED REFERENCES


10.8. ACI Committee 318, Building Code Requirements for Structural Concrete (ACI 318-05) and Commentary (ACI 318R-05), American Concrete Institute, Farmington Hills, MI, 2005, 444 pp.


PROBLEMS FOR SOLUTION

10.1. Calculate the basic development lengths in tension for the following deformed bars embedded in normal-weight concrete. 
   (a) No. 5, No. 8: Given:
   \[ f_s = 5000 \text{ psi} (34.5 \text{ MPa}) \]
   \[ f_y = 60,000 \text{ psi} (413.7 \text{ MPa}) \]

   (b) No. 14, No. 18: Given:
   \[ f_s = 4000 \text{ psi} \]
   \[ f_y = 60,000 \text{ psi} \]
   \[ f_y = 80,000 \text{ psi} \]

10.2. Calculate the total embedment length for the bars in Problem 10.1 if they are used as compression reinforcement and the concrete is normal-weight.

10.3. Design the end length for the continuous beam in Ex. 10.4 if eight No. 8 bars are used instead of six No. 9 bars.
10.4. Design the compression lap splice for a column section 8 in. x 10 in. (203 mm x 254 mm) reinforced with eight No. 9 bars (eight bars of diameter 28.7 mm) equally spaced around all faces.
(a) \( f' = 5000 \text{ psi} \) (34.5 MPa)
    \[ f = 60,000 \text{ psi} \) (414 MPa)
(b) \( f' = 7000 \text{ psi} \) (48.3 MPa)
    \[ f = 80,000 \text{ psi} \) (551 MPa)

10.5. An 8-ft (2.4-m) normal-weight concrete cantilever beam is subjected to a factored \( M_u = 3,300,000 \) in-lb (386 kN-m) and a factored shear \( V_s = 32,400 \text{ lb} \) (44 kN) at the face of the support. Design the top reinforcement and the appropriate embedment of \( W^d \) hook into the concrete wall to sustain the external shear and moment. Given:
    \[ f_s = 4300 \text{ psi} \]
    \[ f = 60,000 \text{ psi} \]

10.6. Design the beam reinforcement in Problem 10.5 if it was simply supported having a span \( L_s = 36 \text{ ft} \) (10.97 m) and subjected to the same factored \( M_u \) value at midspan and the shear \( V_s \) at the face of the support. Evaluate the required embedment length at the support to ensure that no bond failure due to slippage can develop. Assume (a) confining beam tension and (b) beam not monolithic with its support.
11 DESIGN OF TWO-WAY SLABS AND PLATES

11.1 INTRODUCTION: REVIEW OF METHODS

Except for post-tensioned slabs, supported floor systems are usually constructed of reinforced concrete cast in place. Two-way slabs and plates are those panels in which the dimensional ratio of length to width is less than 2. The analysis and design of framed floor slab systems represented in Figure 11.1 encompasses more than one aspect. The present state of knowledge permits reasonable evaluation of (1) the moment capacity, (2) the slab-column shear capacity, and (3) serviceability behavior as determined by deflection control and crack control. Flat plates are slabs supported directly on columns without beams, as shown in Figure 11.1a, compared to Figure 11.1b for slabs on beams, or Figure 11.1c for waffle slab floors. Lift slabs are another form of construction but mostly in pre-stressed concrete.

The evolution of the state of knowledge in slab design in the last 50 years will be briefly reviewed. The analysis of slab behavior in flexure up to the 1940s and early 1950s followed the classical theory of elasticity, particularly in the United States. The small deflections theory of plates, assuming the material to be homogeneous and isotropic, formed the basis of ACI Code recommendations with moment coefficient tables. The work, principally by Westergaard, that empirically allowed limited moment redistribution

Photo 11.1 Sydney Opera House, Sydney, Australia. (Courtesy of Australian Information Service.)
11.1 Introduction: Review of Methods

guided the thinking of the code writers. Hence the elastic solutions, complicated even for simple shapes and boundary conditions when no computers were available, made it mandatory to idealize and sometimes empiricize conditions beyond economic bounds.

In 1943, Lehmann presented his yield-line theory for evaluating the collapse capacity of slabs. Since that time, extensive research into the ultimate behavior of reinforced concrete slabs has been undertaken. Studies by many investigators, such as those of Ock-
11.1.1 Semielastic ACI Code Approach

The ACI approach gives two alternatives for the analysis and design of a framed two-way action slab or plate system: the direct design method and the equivalent frame method. Both methods are discussed in more detail in Sections 11.3 and 11.6.

11.1.2 Yield-line Theory

Whereas the semielastic code approach applies to standard cases and shapes and has an inherent, excessively large safety factor with respect to capacity and the yield-line conditions; the yield-line theory is a plastic theory that is easy to apply to irregular shapes and boundary conditions. Provided that serviceability constraints are applied, Johansen's yield-line theory is the simplest approach that the designer can use, representing the true behavior of reinforced concrete slabs and plates. It permits evaluation of the bending moments from an assumed collapse mechanism that is a function of the type of external load and the shape of the floor panel. This topic will be discussed in more detail in Section 11.9.

11.1.3 Limit Theory of Plates

The interest in developing a limit solution became necessary due to the possibility of finding a variation in the collapse field that can give a lower failure load. Hence an upper-bound solution requiring a valid mechanism when supplying the work equation was sought, as well as a lower-bound solution requiring that the stress field satisfies everywhere the differential equation of equilibrium:

$$\frac{\partial M_x}{\partial x} - 2 \frac{\partial M_y}{\partial x} + \frac{\partial M_z}{\partial y} = -w$$  (11.1)

where $M_x$, $M_y$, and $M_z$ are the bending moments and $w$ is the unit intensity of load. Variable reinforcement permits the lower-bound solution still to be valid. Wood, Park, and other researchers have given more accurate semielastic predictions of the collapse load.

For limit-state solutions, the slab is assumed to be completely rigid until collapse. Further work at Rutgers by the author incorporated the deflection effect at high load levels as well as the compressive membrane forces in predicting the collapse load.

11.1.4 Strip Method

This method was proposed by Hilleborg, attempting to fit the reinforcement to the strip fields. Since practical considerations require the reinforcement to be placed in orthogonal directions, Hilleborg set twisting moments equal to zero and transformed the slab into intersecting beam strips; hence the name strip method. Except for Johansen's yield-line theory, most of the other solutions are lower bound. Johansen's upper-bound solution can give the highest collapse load as long as a valid failure mechanism is used in predicting the collapse load.

11.1.5 Summary

Both the direct design method (DDM) and the equivalent frame method (EFM) will be discussed with appropriate examples. Both methods are based on the concept of an equivalent frame, except that the DDM has several limitations, is less refined, and is suitable for gravity loads only, whereas the EFM is more general, can be utilized for horizontal loading and is more viable for computer programming.
11.2 FLUID BEHAVIOR OF TWO-WAY SLABS AND PLATES

11.2.1 Two-way Action

A single rectangular panel supported on all four sides by unyielding supports such as head walls or stiff beams is first considered. The purpose is to visualize the physical behavior of the panel under gravity load. The panel will deflect in a dish-like form under the external load, and its corners will lift if it is not monolithically cast with the supports. The contours show, in Figure 11.2a, that the curvatures and consequently the moments at the central area C are more severe in the shorter direction y with its steep contours than in the longer direction x.

Evaluation of the division of moments in the x and y directions is extremely complex because the behavior is highly statically indeterminate. The discussion of the simple case of the panel in Figure 11.2a is expanded further by taking strips AB and DE at midspan as in Figure 11.2b such that the deflection of both strips at central point C is the same.

The deflection of a simply supported uniformly loaded beam is $w = P/48EI$, that is, $\Delta = k \Delta u$, where k is a constant. If the thickness of the two strips is the same, the deflection of strip AB would be $w_{AB} = \frac{k}{16} S^4$, and the deflection of strip DE would be $w_{DE} = \frac{k}{16} S^4$, where $w_{AB}$ and $w_{DE}$ are the positions of the total load intensity transferred to strips AB and DE, respectively; that is, $w = w_{AB} + w_{DE}$. Equating the deflections of the two strips at the central point C, we get

$$w_{AB} = \frac{w_{DE}}{L^4 + S^4}$$

(11.2e)

Figure 11.2 Deflection of panels and strips: (a) curvature and deflection contours in a floor panel; (b) central strip in a two-way slab panel.
and

$$w_{eff} = \frac{wL^2}{L^2 + S^2} \quad (11.20)$$

It is seen from the two relationships $w_{x,y}$ and $w_{xy}$ in Eqs. 11.1a and 11.1b that the shorter span $L$ of strip DE carries the heavier portion of the load. Hence the shorter span of such a slab panel on unyielding supports is subjected to the larger moment, supporting the foregoing discussion of the steepness of the curvature contours in Fig. 11.2a.

Note that while the ACI 318-05 Code proposes using "q" for the unit intensity of load for two-way slabs, it is preferable to continue the traditional practice of identifying the load intensity as "w," as is done in this chapter.

11.2.2 Relative Stiffness Effects

Alternatively, we must consider a slab panel supported by flexible supports such as beams and columns or flat plates supported by a grid of columns. The distribution of moments in the short and long directions is considerably more complex. The complexity arises from the fact that the degree of stiffness of the yielding supports determines the intensity of steepness of the curvature contours in Figure 11.2a in both the $x$ and $y$ directions and the redistribution of moments.

The ratio of the stiffness of the beam supports to the slab stiffness could be such that it could result in curvatures and moments in the long direction larger than those in the short direction, because the total floor behaves as an orthotropic plate supported on a grid of columns without beams. The moment values in the long and short directions in Eqs. 11.1 and 11.2 arithmetically illustrate this discussion. If the long span $L$ in such floor systems of slab panels without beams is considerably larger than the short span $S$, the maximum moment at the center of a plate panel would approximate the moment at the middle of an uniformly loaded strip of span $L$ and clamped at both ends.

In summary, because slabs are flexible and highly underreinforced, redistribution of moments in both the long and short directions depends on the relative stiffnesses of the supports and the supported slabs. Overstress in one region is reduced by such redistribution of moments to the lesser stressed regions.

11.3 THE DIRECT DESIGN METHOD

The following discussion of the direct design method (DDM) of analysis for two-way systems summarizes the ACI Code approach for evaluation and distribution of the total moments in a two-way slab panel. The various moment coefficients are taken directly from the ACI Code provisions. An assumption is made that vertical planes cut through an entire rectangle in plan of a multistory building along lines $AB$ and $CD$ in Figure 11.3 mid-way between columns. A rigid frame results in the $x$ direction. Similarly, vertical planes $EF$ and $HG$ result in a rigid frame in the $y$ direction. A solution of such an idealized frame consisting of horizontal beams or equivalent slabs and supporting columns enables the design of the slab as the beam part of the frame. Approximate determinations of the moments and shears using simplified coefficients are presented throughout the direct design method. The equivalent frame method treats the idealized frame in a manner similar to an actual frame, and hence is more exact and has fewer limitations than the direct design method. It basically involves a full moment distribution of many cycles, compared to the direct design method, which involves a one-cycle-moment distribution approximation.
11.3.1 Limitations of the Direct Design Method (DDM)

The following are the limitations of this method:

1. A minimum of three continuous spans are necessary in each direction.
2. The ratio of the longer to the shorter span within a panel should not exceed 2.0.
3. Successive span lengths in each direction should not differ by more than one-third of the longer span.
4. Columns may be offset a maximum of 10% of the span in the direction of the offset from either axis between center lines of successive columns.
5. All loads shall be due to gravity only and uniformly distributed over the entire panel. The live load shall not exceed two times the dead load.
6. If the panel is supported by beams on all sides, the relative stiffness of the beams in two perpendicular directions shall not be less than 0.2 nor greater than 5.0.
7. No moment redistribution in continuous panels is permitted in the direct design method.

It should be noted that the majority of normal floor systems satisfy these conditions.

11.3.2 Determination of the Factored Total Statical Moment \( M_0 \)

There are basically four major steps in the design of the floor panels.

1. Determine the total factored statical moment in each of the two perpendicular directions.
2. Distribute the total factored design moment to the design of sections for negative and positive moment.
Figure 11.3 Continued

Photo 11.2 Sydney Opera House during construction.
3. Distribute the negative and positive design moments to the column and middle strips and to the panel beams, if any. A column strip is a width = 25% of the equivalent frame width on each side of the column center line, and the middle strip is the balance of the equivalent frame width.

4. Proportion of the size and distribution of the reinforcement in the two perpendicular directions.

The correct determination of the values of the distributed moments becomes a principal objective. Consider typical interior panels having center-line dimensions \( l \), in the direction of the moment being considered and dimensions \( l' \), in the direction perpendicular to \( l \), as shown in Fig. 11.4b. The clear span \( l' \) extends from face to face of columns, capitals, or walls. Its value should not be less than \( 0.6l \), and circular supports should be treated as square supports having the same cross-sectional area. The total statical moment of a uniformly loaded simply supported beam as a one-dimensional member is \( M = \frac{wL^2}{8} \). In a two-way slab panel as a two-dimensional member, the idealization of the structure through conversion to an equivalent frame makes it possible to calculate \( M' \), once in the \( x \) direction and again in the orthogonal \( y \) direction. If we take as a free-body diagram the typical interior panel shown in Figure 11.4a, symmetry reduces the shears and twisting moments to zero along the edges of the cut segment. If no restraint existed at ends \( A \) and \( B \), the panel would be considered simply supported in span \( l' \) direction. If we cut at midspan as in Figure 11.4b and consider half the panel as a free-body diagram, the moment \( M'_x \) at midspan would be

\[
M'_x = \frac{wLd_1}{2} \quad \text{or} \quad M'_x = \frac{wL^3}{8}
\]

(11.3)

![Figure 11.4](image)

Figure 11.4 Simple moment \( M'_x \) acting on an interior two-way slab panel in the \( x \) direction: (a) moment on panel; (b) free-body diagram.
Due to the actual existence of restraint at the supports, \( M_y \) in the \( x \) direction would be distributed to the supports and midspan such that

\[ M_x = M_x + \frac{1}{2} (M_y + M_z) \] (11.4)

The distribution would depend on the degree of stiffness of the support. In a similar manner, \( M_y \) in the \( y \) direction would be the sum of the moments at midspan and the average of the moments at the supports in that direction.

The distribution of the statical factored moment \( M_y \) to the column strip of the equivalent frame leads to the proportioning of the reinforcement in these strips.

### 11.4 DISTRIBUTED FACTORED MOMENTS AND SLAB REINFORCEMENT BY THE DIRECT DESIGN METHOD

#### 11.4.1 Negative and Positive Factored Design Moments

From Figure 11.5a, the negative factored moment factor in interior spans is 0.65 and the positive factor is 0.35 of the total statical moment \( M_y \). For end spans of flat-plate floor panels, the \( M_y \) factors are given in Table 11.1.

#### 11.4.2 Factored Moments in Column Strips

A column strip is a design strip with a width on each side of the column equal to 0.25\( L \) or 0.25\( L \), whichever is less, as shown in Figures 11.3b and 11.5. The strip includes beams, if any. The middle strip is a design strip bounded by the two column strips of the panel being analyzed.

#### 11.4.2.1 Interior Panels

For interior negative moments, column strips have to be proportioned to resist the following portions in percent of the interior negative factored moments, with linear interpolation made for intermediate values.

| Table 11.1 Moment Factors for \( M_y \) Distribution in Exterior Spans |
|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
|                             | Slab without Beams between Interior Supports | Slab with Beams between All Supports |
|                             | Exterior Edge Unrestrained | Slab with Beams Without Edge Beam | Slab with Beams With Edge Beam | Exterior Edge Fully Restrained |
| Interior negative factored moment | 0.75                        | 0.70                          | 0.70                        | 0.70                          | 0.65                        |
| Positive factored moment     | 0.65                        | 0.57                          | 0.52                        | 0.50                          | 0.35                        |
| Exterior negative factored moment | 0                          | 0.18                          | 0.26                        | 0.30                          | 0.65                        |
Figure 11.8 Distribution of the service factored moments $M_s$ for slab without beams into negative and positive moments: (a) moment coefficients for multi-spans; (b) slab area for which $M_s$ is calculated.

<table>
<thead>
<tr>
<th>$\lambda_i$</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_f/(\lambda_i)$ = 0</td>
<td>75</td>
<td>75</td>
<td>75</td>
</tr>
<tr>
<td>$a_f/(\lambda_i)$ &gt; 1.0</td>
<td>60</td>
<td>75</td>
<td>45</td>
</tr>
</tbody>
</table>

$\lambda_i$ in these tables is $\lambda$ in the direction of span $l$, for cases of two-way slabs on beams and is equal to the ratio of flexural stiffness of the beam section to the flexural stiffness of a width of slab divided by the center lines of adjacent panels. If any, on each side of the beam $m_i = E_i d_i F_i/\lambda_i$, where $F_i$, and $E_i$ are the modulus values of concrete and $l_i$ and $d_i$ are the moments of inertia of the beam and the slab, respectively. The factored moments in beams between supports have to be proportioned to resist 85% of the column slab moment when $a_f/(\lambda_i) \geq 1.0$. Linear interpolation between 85% and 0% needs to be made for cases where $a_f/(\lambda_i)$ varies between 1.6 and 0. $a_f$ is in direction $-1$ and $\lambda_i$ is in direction $-2$. 
11.4.2 Exterior panels. For exterior negative moments, the column strips should be proportioned to resist the following portions in percent of the exterior negative factored moments with linear interpolation made for the intermediate values, where \( \beta \) is the torsional stiffness ratio. \( \beta_t = \text{ratio of torsional stiffness of the edge beam section to the flexural stiffness of a width of slab equal to the span length of beam center to center of supports.} \)

\[
\begin{array}{c|ccc}
\beta_t & 0.5 & 1.0 & 2.0 \\
\hline
\alpha_t & 0 & 100 & 100 & 100 \\
\beta_t \geq 2.5 & 75 & 75 & 75 \\
\beta_t \geq 10 & 75 & 75 & 45 \\
\end{array}
\]

Torsional stiffness ratio \( \beta_t \) is calculated from \( \beta_t = \frac{E_0L}{2E_0/I} \), where \( C = \Sigma(1 - 0.63 x/y) \frac{x^2}{3} \).

11.4.3 Positive moments. For positive moments, the column strips have to be proportioned to resist the following portions in percent of the positive factored moments with linear interpolation being made for intermediate values.

\[
\begin{array}{c|ccc}
\beta_t & 0.5 & 1.0 & 2.0 \\
\hline
\alpha_t & 0 & 60 & 60 & 60 \\
\beta_t \geq 1.0 & 75 & 75 & 45 \\
\end{array}
\]

For a panel with beams between supports on all sides, the relative stiffness of beams in two perpendicular directions should be within the range

\[
0.2 \leq \frac{\alpha_t}{\alpha_t'} \leq 5.0
\]

The values of the stiffness ratio of \( \alpha_t \) and \( \alpha_t' \) are calculated in accordance with

\[
\alpha_t = \frac{E_0L}{E_0'/I}
\]

(11.5b)

11.4.3 Factored Moments in Middle Strips

That portion of the negative and positive factored moments not resisted by the column strips would have to be proportionately assigned to the corresponding half of the middle strips. Adjacent spans do not necessarily have to be equal so that the two halves of the column strip flanking a row of columns need not be equal in width. Hence each middle strip has to be proportioned to resist the sum of the moments assigned to its two half middle strips. A middle strip adjacent to and parallel with an edge supported by a wall must be proportioned to resist twice the moment assigned to the half middle strip corresponding to the first row of interior columns.

11.4.4 Pattern Loading Consideration

In the analysis of continuous members, pattern loading has to be considered. As a result, the maximum moments are obtained. Pattern loading can cause reversal of stress, as seen in Figure 11.6, as a function of the relative stiffness of the beams and columns intersecting at the joint. Pattern loading analysis is cumbersome. By limiting the applicability of
the Direct Design Method of slabs with "Live load not exceeding two times dead load", there is no longer a need to check for pattern load effects. Slab and column dimensions for virtually all practical cases will meet the values for \( a_{	ext{max}} \) specified in Table 11.2.

### 11.4.5 Shear-Moment Transfer to Columns Supporting Flat Plates

#### 11.4.5.1 Shear strength

The shear behavior of two-way slabs and plates is a threedimensional stress problem. The critical shear failure plane follows the perimeter of the loaded area and is located at a distance that gives a minimum shear perimeter \( b_0 \). Based on extensive analytical and experimental verification, the shear plane should not be closer than a distance \( d/2 \) from the concentrated load or reaction area.

If no special shear reinforcement is provided, the maximum allowable nominal shear strength \( V_c \) of the section as required by the ACI is the smallest of the values from Eqs. 11.5:

\[
V_c = \left( 2 + \frac{a}{b_0} \right) \sqrt{f'_t} \cdot b_0 d
\]

where \( b_0 \) is the ratio of the long side to the short side of the column, concentrated load, or reaction area, and \( b_0 \) is the perimeter of the critical section:

\[
V_c = \left( \frac{a_{	ext{max}}}{b_0} + 2 \right) \sqrt{f'_t} \cdot b_0 d
\]

where \( a_{	ext{max}} \) is 40 for interior columns, 30 for edge columns, and 20 for corner columns and

\[
V_c = 4 \sqrt{f'_t} \cdot b_0 d
\]
Table 11.2 Values of \( \mu_{(m)} \)

<table>
<thead>
<tr>
<th>( \mu_{(m)} )</th>
<th>Aspect Ratio, ( \frac{L}{d} )</th>
<th>Relative Beam Stiffness, ( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.5-2.0</td>
<td>0.5-2.0</td>
</tr>
<tr>
<td>2.0</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1.0</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>1.25</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>1.5</td>
<td>1.2</td>
<td>1.2</td>
</tr>
<tr>
<td>2.0</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>2.5</td>
<td>1.8</td>
<td>1.8</td>
</tr>
<tr>
<td>3.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
<tr>
<td>3.5</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>4.0</td>
<td>2.8</td>
<td>2.8</td>
</tr>
</tbody>
</table>

\( \mu_{(m)} \) unfactored dead load per unit area

\( \alpha = \frac{L(h_{b} + h_{c})}{L(h_{b} + h_{c})} \)

*sums of stiffness of columns above and below slab

\( L \) sum of stiffness of beams and slabs spanning into the joint in the direction of the span for which the moments are being determined.

Equations 11.6(a) and (b) are the results of tests that indicate that as the ratio of height of a beam to depth increases the available nominal shear strength \( V_b \) decreases. It is clear from Eq. 11.6(c) that the shear strength provided by the plain concrete cannot exceed \( 4V_f c \), which is almost double the shear strength allowed in one-way members such as beams and one-way slabs. If special shear reinforcement is provided, the maximum nominal shear strength \( V_b \) cannot exceed \( 5V_f b_d \), provided that the value used for \( V_f \) in the term \( V_f = V_f \), does not exceed \( 2V_f b_d \).

11.4.5.2 Shear-Moment Transfer. The unbalanced moment at the column face support of a slab without beams is one of the more critical design considerations in proportioning a flat plate or a flat slab. To ensure adequate shear strength requires consideration of moment transfer to the column by flexure across the perimeter of the column and by eccentric shearing stress, as approximately 60% is transferred by flexure and 40% by shear.

The fraction \( \gamma_c \) of the moment transferred by eccentricity of the shear stress decreases as the width of the face of the column section resisting the moment increases such that

\[
\gamma_c = 1 - \frac{1}{1 + \frac{2}{3} \frac{b_d}{b}}
\]  
(11.7a)
where $b_i = c_i + \frac{d}{2}$ is the width of the face of the critical section resisting the moment and $b_i = c_i + d$ for interior columns and $b_i = c_i + 2d$ for edge columns and $b_i = c_i + d/2$ and $b_i = c_i + d/2$ for corner columns. $b_i$ is the width of the face at right angles to side $b_i$.

The remaining portion $\gamma_i$ of the unbalanced moment transferred by flexure is given by

$$\gamma_i = \frac{1}{1 + \frac{2}{3} \sqrt{b_i/b_i^2}}$$

or

$$\gamma_i = 1 - \gamma_i$$ (11.7b)

acting on an effective slab between lines that are 1.1 times the total slab thickness $k$ on both sides of the column support.

The ACI Code provides for a simplified method where $\gamma_i = 1.0$ can be used for the unbalanced moment under the following conditions for shear about an axis parallel to the edge:

- For the exterior support, provided that $V_e$ at the support is less than $0.75 \cdot V_e$, the value of $\gamma_i$ can be increased to 1.0.
- At the interior support, $\gamma_i$ can be increased by 25% provided that $V_e$ is equal to or less than $0.44 \cdot V_e$ and $\alpha$ is equal to or less than 0.75 $\alpha_i$.

However, this simplification using $\gamma_i = 1.0$ and $\gamma_i = 0$ might not be on the safe side, as some tests have shown.

The distribution of shear stresses around the column edges is as shown in Figure 11.8. It is considered to vary linearly about the centroid of the critical section. The factored shear force $V_e$ and the unbalanced factored moment $M_e$ both assumed acting at the column face, have to be transferred to the centroidal axis $c-c$ of the critical section. Thus, the axis position has to be located, thereby obtaining the shear force arm $g$ (distance from the column face to the centroidal axis plane) of the critical section $c-c$ for the shear moment transfer.

For calculating the maximum shear stress sustained by the plate in the edge column region, the ACI Code requires using the full nominal moment strength $M_e$ provided by the column strip in Eqs. 11.8 as the unbalanced moment, multiplied by the transfer fraction factor $\gamma_i$. This unbalanced moment $M_e \cdot \gamma_i$ is composed to two parts: the negative end panel moment $M_{en} = M_e/\gamma_i$ at the face of the column and the moment $(V_e/\phi)\gamma_i$ due to the eccentric factored perimetric shear force $V_e$. The limiting value of the shear stress intensity is expressed as

$$\frac{v_{\text{lim}}}{\phi} = \frac{V_e}{\phi A_i} = \frac{\gamma_i M_{en}}{\phi}$$ (11.8a)

$$\frac{v_{\text{lim}}}{\phi} = \frac{V_e}{\phi A_i} = \frac{\gamma_i M_{en}}{\phi}$$ (11.8b)

where the nominal shear strength intensity is

$$v_n = \frac{v_e}{\phi}$$ (11.8c)

and where $A_i$ = area of concrete of assumed critical section

- $= 2(c_i + c_i + 2d)$ for an interior column

- $J_i$ = property of assumed critical section analogous to polar moment of inertia

The value of $J_i$ for an interior column is

$$J_i = \frac{d(c_i + d)^3}{6} + \frac{d(c_i + d)^3}{6} + \frac{d(c_i + d)^3}{6}$$ (11.8d)

The value of $J_i$ for an edge column with bending parallel to the edge is

$$J_i = \frac{(c_i + d)(2d)^3}{6} + \frac{2(c_i + d)(2d)^3}{6} (c_i + c_i) + \frac{(c_i + d)(2d)^3}{6}$$ (11.8e)
Figure 11.7 Shear stress distribution around column edges: (a) interior column; (b) end column; (c) critical surface; (d) transfer nominal moment strength $M_o$. 
where the second part of the term is the shear stress resulting from torsional moment at the column face.

If the nominal moment strength \( M_0 \) of the shear moment transfer zone after the design of the reinforcement is in a larger value than \( M_{cr} \), the value of the torsional moment should be used in Eqs. 11.8a or b for a shear moment zone of \( M_{cr} \). In such a case, where the moment strength value \( M_{cr} = M_{cr} + (\psi \gamma k) \) is increased because of the use of flexural reinforcement in excess of what is needed to resist \( M_{cr} \), the slab stiffness is raised, thereby increasing the transferred shear stress \( v_s \) calculated from Eqs. 11.7a and b for development of full moment transfer. Consequently, it is advisable to maintain a design moment \( M_{cr} \) with a value close to the factored moment value \( M_{cr} \) on an increase in the shear stress due to additional moment transfer needs to be avoided and a possible resulting need for additional increase in the plate design thickness prevented.

Numerical Ex. 11.1 illustrates the procedure for calculating the limit perimeter shear stress in the plate at the edge column region.

A higher perimeter shear stress \( v_s \) can occur than evaluated by Eq. 11.8a or b when an adjoining span of unequal span or equally loaded in the case of an interior column. The ACI Code stipulates that the slab section consisting of factored moments in columns and walls that the supporting element, such as a column or a wall, has to resist an unbalanced moment \( M' \), such that

\[
M' = 0.07 \left[ (v_{cr} + 0.5w_{sym} \gamma f_c) \right] - w_{cr} f_c (S_c)^2 \tag{11.9a}
\]

where \( w_{cr} \) and \( f_c \) refer to the shorter span.

The column at the column-slab monolithic joint would have to be designed to resist the unbalanced moment \( M' \) in direct proportion to their stiffnesses unless a general frame analysis is used. Thus, if the upper and lower columns in the joint have the same weight and stiffness, the column is designed to carry one-half the unbalanced moment \( M' \) in combination with the axial load. In Eq. 11.9a, the one-half live load intensity is applied on the longer span and the dead load only is applied to the shorter span.

Hence, an additional term is added to Eq. 11.8a or b in such cases so that

\[
v_s = \frac{V_s}{A_v} + \frac{\gamma M_{cr} + \gamma M_{cr} + \gamma M_c}{f_c} \tag{11.9b}
\]

where \( f_c \) is the polar moment of inertia with moment of inertia taken in a direction perpendicular to that used for \( f_c \).

If the two adjacent spans are equal and equally loaded with live load, Eq. 11.9(a) for the unbalanced moment of the shear-moment transfer in the slab at the column joint becomes

\[
M' = 0.07 \left[ (0.5w_{sym} f_c) \right] \tag{11.9c}
\]

11.4.6 Deflection Requirements for Minimum Thickness: An Indirect Approach

The serviceability of a slab system can be maintained through deflection control and crack control. Since deflection is a function of the stiffness of the slab as a measure of its thickness, a minimum thickness has to be provided irrespective of the flexural thickness requirement. Table 11.3 gives the minimum thickness of slabs without interior beams. This occurs when \( L_{cr} = 0 \). Table 11.4 gives the maximum permissible computed deflections to safeguard against plastic cracking and to maintain aesthetic appearance. Deflec-
### Table 11.3 Minimum Thickness of Slab Without Interior Beams

<table>
<thead>
<tr>
<th>Yield Stress, $f_y$ (ksi)</th>
<th>Without Drop Panels&lt;sup&gt;a&lt;/sup&gt;</th>
<th>With Drop Panels&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exterior Panels</td>
<td>Exterior Panels</td>
</tr>
<tr>
<td></td>
<td>Without Edge Beams</td>
<td>With Edge Beams&lt;sup&gt;a&lt;/sup&gt;</td>
</tr>
<tr>
<td>40,000</td>
<td>$\frac{L}{33}$</td>
<td>$\frac{L}{36}$</td>
</tr>
<tr>
<td>60,000</td>
<td>$\frac{L}{30}$</td>
<td>$\frac{L}{33}$</td>
</tr>
<tr>
<td>75,000</td>
<td>$\frac{L}{28}$</td>
<td>$\frac{L}{31}$</td>
</tr>
</tbody>
</table>

<sup>a</sup>Drop panel as defined by the ACI Code.
<sup>b</sup>For values of reinforcement yield stress between the values given in the table, minimum thickness shall be obtained by linear interpolation.
<sup>c</sup>Slabs with beams between columns along exterior edges. The value of $\sigma$ for the edge beam shall not be less than 0.8.

### Table 11.4 Minimum Permissible Ratios of Span (l) to Deflection (d) (l = Longer Span)

<table>
<thead>
<tr>
<th>Type of Member</th>
<th>Deflection, a. to Be Considered</th>
<th>(%)$h_{min}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat roofs not supporting and not attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load $L$</td>
<td>180&lt;sup&gt;e&lt;/sup&gt;</td>
</tr>
<tr>
<td>Floors not supporting and not attached to nonstructural elements likely to be damaged by large deflections</td>
<td>Immediate deflection due to live load $L$</td>
<td>300&lt;sup&gt;e&lt;/sup&gt;</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements likely to be damaged by large deflections</td>
<td>That part of total deflection occurring after attachment of nonstructural elements; sum of long-term deflection due to all sustained loads (dead load plus any sustained portion of live load) and immediate deflection due to any additional live load&lt;sup&gt;g&lt;/sup&gt;</td>
<td>480&lt;sup&gt;e&lt;/sup&gt;</td>
</tr>
<tr>
<td>Roof or floor construction supporting or attached to nonstructural elements not likely to be damaged by large deflections</td>
<td></td>
<td>240&lt;sup&gt;e&lt;/sup&gt;</td>
</tr>
</tbody>
</table>

<sup>e</sup>Limit not intended to safeguard against pounding. Pounding should be checked by suitable calculations of deflections, including short-term deflections due to imposed loads and considering long-term effects of all sustained loads, carbon, construction shrinkage, and reliability of provisions for drainage.
<sup>g</sup>Long-term deflection has to be determined, but may be reduced by the amount of deflection calculated to occur before attachment of nonstructural elements. This reduction is made on the basis of accepted engineering data relating to time-deflection characteristics of members similar to those being considered.
<sup>h</sup>Ratio limit may be lower if adequate measures are taken to prevent damage to supported or attached elements, but should not be lower than tolerance of nonstructural elements.
tion computations for two-way-action slabs can be made using the analytical procedures described in Section 11.8 in order to determine whether the analysis gives long-term deflections within the limitations of Table 11.4.

Approximate empirical limitation on deflection through determination of the minimum thickness of the slab on beams or drop panels or bands can be obtained from Table 11.3 if the stiffness ratio \( \alpha_{\text{st}} < 0.2 \).

For \( \alpha_{\text{st}} > 0.2 \) but not greater than 2.0,

\[
h = \frac{l_0 \left(0.8 + f_c/200,000\right)}{36 + 5\left(\alpha_{\text{st}} - 0.8\right)}
\]

(11.10a)

and need not be less than 5 in., where \( \alpha_{\text{st}} \) = average value of \( \alpha \) for all beams on edge of panel.

For \( \alpha_{\text{st}} > 2.0 \),

\[
h = \frac{l_0 \left(0.8 + f_c/200,000\right)}{36 + 9\beta}
\]

(11.10b)

where \( l_0 \) is the length of clear span in the long direction for deflection determination. For moment computation, \( l_0 \) is the length of the clear span in the direction that moments are being computed.

For slabs without beams, but with drop panels having a width in each direction from center line of support a distance not less than one-sixth the span length in that direction measured center to center of supports and a projection below the slab at least one-fourth the slab thickness beyond the drop, the thickness required by Eq. 11.10a or 11.10b may be reduced by 10%. At discontinuous edges, an edge beam shall be provided with a stiffness ratio \( \alpha \) not less than 0.8 or the minimum thickness required by Eq. 11.10a or 11.10b shall be increased by at least 10% in the panel with a discontinuous edge.

Figure 11.8 gives a plot of the thickness ratio \( h/l_0 \) as a function of aspect ratio \( \beta \) for the two equations for various stiffness ratios \( \alpha \). Note from the plots that Eq. 11.10a is an upper-limit expression applicable to limited conditions when the stiffnesses of the panel beam sup-

Figure 11.8 Thickness ratio versus aspect ratio for deflection control of two-way slabs on beams, Eqs. 11.10a and 11.10b limits.
ports are so low that the stiffness ratio \( n \) has a value close to 0, gradually approaching the condition of a flat plate. \( n \) is not applicable when \( n = 0 \). If it is to be used in the later condition, however, we can assume that part of the slab in the column region acts as a beam. Thickness \( h \) cannot be less than the following values:

- Slabs without beams or drop panels: 5 in.
- Slabs without beams, but with drop panels: 4 in.
- Slabs with beams on all four edges with a value of \( \alpha_{dr} \) at least equal to 2.0: 3.5 in.

\( h \) also has to be increased by at least 10% for flat-plate floors if the end panels have no edge beams and by 45% for corner panels.

In addition, in the equations above,

\[ a = \text{ratio of flexural stiffness of beam section to flexural stiffness of a width of slab bounded laterally by the center line of the adjacent panel (if any) on each side of the panel} \]

\[ \alpha_{be} = \text{average value of} \ a \ \text{for all beams on edges of a panel} \]

\[ \beta = \text{ratio of clear spans in long to short direction of two-way slabs} \]

It has to be emphasized that a deflection check is critical for the construction loading condition. Shoring and restripping patterns can result in dead-load deflection in excess of the normal service-load state at a time when the concrete has only a 7-day strength or less and not the normal design 28-day strength. The stiffness \( E' \) in such a state is less than the design value. Flexural cracking lowers further the stiffness values of the two-way slab or plate, with a possible increase in long-term deflection several times the anticipated design deflection. Consequently, reinforced concrete two-way slabs and plates have to be constructed with a camber of \( \frac{1}{16} \) in. in 10-ft span or more and crack control exercised as in Section 11.9 in order to counter the effects of excessive deflection at the construction loading stage. An analysis for the construction load stresses and deflections is important in most cases.

It should be noted that while the ACI Code stipulates the use of \( f' = 0.75f' \) for the modulus of rupture in computing the cracking moment, \( M_{cr} \), it is advisable that a lower value than 0.75 in the expression for \( f' \) be used, such as 0.4-0.6. In this manner, the possibility is avoided of unanticipated deflection of a two-way slab larger than what the ACI deflection tables present.

### 11.5 DESIGN AND ANALYSIS PROCEDURE: DIRECT DESIGN METHOD

#### 11.5.1 Operational Steps

Figure 11.9 gives a logic flowchart for the following operational steps:

1. Determine whether the slab geometry and loading allow the use of the direct design method as listed in Section 11.3.1.

2. Select slab thickness to satisfy deflection and shear requirements. Such calculations require knowledge of the supporting beam or column dimensions. A reasonable value of such a dimension of columns or beams would be 8 to 15% of the average of the long and short spans. Also, \( \frac{d+2}{d} \). For shear check, the critical section is at a distance \( d/2 \) from the face of the support. If the thickness shown for deflection is not adequate to carry the shear, use one or more of the following:

   (a) Increase the column dimension.
   (b) Increase concrete strength.
   (c) Increase slab thickness.
11.5 Design and Analysis Procedure: Direct Design Method

Start

Given design data: arrangement of panels, proof dimensions, live load superimposed dead load, etc.

Carry out the geometry checks for use of direct design method; observe the limitations of the design method

Determine the minimum slab thickness for deflection requirements using Eq. 11.19a and 11.19b

Assume an effective width for the slab, calculate the self-weight of the slab and

Check if the assumed thickness satisfies the shear requirements. If the beam and column dimensions are not known, assume 5% of the average of the span and design for the worst case of 1% of the average of $f_{c}$ and $f_{c}$. Select the critical section for shear to be at 0.5 from the face of the support

Yes

Assumed section satisfies shear requirements

Check if there are any other shear requirements

No

Make adjustments by increasing column dimensions, concrete strength, or shear links, or special shear reinforcement

Divide the structure into the appropriate design areas based on the criteria of panels (Section 11.3)

Compute the total resisting factored moment, $M_{R} = M_{e} + M_{u} + M_{rej}$

Select the distribution factors (Tables 11.1 and 11.2 and Fig. 11.3); assign critical negative and positive moments

For key plates, check the shear requirements (taking into account the moment to be transferred to shear to the column face)

No

Shear requirements are satisfied

Yes

Design the transverse reinforcement for the critical sections

Select the bar size and detail the arrangement of reinforcement

End

Figure 11.9 Flowchart for design sequence in two-way slabs and plates by the direct design method.
(d) Use special shear reinforcement.
(e) Use drop panels or column capitals to improve shear strength.

3. Divide the structure into equivalent design frames bound by center lines of panels on each side of a line of columns.

4. Compute the total statical factored moment \( M_s = (wJ_m)^{1/8} \).

5. Select the distribution factors of the negative and positive moments in the exterior and interior columns and spans as in Figure 11.3 and Table 11.1 and calculate the respective factored moments.

6. Distribute the factored equivalent frame moments from step 5 to the column and midspan strips.

7. Determine whether the trial slab thickness chosen is adequate for moment-shear transfer in the case of flat plates at the column junction computing that portion of the moment transferred by shear and the properties of the critical shear section at distance \( d_2 \) from column face.

8. Design the flexural reinforcement to resist the factored moments in step 6.

9. Select the size and spacing of the reinforcement to fulfill the requirements for crack control, bar development lengths, and shrinkage and temperature stresses.

11.5.2 Example 11.1: Design of Flat Plate without Beams

by the Direct Design Method

A three-story building is four panels by four panels in plan. The clear height between the floors is 12 ft, and the floor system is a reinforced concrete flat-plate construction with no edge beams. The dimensions of the and panels as well as the size of the supporting columns are shown in Figure 11.13. Given:

![Diagram of a three-story building with floor plan of end panels.]
11.5 Design and Analysis Procedure: Direct Design Method

Live load = 70 psf (3.5 kPa)

\( f'_c = 4000 \text{ psi (27.6 MPa)} \)

\( f'_c = 6000 \text{ psi (41 MPa)} \)

The building is not subject to earthquakes; consider gravity loads only. Design the end panel and the size and spacing of the reinforcement needed. Consider floor weight to be 10 psf in addition to the floor self-weight.

Solution: Geometry check for use of direct design method (step 1)

(a) Ratio longer span/shorter span = 24/12 = 1.33 1.20, hence two-way action

(b) More than three spans in each direction and successive spans in each direction the same and columns are not offset.

(c) Assume a thickness of 9 in. and flooring of 10 psf.

\[ w_d = 10 + \frac{9}{12} \times 150 = 125.0 \text{ psf} \]

\[ 2w_d = 245 \text{ psf} \]

\[ w_f = 70 \text{ psf} < 2w_d \text{ O.K.} \]

Hence the direct design method is applicable.

Minimum slab thickness for deflection requirement (step 2)

E-W direction \( L_2 = 24 \times 12 - \frac{18}{2} - \frac{20}{2} = 269 \text{ in. (6.8 m)} \)

N-S direction \( L_3 = 18 \times 12 - \frac{18}{2} - \frac{20}{2} = 196 \text{ in. (4.98 m)} \)

ratio of longer to shorter shear span \( \beta = \frac{269}{196} = 1.37 \)

Minimum preliminary thickness \( h \) from Table 11.3 for a flat plate without edge beams or drop panels using \( f'_c = 6000 \text{ psi steel} in h = 1.50 \) to be increased by at least 10% when no edge beams is used.

E-W: \( L_2 = 269 \text{ in.} \)

\[ h = \frac{269}{30} \times 1.1 = 9.86 \text{ in.} \]

Try a slab thickness \( h = 10 \text{ in.} \). This thickness is larger than the absolute minimum thickness of 5 in. required in the code for flat plates; hence O.K. Assume that \( f' = h - 1 \text{ in.} = 9 \text{ in.} \)

new \( w_d = 10 + \frac{10}{12} \times 150 = 135.0 \text{ psf} \)

Therefore,

\[ 2w_d = 270 \text{ psf} \]

\[ w_f = 70 \text{ psf} < 2w_d \text{ O.K.} \]

Shear thickness requirement (step 2)

\[ w_s = 1.6L + 1.2D = 1.6 \times 70 + 1.2 \times 135 = 274 \text{ psf (13.12 kPa)} \]

Interior column: The controlling critical plane of maximum perimetric shear stress at a distance \( d/2 \) from the column face; hence, the net factored perimetric shear force is
Chapter 11 Design of Two-Way Slabs and Plates

\[ V_s = (t_1 \times d_1) - (t_1 + d_1)(c_2 + d_2) \times n_0 \]
\[ = \left( 18 \times 24 - 20 + 9 \times 20 + 9 \right) \times 124 = 116,768 \]
\[ V_s = 116,768 \times \frac{1}{0.75} = 155,691 \text{ lb} \]

From Figure 11.11, the perimeter of the critical shear failure surface is
\[ b_p = 2(t_1 + d_1 + c_2 + d_2) = 2(c_1 + c_2 + 2d) \]

perimetric shear surface \[ A_p = b_p d = 2d(c_1 + c_2 + 2d) = 2 \times 9(20 + 20 + 18) \]
\[ = 116 \times 9 = 1044 \text{ in.}^2 (875,400 \text{ mm}^2) \]

Since moments are not known at this stage, only a preliminary check for shear can be made.

\[ \beta = \text{ratio of longer to shorter side of columns} = \frac{20}{20} = 1.0 \]

Available nominal shear \( V_s \) is the least of
\[ V_s = \left( \frac{2 + \frac{16}{8}}{3} \right) \sqrt{b_p d} = \left( \frac{2 + \frac{4}{4}}{3} \right) \sqrt{4000 \times 1044} = 369,170 \text{ lb (1,64 \times 10^6 \text{ kN})} \]
or
\[ V_s = \left( \frac{8d + 2}{b_p} \right) \sqrt{b_p d} = \left( \frac{8 \times 9 + 2}{116} \right) \sqrt{4000 \times 1044} = 336,972 \text{ lb (1.5 \times 10^6 \text{ kN})} \]
or
\[ V_s = 4 \sqrt{b_p d} = 4 \sqrt{4000 \times 1044} = 264,113 \text{ lb (1.16 \times 10^6 \text{ kN})} \]

controlling \( V_s = 264,113 \text{ lb} \geq \text{required} \) \( V_s = 155,691 \text{ lb} \)

Hence the floor thickness is adequate. Note that because the preliminary forging check for the shear \( V_s \) at this stage does not take into account the shear transmitted by moment, it is prudent to recognize that the chosen trial slab thickness would have to be larger than what the gravity \( V_s \) requires. As a guideline, in the case of interior columns, a thickness based on about

![Figure 11.11 Critical plane for shear moment transfer in Ex 11.1 Interior column (line B-B, Fig. 11.10).](image-url)
1.2 $V_e$, applies in the case of interior columns. For end columns, a recommended multiplier for $V_e$ might have to be as high as 1.6–1.8, and for corner columns, a higher value is applicable. Often, shear head or drop panels are necessary in interior columns to overcome too large a required thickness of the slab. As the serviceability tabulated values in Table 11.3 for minimum thickness of slabs apply only to the interior column zones, to be augmented by 15–15% for end columns and almost 90% for corner columns, they indirectly take into account the increase stipulations for choosing slab thickness based on augmenting $V_e$, as was done at the outset in basing the choice of the slab thickness on augmenting the Table 11.3 value by 10 percent.

**Exterior column.** Include weight of exterior wall, assuming its service weight to be 270 psf. Net factored perimeter shear force is

$$V_e = \left[ \frac{18 \times \frac{24}{2} + 18}{2 \times \frac{12}{2}} \right] \left( \frac{18 + 450 \times 20 + 9.0}{144} \right) \frac{134}{134}$$

$$+ \left( \frac{18 \times \frac{30}{2}}{12} \right) \times 270 \times 1.2 = 60,993 \text{ lb}$$

$$V_e = 66,933 \times \frac{0.75}{0.75} = 90,244 \text{ lb}$$

Consider the line of action of $V_e$, to be at the column face $LM$ in Figure 11.12 for shear moment transfer to the centroidal plane $c-c$. This approximation is adequate since $V_e$ acts perpendicularly around the column faces and not along line $AB$ only. From Figure 11.12,

$$A_c = d(2h_1 + c_1 + 2d) = 9.0(2 \times 18 + 20 + 18) = 9 \times 74$$

$$= 666 \text{ in}^2 (420700 \text{ mm}^2)$$

Available nominal shear $V_e$ is the least of

$$V_e = \left( 2 + \frac{4}{2} \right) \sqrt{b_d f_d} = \left( 2 + \frac{4}{20/18} \right) \sqrt{4000 \times 666} = 233,881 \text{ lb} (1034 \text{ kN})$$

or

$$V_e = \left( \frac{a_d + 2}{b_0 + 2} \right) \sqrt{b_d f_d} = \left( \frac{30 \times 9}{74} + 2 \right) \sqrt{4000 \times 666} = 227,930 \text{ lb} (1007 \text{ kN})$$

---

**Figure 11.12** Centroidal axis for shear moment transfer in Ex. 11.1 and column (line A–A or 1–1, Fig. 11.10).
where \( c = 30 \) for edge column

or

\[
V_y = 4 \sqrt{f_y d} = 4 \sqrt{4000 \times 665} = 168,486 \text{ lb (0.75 \times 10^3 \text{ kN})}
\]

> required \( V_y = 89,244 \text{ lb} \quad \text{O.K.}
\]

Statistical moment computations (steps 3 to 5)

<table>
<thead>
<tr>
<th>E-W:</th>
<th>( I_{y1} = 269 \text{ in.}^2 = 22.42 \text{ ft}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>N-S:</td>
<td>( I_{y2} = 196 \text{ in.}^2 = 16.33 \text{ ft}^2 )</td>
</tr>
</tbody>
</table>

\[
0.65I_1 = 0.65 \times 24 = 15.6 \text{ ft} \quad \text{Use } I_{y1} = 22.4 \text{ ft}^2
\]

\[
0.65I_2 = 0.65 \times 18 = 11.7 \text{ ft} \quad \text{Use } I_{y2} = 16.33 \text{ ft}^2
\]

(a) \( E-W \) direction

\[
M_x = \frac{wL_1^2I_1}{8} = \frac{274 \times 19(22.42)^2}{8} = 356,808 \text{ ft-lb (420 kN-m)}
\]

For end panel of a flat plate without end beams, the moment distribution factors as in Table 11.1 are

- \( M_x \) at first interior support = 0.7M

- \( M_x \) at midspan of panel = 0.5M

- \( M_x \) at exterior face = 0.2M

Negative design moment \(-M_x = 0.7 \times 356,808 = 251,762 \text{ ft-lb (295 kN-m)}

Positive design moment \( +M_x = 0.5 \times 356,808 = 178,404 \text{ ft-lb (218 kN-m)}

Negative moment at exterior \( -M_x = 0.26 \times 356,808 = 90,171 \text{ ft-lb (106 kN-m)}

(b) \( N-S \) direction

\[
M_y = \frac{wL_2^2I_2}{8} = \frac{274 \times 24(16.33)^2}{8} = 219,202 \text{ ft-lb (298 kN-m)}
\]

Negative design moment \(-M_y = 0.7 \times 219,202 = 153,441 \text{ ft-lb (208 kN-m)}

Positive design moment \( +M_y = 0.52 \times 219,202 = 113,985 \text{ ft-lb (146 kN-m)}

Negative design moment at exterior face \( -M_y = 0.26 \times 219,202 = 46,513 \text{ ft-lb (57 kN-m)}

Note that the smaller moment factor 0.35 could be used for the positive factored moment in the \( N-S \) direction in this example if the exterior edge is fully restrained. For the \( N-S \) direction, panel BC12 was considered.

Moment distribution in the column and wide ribs (steps 6 and 7)

At the exterior column, there is no torsional edge beam; hence the torsional stiffness ratio \( \beta \) of an edge beam to the columns is zero. Hence \( \alpha = 0 \). From the exterior factored moments tables for the column span in Section 11.4.2, the distribution factor for the negative moment at the exterior support is 100%, the positive midspan moment is 90%, and the interior negative moment is 75%. Table 11.5 gives the moment values resulting from the moment distributions to the column and wide ribs.
<table>
<thead>
<tr>
<th>Column step</th>
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<td>$P_0$</td>
<td>$P_{10}$</td>
</tr>
</tbody>
</table>

Table 11: Measured Distribution Corresponding Table
Check the shear moment transfer capacity at the exterior column supports

- $M_e$ at interior column $2-B = 216,922$ ft-lb
- $M_e$ at exterior column $2-A = 80,571$ ft-lb

$V_e = 56,823$ lbs acting at the face of the column

The ACI Code stipulates that the nominal moment strength be used in evaluating the unbraced transfer moment at the edge column, that is, use $M_e$ based on $-M_e = 80,571$ ft-lb. Factor shear force at the edge column adjusted for the interior moment is

$$V_e = \frac{60,823}{24} = 2534$$

$V_e = 60,823 \times 0.75 = 81,133$ lbs, meaning that the design $M_e$ has the same value as the factored $M_e$.

From Figures 11.7c and 11.12, taking the moment of area of the critical plane about axis $AB$,

$$d(2c_i + 2d) = d \left( c_i + \frac{d}{2} \right)$$

where $d$ is the distance to the centroid of the critical section of

$$\left( 2 \times 18 + 20 + 18 \right) \frac{9.0}{2} = \left( 18 + \frac{9.0}{2} \right)$$

$$\frac{506.33}{74} = 6.84 \text{ in.}$$

and $g = 6.84 - 9.02 = 2.34$ in. where $g$ is the distance from the column face to the centroidal axis of the section.

To transfer the shear $V_e$ from the face of column to the centroid of the critical section adds an additional moment to the value of $M_e = 80,571$ ft-lb. Therefore, the total external factored moment $M_{ext} = 80,571 + \frac{6350(2.3412)}{2.34} = 92,437$ ft-lb. Total required minimum unbalanced moment strength is

$$M_e = \frac{M_{ext}}{k_s} = 92,437 = 102,708 \text{ ft-lb}$$

The fraction of nominal moment strength $M_e$ to be transferred by shear is

$$\gamma = 1 - \frac{1}{1 + \frac{1}{2} \sqrt{b_2/b_1}} = 1 - \frac{1}{1 + 0.59} = 0.37$$

where $b_1 = c_i + d_2 = 18 + 4.5 = 22.5$ in. and $b_2 = c_i + d = 20 + 9 = 29$ in. It should be noted that the dimensions $c_i$ or $d$ for the end column in the above expression becomes $c_i + d_2$. Hence $M_{ext} = 0.37 M_e$. Moment of inertia of sides parallel to the moment direction about N-S axis is

$$I_1 = \left( \frac{6b^3}{12} + Ar^2 + \frac{hb_2}{12} \right)$$

for both faces

$$I_1 = \left( \frac{9.0 \times 22.5^3}{12} + \frac{9.0 \times 22.5}{2} \times 6.84 \right) \times \frac{22.5\times 9.0}{12} = 27,696 \text{ in.}^4$$

Moment of inertia of sides perpendicular to the moment direction about N-S axis is

$$I_2 = A(x)^2 = \left( 200 + 9.0 \times 0.8 \times 0.66 \right) = 12,211 \text{ in.}^4$$

Therefore,

$$\text{torsional moment of inertia} I_3 = 27,696 + 12,211 = 40,907 \text{ in.}^4$$
If Eq. 11.7a is used instead of first principle calculations, as shown above, the same value \( I = 39.607 \text{ in.}^4 \) is obtained.

Shearing stress due to perimeter shear stress of \( M_p \) and weight of wall is

\[
\sigma_v = \frac{V}{4d} + \frac{6000}{666} \times \frac{M_p}{I} = \frac{60.850}{0.75} \times 666 \times \frac{9.37 \times 6.81 \times 192.74 \times \frac{\pi}{4}}{36.907} \]

\[
= 121.8 \times 78.2 = 9600 \text{ psi}
\]

From before, maximum allowable \( \sigma_v = 4\sqrt{f_c} = 4 \times \sqrt{3000} = 253.2 \text{ psi} \) and

\( \sigma_v < \sigma_c \)

Therefore, acceptable thickness. For the center column, spacial shear head provision or an enlarged column capital might be needed to resist the high shear provisions that location.

Check for shear-moment transfer at the interior column joints.

\[ u_w = 1.8 \times f_c = 112 \text{ psi} \]

From Eq. 11.2.3

\[ M_a = 0.07 (0.5 \times 112 \times 165(22.42)) = 35.667 \text{ ft-lb} \]

\[ V_c = \frac{M_a}{0.75} = \frac{35.667}{0.75} = 47.556 \text{ lb at the face of the column} \]

\[ I = \frac{9(2)^4}{6} + \frac{9 \times 2 \times 28(2)^3}{2} = 49,858 \text{ in.}^4 \]

\( g = 20/2 = 10/\text{in.} \)

Total unbalanced moment \( M_e = \frac{35.667 \times 12}{0.9} + 163.012 \times 10 = 2,120.01 \text{ in.-lb} \)

\[ y = \frac{1}{1 + \frac{2/3}{2}} = 0.4 \]

Hence actual \( u_w = \frac{183.940}{1044} + \frac{1.40 \times 2,120.01 \times 29(2)}{149,848} = 156.9 + 81.7 = 238.6 \text{ psi} \leq 253 \text{ psi} \)

O.K.

This process in the same manner as that used for the end columns for showing the concentrated reinforcement in the column zone at the slab top. To account for the \( \gamma M_e \) moment to be transferred in fixture. Use straight bar over the column extending over the two adjacent spandrel beams full development length.

Design of reinforcement in the slab area of column face for the unbalanced moment transferred to the column by fixture

From Eq. 11.8b

\[ \gamma = 1 - \gamma = 0.37 = 0.63 \]

\[ M_e = \gamma M_e = 0.63 \times 163.012 = 100.47 \text{ in.-lb} \]

This moment must be transferred within 1.36 ft each of the column as in Figure 11.7b.

Section width = 3.5 \times 1015 + 20 = 50.0 \text{ in.}

\[ M_e = A_f \left( d - \frac{a}{2} \right) \]

Assume that \( d - \frac{a}{2} = 0.9 \text{ in.} \)
or \( 776.427 = A_r \times 60,000(9.0 \times 0.9) \) gives:

\[
A_r = 1.60 \text{ in.}^2 \text{ over a strip width = 30.0 in.}
\]

Verifying \( A_r \):

\[
A_r = 0.85 \times 60,000 = 0.56 \text{ in.}
\]

Therefore:

\[
\frac{776.427}{A_r} = 9.0 \times 0.56 = 5.06 \text{ in.}
\]

For the total negative \( M_r \) assigned totally to the negative strip - see Table 11.6, \( M_r = 80.571 \text{ ft-lb}. \) The computed \((V \times d)/A_r = 8.8 \text{ in.} \) Hence \( A_r = 89.523 \times 12 / (88.000 \times 8.8) = 2.08 \text{ in.}^2 \). Consequently, reinforcement outside the column moment-transfer 30 in. wide zone in \( A_r = 2.08 \times 1.48 = 0.60 \text{ in.}^2 \) over a segment = 9 ft less 59 in. = 8.3 ft. Minimum temperature reinforcement = 0.0018 lb ft/ft = 0.0018 \times 12 \times 0.0216 = 0.036 \text{ in.}^2 \); requiring 6 No. 4 bars at the right and left sides of the end column. The 3 No. 5 bars for the \( M_r \) portion of the total end negative moment is concentrated within the 20 in. column width in order to efficiently develop the required development length (see Fig. 11.13).

Checks have to be made in a similar manner for the shear-moment transfer at the face of the interior column C. As also described in Section 11.4.5.2, checks are sometimes necessary for patterns loading conditions and for cases where adjoining spans are not equal or not equally loaded.

Proportionaling of the plate reinforcement (steps 8 and 9)

(a) E-W direction (long span)

1. Summary of moments in column strip (b-lb)

\[
\begin{align*}
\text{interior column negative} & : M_r = 162.649 \text{ ft-lb} \quad \phi = 0.9 \quad \frac{162.649}{0.9} = 180.769 \\
\text{midspan positive} & : M_r = 96.685 \quad \frac{96.685}{0.9} = 107.428 \\
\text{exterior column negative} & : M_r = 89.571 \quad \frac{89.571}{0.9} = 99.523
\end{align*}
\]

2. Summary of moments in middle strip (b-lb)

\[
\begin{align*}
\text{interior column negative} & : M_r = 154.230 \text{ ft-lb} \quad \phi = 0.9 \quad \frac{154.230}{0.9} = 171.366 \\
\text{midspan positive} & : M_r = 64.457 \quad \frac{64.457}{0.9} = 71.619 \\
\text{exterior column negative} & : M_r = 0
\end{align*}
\]

\[\text{Design of reinforcement for column strip} \]

\[-M_r = 180.769 \text{ ft-lb acts on a strip width of } 2(0.25 \times 12) = 9.0 \text{ ft} \]

\[
\text{unit } M_r \text{ per 12-in.-wide strip} = \frac{180.769 \times 12}{9.6} = 214.025 \text{ in.-lb}
\]

\[
\text{unit } M_r = \frac{107.428 \times 12}{0.0018} = 1,342.27 \text{ in.-lb/12 in.-wide strip}
\]

\[
\text{minimum } A \text{ for two-way plates using } f_r = 0.10 \text{ ksi} = 0.0018 \\
\text{Negative steel: } M_r = A_f d \left( \frac{d}{2} \right) \text{ or } 214.025 = A_f \times 0.10 \times (9.0 - \frac{d}{2})
\]
Assuming that moment arm \( d = 0.8 \times \text{bar} \) for first trial and \( h = 8 \) in., the diameter of bar = 0.6 in. for all practical purposes. Therefore,

\[
A_1 = \frac{241.025}{10,000 \times 0.8 \times 0.6} = 5.02 \text{ in.}^2
\]

\[
\sigma = \frac{F_1}{A_1} = \frac{60,000}{0.625 \times 5.02} = 0.74 \text{ in.}
\]

For the second trial and adjustment cycle,

\[
241.025 = A_1 \times 60,000 \left( \sqrt{\frac{9.0 - 0.74}{2}} \right)
\]

Therefore, required \( A_1 \) per 12-in. wide strip = 0.47 in.². Try No. 5 bars (area per bar = 0.305 in.²).

\[
\text{Spacing} = \frac{\text{area of one bar}}{\text{required } A_1 \text{ per 12 in. strip}}
\]

Therefore,

\[
s \text{ for negative moment} = \frac{2.305}{0.47/12} = 7.71 \text{ in. c-c (194 mm)}
\]

\[
r \text{ for positive moment} = 7.50 \times \frac{241.025}{10,000} = 13.11 \text{ in. c-c (332 mm)}
\]

The maximum allowable spacing = \( 2d = 10 \times 0.6 = 6.0 \) in. (150 mm). Try No. 4 bars for positive moment (\( A_1 = 0.29 \) in.²).

\[
A_1 = \frac{143.237}{241.025} = 0.58 \text{ in.² per 12-in. strip}
\]

\[
\text{Minimum temperature reinforcement} = 0.0015 \times 0.6 = 0.0008 \text{ in. x 12 x 10}
\]

\[
= 0.256 \text{ in.}^2 < 0.29 \text{ in.}^2 \text{ O.K.}
\]

\[
x = \frac{9.0}{0.305/12} = 35.57 \text{ in. c-c (907 mm)}
\]

For an external negative support, use No. 4 bars.

\[
\text{Moment} = \frac{80,341 \times 12}{90} = 115,451 \text{ in.-lb}
\]

\[
x = \frac{8.27 \times 143.237}{19,354} = 0.27 \text{ in. c-c}
\]

Use 14 No. 4 bars at 7 in. center to center for negative moments as upper column side; 12 No. 4 bars at 8 in. center to center for positive moments; and 10 No. 4 bars at 10 in. center to center for the external negative moment \( M_e \), with 8 of these bars to be placed outside the shear reinforcement band with 5 in. c-c as seen in Fig. 11.5b.

4. Design of reinforcements for middle strip

unit \(-M_e = \frac{15,120}{12} = 1,260 \) ft-in. on 1 strip width of 7.5 - 9.0 = 9.0 ft,

unit \( M \) per 12 in. width strip = \( \frac{1,260}{12} = 105 \) ft-lb per in.

80,341 = \( A_1 \times 60,000 \) ft-lb

\[
A_1 = \frac{80,341}{60,000} = 0.71 \text{ in.²}
\]

\[
A_1 = \frac{0.71 \times 0.305}{12} = 0.025 \text{ in.²}
\]

Second cycle.

\[
80,341 = A_1 \times 60,000 \left( \sqrt{\frac{9.0 - 0.25}{2}} \right)
\]
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Figure 11.13 Shear moment transfer zone: (a) effective bandwidth; (b) reinforcing details.

\[ A_e = 0.15 \text{ in.}^2/12\text{-in. strip} \]
\[ \text{Minimum } A_e = 0.216 \text{ in.}^2/12\text{-in. strip} \]

Try No. 3 bars \( A_e = 0.11 \text{ in.}^2 \) per bar:
\[ \text{unit } \times M = 64.457 \times 12 = 953.441 \text{ in.-lb per 12-in. strip} \]
\[ A_e = \frac{953.441}{60000} = 0.16 \text{ in.}^2 \]

Hence use negative and positive stirrup spacing = 0.11 x 2(16/12) = 0.17 in. (No. 3 at 6 in. centers to center).

(b) N-S direction (short span): The same procedure has to be followed as for the E-W direction. The width of the column strip on one side of the column = 0.25 x 6 = 1.5 ft.
### Table 11.6  Moments, Bar Sizes, and Spacing

<table>
<thead>
<tr>
<th>Strip</th>
<th>Moment Type</th>
<th>Moment (lb-in./12 in.)</th>
<th>$A_p$ Req'd</th>
<th>Bar Size and Spacing</th>
<th>Moment (lb-in./12 in.)</th>
<th>$A_p$ Req'd</th>
<th>Bar Size and Spacing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coloum</td>
<td>Interior negative</td>
<td>241.025</td>
<td>0.47</td>
<td>No. 5 at 75</td>
<td>138.890</td>
<td>0.31</td>
<td>No. 4 at 60</td>
</tr>
<tr>
<td></td>
<td>Exterior negative</td>
<td>119.564</td>
<td>0.25</td>
<td>No. 4 at 80</td>
<td></td>
<td>84.044</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>Midspan positive</td>
<td>143.277</td>
<td>0.71</td>
<td>No. 4 at 82</td>
<td>84.472</td>
<td>0.22</td>
<td>No. 3 at 60</td>
</tr>
<tr>
<td>Middle</td>
<td>Interior negative</td>
<td>85.341</td>
<td>0.15</td>
<td>No. 3 at 60</td>
<td>34.006</td>
<td>0.07</td>
<td>No. 3 at 60</td>
</tr>
<tr>
<td></td>
<td>Exterior negative</td>
<td>0</td>
<td>0.22</td>
<td>No. 3 at 60</td>
<td>0</td>
<td>0.22</td>
<td>No. 3 at 60</td>
</tr>
<tr>
<td></td>
<td>Midspan positive</td>
<td>95.492</td>
<td>0.18</td>
<td>No. 3 at 60</td>
<td>40.509</td>
<td>0.01</td>
<td>No. 3 at 60</td>
</tr>
</tbody>
</table>

---

1. Minimum temperature steel = 0.0001 lbs/°F when h controls.
2. For panel D2.22 see comments on page 494.
3. 10% of this moment is used for the shear-moment transfer negative reinforcement $M_u$ of the end column zone (see Fig. 11.12).
which is greater than 0.25% = 4.5 ft; hence a width of 4.5 ft controls. The total width of the column strip is the N-S direction = 2 x 4.5 = 9.0 ft. The width of the middle strip = 24.0 - 9.0 = 15.0 ft. Also, the effective depth d' would be smaller; d' = (b - 1 in. cover = 0.5 in. - 0.52) = 8.5 in. The moment values and the bar size and distribution for the panel in the N-S direction as well as the E-W directions are listed in Table 11.6. It is recommended for crack-control purposes that a minimum of No. 3 bars at 12 in. center to center be used and that bar spacing not exceed 12 in. center to center. In this case, the maximum reinforcement required by the ACI Code for slabs reinforced with f' = 60,000 psi is equal to 0.0116 b = 0.3 ft at 63 in. center to center. Spans at 6 in. center to center.

The choice of size and spacing of the reinforcement is a matter of engineering judgment. As an example, the designer could have chosen for the positive moment in the middle strip No. 4 bars at 12 in. center to center, instead of No. 3 bars at 6 in. center to center, as long as the maximum permissible spacing is not exceeded and practical bar sizes are used for the middle strip.

The placing of the reinforcement is schematically shown in Figure 11.14. The minimum cutoff of reinforcement for bond requirements in flat-plate floors is given in Figure 11.15. The exterior panel negative steel at outer edges, if no edge beams are used, has to be bent into full hook in order to ensure sufficient anchorage of the reinforcement. The floor reinforcement plan gives the E-W steel for panel A823 and N-S steel for panel BC12 of Figure 11.10.

11.5.3 Example 11.2 Design of Two-way Slab on Beams by Direct Design Method (DDM)

A two-story factory building is three panels by three panels in plan, monolithically supported on beams. Each panel is 18 ft (5.49 m) center to center in the N-S direction and 34 ft (10.37 m) center to center in the E-W direction, as shown in Figure 11.16. The clear height between the floors is 16 ft. The dimensions of the supporting beams and columns are also shown in Figure 11.16, and the building is subject to gravity loads only. Given:

- live load = 125 psf (600 kPa)
- f' = 4000 psi (27.6 MPa), normal-weight concrete
- f = 60,000 psi (415 MPa)
Assume $f_t > 2.5$.

Design the interior panel and the size and spacing of reinforcement needed. Consider:
- Flooring weight to be 14 psf in addition to the dead-load weight.

**Solution:** Geometry check for use of direct design method (step 1)
(a) Ratio longer span/shorter span = $14/18 = 0.78 > 0.5$, hence two-way action.
(b) More than three panels in each direction.
(c) Assume a thickness of 7 in.

\[
\begin{align*}
\bar{w} &= 14 + \frac{7}{12} \times 18 = 101.5 \text{ psf} \\
2\bar{w} &= 203 \text{ psf} \\
\bar{w} &= 135 \text{ psf} < 2\bar{w},
\end{align*}
\]

Hence the direct design method is applicable.

**Minimum slab thickness for deflection requirement (step 2)**
\[
\begin{align*}
l_1 (E-I) &= 24 \times 12 - 2 \times 8 = 294 \text{ in.} \\
l_2 (E-I) &= 18 \times 12 - 2 \times 8 = 200 \text{ in.} \\
\frac{k}{200} &= 1.36
\end{align*}
\]

For a preliminary estimate of the required total thickness $h$, using Eq. 11.10,
\[
h = \frac{h_0 (0.03 + 0.001)}{38 + 0.03}
\]


<table>
<thead>
<tr>
<th>Drop Location</th>
<th>Minimum percent $A_y$ at section</th>
<th>Without drop panels</th>
<th>With drop panels</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top</td>
<td>50 Remainder</td>
<td>0.30$w$</td>
<td>0.33$w$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.30$L$</td>
<td>0.33$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.28$L$</td>
<td>0.32$L$</td>
</tr>
<tr>
<td></td>
<td>50 Remainder</td>
<td>0.19$L$</td>
<td>0.22$L$</td>
</tr>
<tr>
<td></td>
<td>24 bar dia. or 12” min. all bars</td>
<td>3” max.</td>
<td>Max. 0.125$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Max. 0.125$L$</td>
<td>6”</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6” max.</td>
<td>Max. 0.15$L$</td>
</tr>
<tr>
<td></td>
<td>At least 2 bars continuous or anchored as required</td>
<td>Edge of drop</td>
<td>6”</td>
</tr>
<tr>
<td>Bottom</td>
<td>50 Remainder</td>
<td>0.22$L$</td>
<td>0.25$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.22$L$</td>
<td>0.25$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.23$L$</td>
<td>0.25$L$</td>
</tr>
<tr>
<td>Middle</td>
<td>Top</td>
<td>0.22$L$</td>
<td>0.25$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.22$L$</td>
<td>0.25$L$</td>
</tr>
<tr>
<td></td>
<td>50 Remainder</td>
<td>0.15$L$</td>
<td>0.16$L$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6”</td>
<td>6”</td>
</tr>
</tbody>
</table>

Figure 11.15 Minimum extensions for reinforcement in slabs without beams.
To check $\delta$ from Eq. 11.9, the stiffness ratio $a_2$ is needed. Since this is an interior panel, the end and corner adjacent panel would necessitate larger thickness. Try $\delta = 7$ in.

To locate the beam centroid for the section in Figure 11.17,
\[ (38 \times 7)^2 + 3.5) = \frac{12(7)^2}{2} = \frac{12(13 - y)^2}{2} \]

\[ P = 0.20 \text{ in.} \]

\[ l_y = \frac{1}{12} \times 12(0.20)^2 + \frac{1}{12} \times 38(7)^2 + 38 \times 7(0.20 + 3.5)^2 + \frac{1}{12} \times 12(13 - 0.20)^2 = 13.1164 \text{ in.}^2 \]

\[ J_y = h/12 \times \text{width of slab bound laterally by the outer line of the adjacent panel on each side of the beam section shown in Figure 11.16} \]

\[ l_y \text{ (N-S)} = \frac{77}{12} \times 24 \times 12 = 8232 \text{ in.}^4 \]

\[ l_y \text{ (E-W)} = \frac{77}{12} \times 23 \times 12 = 6714 \text{ in.}^4 \]

Therefore,

\[ a_{f} = \frac{13,116.4}{8232} = 1.59 \quad a_{r} = \frac{13,116.4}{6714} = 2.12 \]

\[ a_{r} = \frac{1.59 \times 2 + 2.12 \times 3}{4} = 1.86 \]

From Eq. 11.9,

\[ h = \frac{12(0.8) + f_{f}(203.03)}{36 + 50(a_{f} - 0.3)} \]

\[ = \frac{272.08 + 10.04(203.03)}{36 + 5 \times 1.59[1.80 - 0.2]} = 6.29 \text{ in. (16 cm)} \]

Minimum \( h \) in this case from Eq. 11.5 = 6.29 in. Therefore, for deflection, use \( h = 7 \) in. (17.8 cm) assumed at the beginning.

**Static moment computation (steps 3 & 5)**

Given the flooring weights 14 psf

\[ w_e = 1.2D + 1.6L \]

\[ = 1.2 \left( \frac{7}{12} \times 150 + 14 \right) + 1.6 \times 135 = 338 \text{ psf} \]

E-W \( l_y = 272 \text{ in.} = 22.7 \text{ ft} \)

N-S \( l_y = 200 \text{ in.} = 16.7 \text{ ft} \)

0.65L = 154 ft use \( l_y = 22.7 \text{ ft} \)

0.65L = 117 ft use \( l_y = 16.7 \text{ ft} \)

(a) E-W direction

\[ M_y = \frac{w_e l_y^2}{8} = \frac{338 \times 18.0(22.7)^2}{8} = 391,878 \text{ lb-ft (532 kN-m)} \]

Moment distribution factors for interior panel from Fig. 11.5 = nn

\[-M_y = M_e = 0.65M_y = 0.65 \times 391,878 = 254,720 \text{ lb-ft} \]

\[ + M_e = 0.35M_y = 0.35 \times 391,878 = 137,157 \text{ lb-ft} \]

(b) N-S direction

\[ M_y = \frac{w_e l_y^2}{8} = \frac{338 \times 24.0(16.7)^2}{8} = 263,704 \text{ lb-ft (364 kN-m)} \]
11.5 Design and Analysis Procedure: Direct Design Method

Moment distribution factors for interior panel from Figure 11.5 or Table 11.1 are:

\[ +M_v = 0.65M_0 = 0.65 \times 282,794 = 183,816 \text{ kN-m} \]
\[ +M_v = 0.35M_0 = 0.35 \times 282,794 = 99,971 \text{ kN-m} \]

Moment distribution in the column and middle strips (steps 5 to 7)

(a) E-W stiffness ratio (long span)

\[ \alpha_y = \frac{EJ_y}{EJ_z} = \frac{13,116.4}{6174} = 2.12 \]
\[ \frac{l_y}{l_z} = \frac{18}{24} = 0.75 \quad \frac{l_y}{l_z} = 1.59 > 1.0 \]

Moment factors for the column strip for this panel from the factored moment coefficient for the column strip of an interior panel (Section 11.4.2. Interior panels and positive moments) are linearly interpolated to give the following:

\[ -M_v: 0.75 + \frac{0.80 - 0.75}{2} = 0.83 \]
\[ +M_v: 0.75 + \frac{0.80 - 0.75}{2} = 0.83 \]

(b) N-S stiffer ratio a

\[ \alpha_y = \frac{EJ_y}{EJ_z} = \frac{13,116.4}{8232} = 1.59 \]
\[ \frac{l_y}{l_z} = \frac{24}{18} = 1.33 \quad \frac{l_y}{l_z} = 2.12 > 1.0 \]

Hence moment factors for the column strip are in this case using the same tables by linear interpolation:

\[ -M_v: 0.75 - (0.75 - 0.45) \frac{1}{3} = 0.65 \]
\[ +M_v: 0.75 - (0.75 - 0.45) \frac{1}{3} = 0.65 \]

Table 11.7 Moment Distribution Operations Table

<table>
<thead>
<tr>
<th>Column Strip</th>
<th>E-W Direction</th>
<th>N-S Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Moment (kN-m)</td>
<td>Moment (kN-m)</td>
</tr>
<tr>
<td></td>
<td>Negative</td>
<td>Positive</td>
</tr>
<tr>
<td>M_y(0-ib)</td>
<td>234,720</td>
<td>137,157</td>
</tr>
<tr>
<td>Distribution factor (%)</td>
<td>83</td>
<td>83</td>
</tr>
<tr>
<td>Total column strip design moment (0-ib)</td>
<td>211,418</td>
<td>113,940</td>
</tr>
<tr>
<td>Beam moment, 85%</td>
<td>179,705</td>
<td>96,764</td>
</tr>
<tr>
<td>Column strip slab moment (0-ib)</td>
<td>31,713</td>
<td>17,476</td>
</tr>
<tr>
<td>Total middle strip design moment (0-ib)</td>
<td>254,064</td>
<td>137,637</td>
</tr>
<tr>
<td>× 0.17</td>
<td>× 0.17</td>
<td>× 0.25</td>
</tr>
<tr>
<td>43,403</td>
<td>22,317</td>
<td>34,336</td>
</tr>
</tbody>
</table>
The distributed moments are then evaluated using the interpolated factors above to produce a moment distribution operations table (Table 11.7). Note from this table that the stiffness ratio of the slab to the supporting beams for the span ratio in this example has resulted in middle strip moments in the N-S direction larger than the moments in the E-W direction.

Check slab thickness for shear capacity:

\[
\frac{h_f}{t} = 1.59 > 1.0
\]

Hence shear will be transferred to the beams surrounding the slab according to a tributary area bound by 45° lines drawn from the corners of the slab and the center line of the panel parallel to the long side.

The largest part of the load has to be carried in the short direction with the largest value at the face of the first interior support. The factored shear on a 12-in.-wide strip spanning in the short direction can be approximated as

\[
V_s = 1.15 \frac{4.0 \times 338.0 \times 16.7 \times 2}{2 \times 12} = 3240 \text{ lb/ft width}
\]

where the value 1.15 is the continuity factor.

slab effective \( d = 7 - 0.75 - 0.25 = 6.0 \text{ in. (152 mm)} \)

\[
\phi V_s = 0.75 \times 2\sqrt{4 \times 8000} \times 12 \times 6 = 6481 \text{ lb}
\]

\( V_s < \phi V_s \quad \text{hence safe}
\]

Proportioning of the slab reinforcement (steps 7 and 8)

Minimum \( A_s \), using \( f_s = 60,000 \text{ psi steel} \) is \( 0.0018 \times 12 \times 7 = 0.15 \text{ in. slab in strip}. \) For No. 3 steel, \( A_s = 0.10 \times (0.19 \times 13) = 8.8 \text{ in. on center}; use No. 3 at 8 in. on center to center. As was done in Ex. 11.1, the moments per 12-in.-wide strip have to be evaluated.

(a) E-W direction

Column strip

\[
M_s = \frac{31.713}{4} = 7.93 = 35.237
\]

\[0.251 = 0.25 \times 18 \text{ ft} = 4.5 \text{ ft} < 0.25 \times 34 \text{ ft}
\]

Hence the half column strip = 4.5 ft controls. The net width of the slab in the column strip on which moments act = \( 2.5 \times 4.5 = 36 \text{ in./12 = 3 ft}. \)

required unit \(- M \) per 12-in. strip = \( \frac{35.237 \times 12}{5.83} = 75.529 \text{ in.-lb} \)

required unit + \( M \) per 12-in. strip = \( \frac{75.529 \times 12}{0.9 \times 5.83} = 39.053 \text{ in.-lb} \)

Middle strip

width of strip = 18 - 9.0 = 9.0 ft

required unit \(- M \) per 12-in. strip = \( \frac{42.403 \times 12}{0.9 \times 9} = 643.01 \text{ in.-lb} \)

required unit + \( M \) per 12-in. strip = \( \frac{643.01 \times 12}{0.9 \times 9.0} = 33.544 \text{ in.-lb} \)

(b) N-S direction (short span): From before, the maximum allowable width of the half

\[ \text{strip} = 4.5 \text{ ft}. \]
11.5 Design and Analysis Procedure: Direct Design Method

Columns strip

net width of slab in column strip on which moments act = 2 x 4.5 - 38/12 = 5.25 ft

required unit - M per 12-in. strip = 1.622 x 12 = 19,688 in-lb

required unit - M per 12-in. strip = 0.9 x 5.83 = 5,272 in-lb

Middle strip

width of strip = 24 - 9 = 15.0 ft

required unit - M per 12-in. strip = 64.36x12 = 57,188 in-lb

required unit - M per 12-in. strip = 22.542 x 12 = 28,062 in-lb

Selection of size and spacing of reinforcement (Step 9)

The maximum unit moment in the negative moment region of the column strip in the E-W direction = 73,520 in-lb per 15-in-wide strip.

\[ M_e = A_f f_y \left( \frac{d}{2} \right) \]

Hence

73,520 = A_f f_y \left( \frac{d}{2} \right)

\[ A_f = \frac{73,520}{60,000 \times 0.6 \times 6.0} = 0.22 \text{ in}^2 \]

Adjustment

\[ \rho = \frac{A_f f_y}{0.85 f_{y,b}} = \frac{22 \times 60,000}{0.85 \times 4000 \times 12} = 0.32 \text{ in} \]

Hence

73,520 = A_f x 60,000 \left( \frac{6.0 - 0.32}{2} \right)

Therefore, required \( A_f = 0.11 \text{ in}^2 \) per 15-in. strip. Try No. 4 bars (0.20 in\(^2\)) (12.7-mm diameter)

\[ \rho = \frac{\text{area of one bar}}{\text{required area per 12-in strip}} = \frac{0.20}{0.21/12} = 11.43 \text{ in} \cdot \text{cp} \]

Hence, use No. 4 bars at 1-in. center to center (12.7-mm diameter at 290 mm center to center.

In the same manner, calculate the area of steel needed in each direction for both the column and middle strips (Table 11.8). Note that the effective depth of the N-S strip does not vary from 7.0 - (0.3 + 0.5 + 0.25) = 5.5 in, since it is assumed in this design that the E-W grid of reinforcement is closest to the concrete surface.

Compare the reinforcement area obtained in this example with those of Ex. 11.1 in conjunction with the discussion in Section 11.2 on two-way action and moment redistribution as a function of stiffness ratios. It should be noted that, when the slab or plate panel is either supported on flexible supports or on columns only, the moments are not necessarily more severe in the shorter direction.

Carry the reinforcement at the same spacing for each respective strip up to the webs of the supporting beams. Also, in the next step, design (analyze) the supporting beams in the usual manner as discussed in Chapter 5.
Table 11.8 Column and Middle Strip Calculations

<table>
<thead>
<tr>
<th>Direction</th>
<th>Column Strip</th>
<th>Support</th>
<th>Midspan</th>
<th>Column Strip</th>
<th>Support</th>
<th>Midspan</th>
</tr>
</thead>
<tbody>
<tr>
<td>E-W (d = 6.0)</td>
<td>0.21</td>
<td>No. 4 at 11</td>
<td>0.11</td>
<td>No. 3 at $|$</td>
<td>0.10</td>
<td>No. 4 at 12</td>
</tr>
<tr>
<td>N-S (d = 5.5)</td>
<td>0.14</td>
<td>No. 3 at 9</td>
<td>0.06</td>
<td>No. 3 at $|$</td>
<td>0.20</td>
<td>No. 4 at 12</td>
</tr>
</tbody>
</table>

*Maximum spacing of bars should not exceed 12 in. to allow for crack control.

11.6 EQUIVALENT FRAME METHOD FOR FLOOR SLAB DESIGN

11.6.1 Applicability

The equivalent frame method for the design of two-way slab and plate systems is a more rigorous form of the direct design method presented in Sections 11.3 and 11.4. It differs only in the means of computing the longitudinal variation of bending moments along the design frame such that it would be applicable to a wide range of applications. Its main features can be summarized in the following:

1. Moments are distributed to critical sections by an elastic analysis such as moment distribution, rather than by general factors. Pattern loadings have to be considered for the most critical loading conditions.
2. There are no limitations on dimensions or loadings.
3. Contrary to the simplifications in the direct design method, variations in the moment of inertia along the axes of members have to be considered, such as the effects of column capitals.
4. Effects of lateral loading can be accounted for in the analysis.
5. In contrast to the direct design method, use of a computer facilitates the analysis in this method through evaluation of the various stiffnesses.
6. Because of the refinement possible in its use, the total statical moment need not exceed the statical moment \(M_s\) required by the direct design method.

11.6.2 Stiffness Coefficients

The structure, divided into continuous frames as shown in Figure 11.3 for frames in both orthogonal directions, would have the row of columns and a wide continuous beam (slab) \(ABCD\). Each floor is analyzed separately whereby the columns are assumed fixed at the floor above and below. To satisfy statics and equilibrium, each equivalent frame must carry the total applied load. Alternate span loading has to be used for the worst live load condition.
It is necessary to account for the rotational restraint of the column at the joint when summing a moment relaxation or distribution except when the columns are very slender, with very low rigidity compared to the rigidity of the slab at the joint. In such cases, such as in lift slab construction, only a continuous beam is necessary. A schematic illustration of the constituent elements of the equivalent frame is given in Figure 11.16.

The slab strips are assumed to be supported by transverse slabs. The column provides a resisting torque $T_r$, equivalent to the applied torsional moment $m$. The exterior ends of the slab strip rotate more than the central section because of the torsional deformation. To account for this rotation and deformation, the actual column and the transverse slab strip are replaced by an equivalent column, such that the flexibility of the equivalent column is equal to the sum of the flexibilities of the actual column and the slab strip. This assumption is represented as follows:

$$\frac{1}{K_c} = \frac{1}{K_c^e} + \frac{1}{K_r}$$

where $K_c = \text{flexural stiffness of the equivalent column, moment per unit rotation}$

$\Sigma K_r = \text{sum of flexural stiffnesses of the upper and lower columns at the joint, moment per unit rotation}$

$K_r = \text{torsional stiffness of the torsional beam, moment per unit rotation}$

Alternatively, the flexibility equation (11.11) can be written as a stiffness equation:

$$K_c = \frac{\Sigma K_r}{1 + \Sigma K_r/K_c^e}$$

The column stiffness for an equivalent frame (Ref. 11.7) can be defined as

$$K_c = \frac{EI}{T_e} \left[ 1 + 3 \left( \frac{\gamma}{\gamma_e} \right)^2 \right]$$

(11.13)

Figure 11.16 Consistent elements of the equivalent frame.
where \( I \) = column moment of inertia
\( I \) = center-line span
\( I' \) = clean span of the equivalent beam

The carry-over factors are approximated by \(-1(1 + 1/\alpha)\). A simpler expression for \( K_v \)
(Ref. 11.8) gives results within 5% of the more refined values from Eq. 11.13 as follows:

\[
K_v = \frac{4EI}{L - 2h} \tag{11.14}
\]

where \( h \) is the slab thickness.

The torsional stiffness of the slab in the column line is

\[
K_T = \sum \frac{9E_cC_c}{L_c\left(1 - \left(c/c_c\right)^2\right)} \tag{11.15}
\]

where the torsional constant is

\[
C = \frac{\sum(1 - 0.633/s)^{1.2}}{3}
\]

and
\( x \) = shorter dimension of rectangular part of cross section at column junction (such as slab depth)
\( y \) = longer dimension of rectangular part of cross section at column junction (such as column width)
\( L_o \) = band width
\( L_m \) = span
\( c \) = column dimensions in direction parallel to the torsional beam

\[
K_T = \frac{4E_cI_c}{L_c - c/2} \tag{11.16}
\]

As the effective stiffness \( K_e \) of the column and the slab stiffness \( K_s \) are established, the analysis of the equivalent frame can be performed by any applicable methods, such as relaxation or moment distribution. The distribution factor for fixed-end moment (FEM) is

\[
DF = \frac{K_e}{\sum K} \tag{11.17}
\]

where \( \sum K = K_e + K_{slab} + K_{carry} \)

For carry-over factors, COF = \( \frac{1}{2} \) can be used without loss of accuracy since the non-prismatic section causes only very small effects on fixed-end moments and carry-over factors. The fixed-end moments for a uniformly distributed load are \( w_d L_c^2/12 \) at the supports, such that after moment redistribution the sum of the negative distributed moment at the support and the midspan would always be equal to the static moment \( M_0 = w_d L_c^2/8 \). A moment redistribution factor in continuous panels can be applied as in Figure 5.7 for the equivalent frame design, provided that it does not exceed the smaller of 20 percent or (1000, e) percent.

11.6.3 Pattern Loading of Spans

Loading all spans simultaneously does not necessarily produce the maximum positive and negative flexural stresses. Consequently, it is advisable to analyze the multigan frame also using alternate span loading patterns for the live load. For a three-span frame, the suggested patterns for the live load are as shown in Figure 11.19. The ACI Code, however, permits the full factored live load to be used on the entire slab system if the live load is less than 75% of the dead load.
11.6 Equivalent Frame Method for Floor Slab Design

11.6.4 Operational Flowchart for the Design (Analysis) of Two-way Floor Slabs by the Equivalent Frame Method

The flowchart for this procedure is shown in Figure 11.20.

11.6.5 Example 11.3: Design of Flat Plate without End Beams by the Equivalent Frame Method (EFM)

A reinforced concrete flat plate apartment floor system without end beams and drop panels is shown in Figure 11.21. The end panel center-line dimensions are 12'-0" x 20'-0" (3.66 m x 6.00 m) and the interior panel dimensions are 24'-0" x 20'-0" (7.32 m x 6.00 m). The floor heights of intermediate floors are typically 8'-0" (2.44 m). Evaluate the required nominal masonry strengths $M_{o}$ for a typical floor panel in the north-south direction (without and with live load $w_{L} = 40$ psf (192.8 kPa) and a superimposed dead load $w_{D} = 10$ psf due to partitions and flooring. Assume that all panels are simultaneously loaded by the live load in your solution. Given:

$f_{c}' = 4000$ psi (27.6 MPa) normal-weight concrete

$f_{c} = 6000$ psi (41.3 MPa)

Solution: Equivalent frame characteristics

Deflection thickness check: From Table 11.3, the minimum thickness $h$ for interior panels using steel having $f_{c}' = 6000$ psi and without drop panels is

$h_{c} = \frac{22.33}{33} = 0.68$ in

Required $h = \frac{22.33}{33} \times 12 = 8.18$ in, say 8.25 in (210 mm)

Thickness of exterior panel without edge beams is $h_{e} = 0.25$ to be increased at discontinuous edges by at least 10%. Exterior panel $l_{o} = 20'-0" = 14$ in, $h = 18.82$ ft, $h = 18.82/30 \times 12 = 12 \times 1.10 = 13.2$ in. Use $h = 8.25$ in (210 mm) for all panels of the floor system.
Figure 11.20 Flowchart for the design (analysis) of reinforced concrete two-way slabs and plates by the equivalent frame method.
Figure 11.21 Flat-plate apartment structure (a) plan; (b) section A-A, N-S.

Take the equivalent frame in the N-S direction whose plan is shown in the shaded portion in Figure 11.21.

\[ w_e = 1.2 \left( \frac{18.25}{13} \times 150 + 16 \right) = 1.6 \times 60 = 96 \text{ psf} \]

Approximate flexural stiffness of column above and below the floor joists (moment per unit rotation), from Eq. 11.14 is

\[ K = \frac{4EI}{L_n^3} \quad \text{where} \quad L_n = 8' - 9'' = 105 \text{ in.} \]
11.6 Equivalent Frame Method for Floor Slab Design

(a) Exterior column (14 in. × 12 in.) stiffness

\[ b = 14 \text{ in.} \quad L = \frac{14(12)}{12} = 2016 \text{ in.}^4 \]

Assume that \( E_u/E_{cu} = E_c/E_{cu} = 1.6 \). Use \( E_u = E_c = 1.0 \) in the calculations as \( E_u \) drops out in
the equation for \( K_u \).

total \( K_u = \frac{4 \times 1 \times 2016}{108 - (2 \times 8.25)} \times 2 \) (for top and bottom columns)

\[ = 192.2 \text{ in.-lb/rad} / E_u \]

Torsional constant \( C \) from Eq. 11.15 is

\[ C = \frac{1}{1 - 0.63 x} \frac{x^2}{3} \]

\[ = \left( \frac{1 - 0.63 \times 8.25}{12} \right) \left( 8.25^3 \right) \times \frac{12}{3} = 1273 \]

Torsional stiffness of the slab at the column line is

\[ K_s = \frac{1}{L} \frac{9E_u C}{2(1 - \alpha L / 4B)^2} \]

\[ = \frac{9 \times 1 \times 1273}{20 \times 12 \left[ 1 - 14(12 \times 20)^3 \right]} \]

\[ = 57.1 \text{ in.-lb/rad} / E_u \]

From Eq. 11.12, the equivalent column stiffness is

\[ K_u = \left( \frac{1}{K_s} \right)^{\frac{1}{1}} \left( \frac{1}{1 + \frac{1}{1273}} \right)^{1} = 43.5 \text{ in.-lb/rad} / E_u \]

(b) Interior column (14 in. × 20 in.) stiffness

\[ b = 14 \text{ in.} \quad L = \frac{14(20)}{12} = 9333 \text{ in.}^4 \]

total \( K_u = \frac{4 \times 1 \times 9333}{108 - (2 \times 8.25)} \times 2 = 843.7 \text{ in.-lb/rad} / E_u \)

\[ C = \left( 1 - 0.63 \times 8.25 \right) \left( 8.25^3 \right) \times 20/3 = 2770 \]

\[ K_s = \frac{9 \times 2770}{20 \times 12 \left[ 1 - 14(12 \times 20)^3 \right]} \]

\[ = 248.8 \text{ in.-lb/rad} / E_u \]

\[ K_u = \left( \frac{1}{9333} \right)^{\frac{1}{1}} \left( \frac{1}{1 + \frac{1}{248.8}} \right)^{1} = 192.1 \text{ in.-lb/rad} / E_u \]

(c) Slab stiffness

\[ h = 8.25 \text{ in.} \]

From Eq. 11.16,

\[ K_s = \frac{4E_u f}{\ell_u \sqrt{c}} \]

where \( \ell_u \) = center-line span

\( c \) = column depth

Slab band width in E-W direction = 20 + 2 × 92 = 20 + 184 = 204.
Chapter 11  Design of Two-Way Slabs and Plates

\[ I = 20 \times \frac{(8.25)^2}{12} = 11.230 \text{ in}^4 \]

Slab at right of exterior column A:

\[ K_e = \frac{4 \times 1 \times 20(8.25)^2}{12 \times 17.5 - 12/2} = 200.2 \text{ in-lb}/\text{rad}/E_{ce} \]

Slab at left of interior column B:

\[ K_e = \frac{4 \times 1 \times 20(8.25)^2}{12 \times 17.5 - 20/2} = 224.6 \text{ in-lb}/\text{rad}/E_{ce} \]

Slab at right of interior column B:

\[ K_e = \frac{4 \times 1 \times 20(8.33)^2}{12 \times 24 - 20/2} = 161.6 \text{ in-lb}/\text{rad}/E_{ce} \]

From Eq. 11.17, slab distribution factor at joint A:

\[ DF = K_{eA} \text{ where} \sum K = K_{eA} + K_{eB} \]

Outer joint A slab:

\[ DF = \frac{220.2}{43.1 + 220.2} = 0.385 \]

Left joint B slab:

\[ LF = \frac{224.6}{192.1 + 224.6 + 161.6} = 0.388 \]

Right joint B slab:

\[ DF = \frac{161.6}{192.1 + 224.6 + 161.6} = 0.379 \]

Moment distribution of factored moment \( M_e \):

From before, \( w_r = 240 \text{ psf} \).

Joint A, span AB:

Factored fixed-end moment \( M_{eA} = \frac{w_r L^3}{12} \)

\[ = \frac{240(17.5)^2 \times 240(24.0)^3}{12} = 73,500 \text{ in-lb}/\text{ft} \]

Joint B, span BC:

Factored fixed-end moment \( M_{eB} = \frac{w_r L^3}{12} \)

\[ = \frac{240(24.0)^2 \times 240(24.0)^3}{12} = 138,240 \text{ in-lb}/\text{ft} \]

Run a moment distribution for the factored moments as shown in Table 11.9.

Factored and required nominal moment reactions \( M_n \):

Joint A (span AB) moment = \( M_n \)
### Table 11.9: Moment Distribution of Factored Loads

<table>
<thead>
<tr>
<th>Location</th>
<th>( M_x ) (in·lb/ft)</th>
<th>( M_y ) (kip-ft)</th>
<th>( M_z ) (kip-ft)</th>
<th>( M_{xy} ) (kip-ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Column</td>
<td>1</td>
<td>0.835</td>
<td>0.388</td>
<td>0.279</td>
</tr>
</tbody>
</table>

---

- **Moment Reduction Factor (MRF)**: Calculated as follows:
  
  \[
  MRF = \frac{M_{factored}}{M_{uncertainties}}
  \]

- **Final Moments**:
  
  \[
  M_x = M_{factored} - M_{uncertainties}
  \]

---

**Joint B (span BC) Moment**:

\[
M_x = 2800 \times \frac{2}{3} = 1866.67 \text{ kip-ft}
\]

---

**Joint C (span CD) Moment**:

\[
M_x = 2800 \times \frac{2}{3} = 1866.67 \text{ kip-ft}
\]
required \( M_x = \frac{M_x}{\phi} = \frac{116,050}{0.9} = 128,944 \text{ in.-lb/ft (47.8 kNm/m)} \)

**Span AB maximum positive moment + \( M_x \). Assume that point of zero shear and maximum moment is \( x \) ft from face \( A \):**

\[
x = \frac{V_{AC}}{V_{AC}} = \frac{1572.1}{240} = 6.54 \text{ ft}
\]

End \( M_x \) at \( A \) (Table 11.9) = 10,800 in.-lb/ft

\[\text{maximum} + M_x = V_{AC} \times x - \frac{w_x x^2}{2} - M_x\]
\[
= 1572.1 \times 6.54 \times 12 - \frac{24(6.54)^2}{2} \times 12 - 10,800
\]
\[
= 133,245 - 61,591 - 10,800 = 51,594 \text{ at 6.54 ft from } A
\]

required \( M_x = \frac{M_x}{\phi} = \frac{51,594}{0.9} = 57,327 \text{ in.-lb/ft (21.3 kNm/m)} \)

**Span BC maximum positive moment + \( M_x \)**

\[V_{BC} = 2680 \text{ lb/ft} \quad x = \frac{L_x}{2} = \frac{24}{2} = 12 \text{ ft}
\]

Simple span midspan moment \( M = V_{AC} \times \frac{L_x}{2} - \left( \frac{w_s \times L}{2} \right) \left( \frac{L}{2} \right) \)
\[
= 2680 \times \frac{24}{2} - \frac{24(24)^2}{8}
\]
\[
= 17,280 \text{ ft-lb/ft} = 207,360 \text{ in.-lb/ft}
\]

Alternatively,

\[
\text{Simple span moment } M_s = \frac{w_s L^2}{8} - \frac{24(24)^2}{8} \times 12
\]
\[
= 207,360 \text{ in.-lb/ft}
\]

\[M_s = M_s - (-M_x)
\]

\[M_x \text{ from Table 11.9} = -155,250 \text{ in.-lb/ft}
\]

required maximum + \( M_x = 207,360 - 155,250 \)
\[
= 72,110 \text{ in.-lb/ft (15.93 kNm/m) at midspan}
\]

required + \( M_x = \frac{M_x}{\phi} = \frac{72,110}{0.9} = 80,222 \text{ in.-lb/ft (18.90 kNm/m)}
\]

Figure 11.22 gives a plot of the required moment strengths \( M_x \) across the continuous span at the column face.

For a complete design, a similar analysis has to be performed in the E-W direction. From there on, the nominal moment strength values are split into column strip moments and middle strip moments in both the N-S and E-W directions in a manner identical to the pro-
11.6.6 Coefficients Method of Ultimate Moment Evaluation for Slabs on Continuous Supports

For slabs continuous over some supports along some edges and discontinuous at others, Fig. 11.23 can be used for a rapid evaluation check of the support moments coefficients at ultimate load. A provision is made in the chart at the discontinuous edge for possible moment restraint of the slab as cenv monolithically with the edge beams.

- Short span:
  - Support moment: \( M_s = -\beta_s w_i d_i^2 \)
  - Span: \( M_s = +\beta_s w_i d_i^2 \)

- Long span:
  - Support moment: \( M_s = -\beta_s w_i d_i^2 \)
  - Span: \( M_s = +\beta_s w_i d_i^2 \)

where \( w_i \) is the unit intensity of factored load per unit area of the slab.

11.7 SI TWO-WAY SLAB DESIGN EXPRESSIONS AND EXAMPLE

1. Material properties

\[
E_s = 70,000 \text{ N/m}^2 = 70,000 \text{ MPa} \\
E_s = 200,000 \text{ N/m}^2 = 200,000 \text{ MPa}
\]
Figure 11.23 Ultimate load moment coefficients in two-way-action slabs and plates. From Ref. 11.3.)
11.7 Two-way Slab Design Expressions and Example

\[ c_s = \frac{600}{600 + f'_c} \]

\[ \beta_s = 0.88 - 0.008 (f'_c - 30) \]

Minimum temperature steel = 0.00106. The value of \( \beta_s \) for strengths above 30 MPa should be reduced at the rate of 0.008 for each 1 MPa in excess of 30 MPa, but \( \beta_s \) cannot be less than 0.65. Maximum spacing of reinforcement = 2 \( f'_c \).

2. Deflection of two-way slab on beams: For \( a_w \leq 0.2 \), use Table 11.1. For \( a_w > 0.2 \), as in Equation 11.9.

\[ h = \frac{I_4}{36 + 40(\alpha f'_c - 0.2)} \]

For \( a_w > 2.0 \),

\[ h = \frac{(0.8 + \frac{f'_c}{1500})}{36 + 9B} \]

3. Shear
(a) Slabs on beams

\[ V_t = \frac{\sqrt{f'_c}}{6} b_d \]

(b) Flat plates: The smaller of

\[ V_r = (2 + 4) \frac{\sqrt{f'_c}}{12} b_d \]

\[ V_r = \left( \frac{\alpha d}{b_0} + 2 \right) \frac{\sqrt{f'_c}}{12} b_d \]

\[ V_r = 4 \frac{\sqrt{f'_c}}{12} b_d \]

where \( b_0 \) = perimeter length of critical section in slabs and footings (at 1/2 from column face in two-way action)

\( \beta = \) ratio of longer side to shorter side of the panel

\( \alpha = 40 \) for interior columns

\( \alpha = 30 \) for edge columns

\( \alpha = 20 \) for corner columns

4. Unbalanced moments

\[ \gamma_s = \frac{1}{1 + 2 \sqrt{b_1/b_2}} \]

\[ \gamma_s = 1 - \gamma_t \]

(a) Exterior supports: The value of \( \gamma_s \) can be increased up to 1.0 if \( \gamma_t \) at support < 0.75 \( \phi \sqrt{f'_c} \).

(b) Interior supports: The value of \( \gamma_s \) can be increased by 25% if \( V_s \leq 0.1 \phi V_c \) and \( p < 0.375p_c \).
11.7.1 Example 11.4: SI Design of Two-Way Slabs and Plates

Solve Ex. 11.1 using SI units.

Data

\[ f' = 27.6 \text{ MPa} \quad \text{MPa} = \frac{N}{m^2} \]
\[ f = 41.4 \text{ MPa} \quad \text{Pa} = \frac{N}{m^2} \]
\[ w_e = 3.40 \text{ kPa} \quad 1 \text{ kg} = 9.81 \text{ N} \]
\[ \text{unit weight of concrete} = 2400 \text{ kg/m}^3 \]

Geometry from Fig. 11.10

\[ l_{es} = 7.32 \text{ m} \]
\[ l_{es} = 3.49 \text{ m} \]
\[ w_{es} = 0.5 \text{ kPa} \]

Solution:

Geometry check for use of the direct design method (step 1)

(a) Ratio larger span/shorter span = 7.32/3.49 < 2.0; hence two-way action.

(b) More than three spans in each direction and sufficient spans in each direction are the same, with columns not offset.

(c) Assume a thickness of 250 mm. Flooring of 500 N/m².

\[ w_e = 300 \times \frac{250}{183} \text{ (m)} \times 2400 \text{ kg/m}^2 \times \frac{9.81 \text{ N}}{1 \text{ kg}} \]
\[ = 6000 \text{ N/m}^2 = 6.0 \text{ kN/m}^2 \]
\[ 2w_e = 12.0 \text{ kN/m}^2 > w_{es} = 3.4 \text{ kN/m}^2 \quad \text{O.K.} \]

From items a, b, and c, the DDDM is applicable.

Minimum slab thickness for deflection requirement (step 2)

E-W direction: \[ l_{es} = 7.32 - 0.225 - 0.225 = 6.84 \text{ (m)} \]
N-S direction: \[ l_{es} = 3.49 - 0.225 - 0.225 = 3.04 \text{ (m)} \]

Ratio of longer to shorter clear span \[ \beta = \frac{6.84}{3.04} = 2.23 \]
Minimum preliminary thickness \( h \) from Table 11.3 for a flat plate without edge beams or drop panels using \( f' = 41.4 \text{ MPa} \) steel is \( h = 1/30 \), to be increased by at least 10% when an edge beam is used.

E-W: \[ l_{es} = 6.84 \text{ m} \]
\[ h = \frac{6.84}{30} \times 1.1 = 0.251 \text{ m} \]

Try a slab thickness \( h = 250 \text{ mm} \). This thickness is larger than the absolute minimum thickness of 5 in. = 120 mm required in the code for flat plates hence O.K. Assume that \( d = h = 30 \text{ mm} \) = 250 mm. New \( w_e = 6.7 \text{ kN/m}^2 (6.7 \text{ kPa}) \). Therefore,

\[ 2w_e = 13.4 \text{ kPa} \]
\[ w_e = 3.4 \text{ kPa} < 2w_e \quad \text{O.K.} \]

Shear thickness requirement (step 3)

\[ w_{es} = 1.6 \ell + 1.2D = 1.6 \times 3.4 + 1.2 \times 1.7 = 13.5 \text{ kPa} \]

Interior columns: The controlling critical plane of maximum perimetric shear stress is at a distance \( d/2 \) from the column line; hence, the net factored perimetric shear force is

\[ V_n = (l_e - l_c - s - f(x_d + y_d))w_e \]
\[ = (7.32 \times 5.49 - 0.74 \times 0.74 \times 13.5 \times 10^3 = 535 \times 10^3 N = 535 \text{ kN} \]
\[ V_n = \frac{535}{0.75} = 713 \text{ kN} \]
From Figure 11.11, perimeter of the critical shear failure surface is

\[ h_0 = 2l_0 + d_1 + d_2 + d = 2l_0 + d_1 + 2d_2 \]
\[ = 2(0.51 + 0.51 + 0.45) = 2.94 \times 10^3 \text{ mm}^2 \]

Perimeter shear force is

\[ A_1 = b_0 = 2.94 \times 0.21 = 0.66 \text{ m}^2 = 0.66 \times 10^6 \text{ mm}^2 \]

Since moments are not known at this stage, a preliminary check for shear can be made

\[ b_0 = \text{ratio of longer to shorter side of columns} = 510 \text{ mm} / 510 \text{ mm} = 1.0 \]

Available nominal shear \( V_s \) is the least of

\[ V_s = \left( \frac{2 + 4}{b_0} \right) \sqrt{\frac{E}{12}} b_0 d = \left( \frac{2 + 4}{2} \right) \sqrt{\frac{210,000}{12}} \times 0.66 \times 10^6 = 1.70 \times 10^9 \text{ N} \]

\[ V_s = \left( \frac{a_d}{b_0} + 2 \right) \sqrt{\frac{E}{12}} b_0 d = \left( \frac{0.6 \times 0.21}{2.24} + 2 \right) \sqrt{\frac{210,000}{12}} \times 0.66 \times 10^6 = 1.50 \times 10^9 \text{ N} \]

\[ V_s = \frac{4 \sqrt{E}}{12} b_0 d = 1.16 \times 10^9 \text{ N} \]

Controlling \( V_s = 1.16 \times 10^9 \text{ N} > \text{required} \ V_s = 0.713 \times 10^9 \text{ N} \); hence the floor thickness is adequate at this design stage.

**Exterior column**: Include weight of exterior wall, assuming its service weight to be 4 kN/m.

Net factored perimeter shear force is

\[ V_s = 5.49 \times \left( \frac{7.32}{2} + \frac{0.457}{2} \right) - \left( \frac{0.457 + 0.3}{2} \right) \left( 0.058 + 0.23 \right) \times 10^9 \]
\[ + (5.49 - 0.598) \times 4 \times 0.43 \times 10^9 \leq 310 \times 10^9 \text{ N} \geq 310 \text{ kN} \]

\[ V_s = \frac{V_s}{4} = 310 \times 0.75 = 232.5 \text{ kN} \]

Consider the line of action of \( V_s \) to be at the column face \( L_M \) in Figure 11.12 for shear moment transfer to the centroidal plane c-c. This approximation is adequate since \( V_s \) acts perpendicularly around the column face and not along face \( L_M \) only. From Figure 11.12,

\[ A_1 = a_0 = 0.21 \times c_1 + d = 0.21 \times 0.457 + 0.508 + 2 \times 0.23 \]
\[ = 0.430 \text{ m}^2 = 0.430 \times 10^6 \text{ mm}^2 \]

\[ \frac{A_1}{b_0} = \frac{0.430}{510} = 0.84 \]

Available nominal shear \( V_s \) is the least of

\[ V_s = \left( \frac{2 + 4}{b_0} \right) \sqrt{\frac{E}{12}} b_0 d = 1.06 \times 10^9 \text{ N} \]

\[ V_s = \left( \frac{a_d}{b_0} + 2 \right) \sqrt{\frac{E}{12}} b_0 d = 1.06 \times 10^9 \text{ N} \]

\[ V_s = \frac{4 \sqrt{E}}{12} b_0 d = 0.75 \times 10^9 \text{ N} > \text{required} \ V_s = 0.413 \times 10^9 \text{ N} \quad \text{O.K.} \]

Available \( V_s = \frac{V_s}{A_1} = 0.430 \times 10^9 \text{ N} = 1.74 \text{ N/mm}^2 \)
Chapter 11  Design of Two-Way Slabs and Plates

Statistical moment computation (steps 3 & 5)

E-W:  
\( c_1 = 6.83 \text{ m} \)
\( c_2 = 4.92 \text{ m} \)
\( 0.65c_1 = 0.65 \times 7.32 = 4.76 \text{ m} \)  Use \( c_1 = 6.83 \text{ m} \)
\( 0.65c_2 = 0.65 \times 5.49 = 3.57 \text{ m} \)  Use \( c_2 = 4.98 \text{ m} \)

(a) E-W direction

\[ M_{e} = \frac{w_{c}c_1^2}{8} = \frac{13.49 \times 10^4 \times 7.32^2 (0.83)^2}{8} = 432 \text{ kN-m} \]

For end panel of a flat plate without end beams, the moment distribution factors as in Table 11.1 are:

- \( M_e \) at first interior support = 0.7\( M_f \)
- \( M_e \) at midpoint of panel = 0.52\( M_f \)
- \( M_e \) at exterior face = 0.26\( M_f \)

- Negative design moment \( -M_e = 0.70 \times 432 = 302 \text{ kN-m} \)
- Positive design moment \( +M_e = 0.52 \times 506 = 159 \text{ kN-m} \)
- Negative moment at exterior \( -M_e = 0.26 \times 432 = 112 \text{ kN-m} \)

(b) N-S directions

\[ M_{e} = \frac{w_{c}c_2^2}{8} = \frac{13.49 \times 10^4 \times 5.49^2 (0.83)^2}{8} = 306 \text{ kN-m} \]

- Negative design moment \( -M_e = 0.70 \times 306 = 214 \text{ kN-m} \)
- Positive design moment \( +M_e = 0.52 \times 306 = 159 \text{ kN-m} \)
- Negative moment at exterior \( -M_e = 0.26 \times 306 = 81 \text{ kN-m} \)

Moment distribution in the column and subfloor strips (steps 6 and 7)

Check the shear-moment transfer capacity at the exterior columns supports:

- \( M_e \) at interior column 2-B = 302 kN-m
- \( M_e \) at exterior column 2-A = 112 kN-m

- \( V_e = 3.10 \text{ kN at the face of the column} \)

Factored shear force at the edge column adjusted for the interior moment is

\[ V_e = 3.10 \times \left( \frac{302 - 112}{7.32 - 0.483} \right) = 282 \text{ kN} \]

\[ V_e = \frac{282}{0.75} = 376 \text{ kN} \]

Assuming that the design \( M_{e} \) has the same value as the factored \( M_{e} \),

\[ A, \text{ from before = } 0.431 \text{ m}^2 \]

From: Figures 11.7c and 11.12, taking the moment of area of the critical plane about axis \( AB \),

\[ d(2c_1 + c_2 + 2d) \bar{x} = d \left( c_1 + \frac{c_2}{2} \right) \]

where \( d \) is the distance to the centroid of the critical section, or

\( (2 \times 0.457 + 0.506 + 0.457) \bar{x} = \left( 0.457 + \frac{0.23}{2} \right) \frac{1}{2} \)

\[ \bar{x} = 0.174 \text{ m} \]
\[ g = 0.174 - 0.232 = 0.049 \text{ m} = 59 \text{ mm}, \] where \( g \) is the distance from the column face to the centroidal axis of the section.

The total external factored moment is
\[ M_e = 112 + 282 \times 0.059 = 129 \text{ kN-m} \]

Total required minimum unbalanced moment strength is
\[ M_u = \frac{M_e}{\phi} = \frac{129}{0.9} = 143 \text{ kN-m} \]

The fraction of nominal moment strength \( M_t \) to be transferred by shear is
\[ v_s = \frac{1}{3.2 \sqrt{6/15}} = 0.37 \]

where \( b_1 = c - d/2 = 0.457 + 0.23/2 = 0.572 \text{ m} \)
\[ b_2 = c + d = 0.508 + 0.23 = 0.738 \text{ m} \]

Moment of inertia of sides parallel to the moment direction about \( N-S \) axis is
\[ I_1 = \frac{b_1^2}{12} A + \frac{b_2}{12}^2 A \]

\[ = \frac{0.23 \times (0.572)^2}{12} \times 12.5 = 0.23 \times 0.572 \times \left( \frac{0.572}{2} - 0.174 \right) + \frac{0.572(0.23)^2}{12} \]
\[ = 356.792 + 169.009 + 57.966/2 = 1.163,450 \text{ cm}^4 \]

Moment of inertia of sides perpendicular to the moment direction about \( N-S \) axis is
\[ I_2 = A(c)^2 \]

\[ = (108.8 + 2322)(17.4)^2 = 513,900 \text{ cm}^4 \]

Therefore, the torsional moment of inertia is
\[ I_t = 1,163,450 + 513,900 = 1,677,350 \text{ cm}^4 \]

Shearing stress due to perimeter shear effect on \( M_e \) is
\[ v_s = \frac{V_s}{\phi A f_c} \]

\[ = \frac{282 \times 0.35 \times 0.435}{1677.350 \times 10^{-3}} = \frac{0.37 \times 0.174}{1.677.350 \times 10^{-3}} = 866 \text{ kPa} + 550 \text{ kPa} \]

\[ = 1.42 \text{ MPa} \]

From before, maximum allowable \( v_s = 1.74 \text{ MPa} > 1.42 \text{ MPa} \)

\[ v_s < v_s \text{ O.K.} \]

Therefore, accept plate thickness.

Design of reinforcement in the slab as it is transferred to the column by flexure

From Eq. 11.6d,
\[ v_f = 1 - v_s = 1 - 0.37 = 0.63 \]
\[ M_{cf} = \gamma_s M_c = 0.63 \times 143 = 90.1 \text{ kN-m} \]

This moment has to be transferred within 1.5 \( f \) on each side of the column as in Figure 11.3d.

\[ f = \frac{0.508}{2 \times 0.235} = 1.08 \text{ in} \]

\[ M_{cf} = A_f \left( d - \frac{d}{2} \right) \text{ assume that } \left( d - \frac{d}{2} \right) = 0.9d \]
or

\[ 90.1 \times 10^3 = A_t \times 414 \times 0.9 \times 230 \]
\[ A_t = 1650 \text{ mm}^2 \text{ over strip width} = 1300 \text{ mm} \]

Verify \( A_t \)

\[ a = \frac{1650 \times 414}{0.03 \times 27.6 \times 1300} = 14.3 \text{ mm} \]

Therefore,

\[ 90.1 \times 10^3 = A_t \times 414 \left( \frac{230 - 14.3}{2} \right) \]
\[ A_t = 1000 \text{ mm}^2 \]

Use five No. 15 M bars (1000 mm) at 100 mm c.c. to be used in the 100 mm column width at the top and center into the column as required for bond length development.

Proportioning of the plate reinforcement (steps 8 and 9)

(a) E-W direction (long span)

1. Summary of moments in column strip
   - Interior column negative \( M_{n} = 0.75 \times 300 - 252 \text{ kN-m} \)
   - Midspan negative \( M_{n} = 0.8 \times 225 = 150 \text{ kN-m} \)
   - Exterior column negative \( M_{n} = 1 \times 112 = 124 \text{ kN-m} \)

2. Summary of moment in middle strip
   - Interior column negative \( M_{n} = 300 - 0.75 \times 300 = 54 \text{ kN-m} \)
   - Midspan positive \( M_{p} = 225 - 0.6 \times 225 = 90 \text{ kN-m} \)
   - Exterior column negative \( M_{n} = 0 \)

3. Design of reinforcement for column strip
   - \( M_{n} = 252 \text{ kN-m} \) acts on a strip width of \( 2(0.25 \times 5.49) = 2.745 \text{ m} \)
   - Unit \(-M_{n}\) per meter-wide strip = \( \frac{252}{2.745} = 91.8 \text{ kN-m} \)
   - Unit \(+M_{p}\) per meter-wide strip = \( \frac{150}{2.745} = 54.6 \text{ kN-m} \)

Minimum temperature steel reinforcement \( A_{st} \) for two-way plates

Using \( f_s = 414 \text{ MPa steel}, A_t = 0.0018 \text{ in}^2 = 0.0028 \times 1000 \times 260 = 668 \text{ mm}^2 \) per 1 m-strip width.

Try No. 10 M bars (\( A_t = 100 \text{ mm}^2 \))

\[ s = \frac{100 \text{ mm}^3}{468 \text{ mm}^2/1000 \text{ mm}} = 215 \text{ mm c.c} \]

**Negative steel**

\[ 91.8 \times 10^3 = A_t \times 414 \times \left( 230 - \frac{a}{2} \right) \]

Assuming \( (d = a/2) = 0.6d \) for the first trial,

\[ A_t = \frac{91.8 \times 10^3 \text{ N-mm}}{414 \times 0.9 \times 230} = 1070 \text{ mm}^2 \]

\[ a = \frac{A_t / 0.6d}{0.85 \times 27.6 \times 1000} = 18.8 \text{ mm} \]
For the second trial and adjustment cycle,

\[ 91.8 \times 10^2 = A_e \times 414 \left( 330 - \frac{15.8}{2} \right) \]

Negative \( A_e = 1010 \text{ mm}^2/\text{m-width strip} \)

Try No. 15 M bars from Table B.2b in the appendix (area per bar = 200 mm\(^2\)).

\[ \text{Spacing} \, s = \frac{200}{1010/1000} = 200 \text{ mm c-c for negative moment} \]

Positive steel

Spacing \( s \) as required by moment = 200 \times 91.8/54.6 = 330 mm c-c. The maximum allowable spacing = 26 = 2 x 260 = 520 mm. Try No. 10 M bars for positive moment (\( A_e = 100 \text{ mm}^2 \)) from Table B.2b in the appendix.

\[ A_e = \frac{54.6}{91.8} \times 1010 = 600 \text{ mm}^2/\text{m strip} \]

\[ s = \frac{100}{400/1000} = 170 \text{ mm c-c} \]

4. Design of reinforcement for middle strip

Unit - \( M_e = 840/0.9 = 93.3 \text{ kN-m acting on a strip width of 2.745 m} \)

Unit - \( M_e \) per meter-width strip = 93.3/2.745 = 34.0 kN-m = 34 \times 10^3 \text{ N-mm} \)

\[ 34 \times 10^3 = A_e \times 414 \times 0.9 \times 230 \]

\[ A_e = 400 \text{ mm}^2/\text{m strip} \]

\[ a = \frac{400 \times 414}{0.85 \times 27.8 \times 1000} = 5.6 \text{ mm} \]

Second cycle using \( a = 5.6 \text{ mm} \)

\[ 34 \times 10^3 = A_e \times 414 \left( 230 - \frac{5.6^2}{2} \right) \]

\[ A_e = 363 \text{ mm}^2/\text{m strip} \]

Photo 11.4  Rectangular concrete slab at rupture. (Tests by Nowy et al.)
From Table B.2b, try No. 10 M bars ($A_e = 100 \, \text{mm}^2$).

\[
\text{unit} = \frac{M_e}{\phi \cdot 2.740} = \frac{90 \, \text{kN} \cdot \text{m}}{0.9 \times 2.740} = 36.4 \, \text{kN} \cdot \text{m} = 36.4 \times 10^6 \, \text{N} \cdot \text{mm/meter strip}
\]

\[
A_e = \frac{36.4 \times 10^6}{41 \times 0.9 \times 330} = 425 \, \text{mm}^2
\]

Minimum $A_e = 0.0018 \times 100 \times 100 = 18 \, \text{mm}^2$, controls.

For positive reinforcement, the spacing using No. 10 M bars is

\[
s = \frac{\text{area of bar}}{\text{required unit area}} = \frac{100}{453/1000} = 213 \, \text{mm c-c}
\]

Use No. 12 M bars at 200 mm c-c for the negative reinforcement and No. 10 M bars at 200 mm c-c for positive reinforcement.

(b) N-S direction (short spans): The same procedure is followed as for the E-W direction for both the negative and positive moments. The effective width $d_e = 260 - 30 - 10 = 220 \, \text{mm}$ is to be used for the N-S direction. A table for moments and bar sizes is then constructed as in Table 11.6 and a plan for schematic distribution of reinforcement in $\Sigma 1$ units is provided similar to Figure 11.14.

11.8 DIRECT METHOD OF DEFLECTION EVALUATION

11.8.1 The Equivalent Frame Approach

As in the equivalent frame method discussed in detail in the preceding sections, the structure is divided into continuous frames centered on the column lines in each of the two perpendicular directions. Each frame would be composed of a row of columns and a broad band of slab together with column line beams, if any, between panel center lines.

By the requirement of statics, the applied load must be accounted for in each of the two perpendicular (orthogonal) directions. In order to account for the torsional deformations of the support beams, an equivalent column is used whose flexibility is the sum of the flexibilities of the actual column and the torsional flexibility of the transverse beam or slab strips (stiffness is the inverse of flexibility). In other words,

\[
\frac{1}{K_{\Sigma}} = \frac{1}{K_c} + \frac{1}{K_t}
\]

where $K_c$ = flexural stiffness of the equivalent column; bending moment per unit rotation

$\Sigma K_t$ = sum of flexural stiffnesses of upper and lower columns; bending moment per unit rotation

$K_t$ = torsional stiffness of the transverse beam or slab strip; torsional moment per unit rotation

The value of $K_c$ would thus have to be known in order to calculate the deflection by this procedure.

The slab beams strips are considered to be supported not on the columns but on transverse slab-beam strips on the column center lines. Figure 11.24a illustrates this point. Deformation of a typical panel is considered in one direction at a time. Thereafter, the contribution in each of the two directions, $x$ and $y$, is added to obtain the total deflection at any point in the slab or plate.

First, the deflection due to bending in the $x$ direction is computed (Figure 11.24b). Then the deflection due to bending in the $y$ direction is found. The midpanel deflection
11.4 Direct Method of Deflection Evaluation

Figure 11.24 Equivalent frame method for deflection analysis: (a) plate panel transferred into equivalent frame; (b) profile of deflected shape at center line; (c) deflected shape of panel.

The deflection of each panel can be considered as the sum of three components:

1. Basic midspan deflection of the panel, assumed fixed at both ends, given by
   \[
   \delta = \frac{Wl^4}{384EI}\frac{1}{l^4}
   \]
This has to be proportioned to separate deflection \( \delta_c \) of the column strip and \( \delta_s \) of the middle strip, such that:

\[
\begin{align*}
\delta_c &= \delta \frac{M_{col,y}E/I_y}{M_{col,x}E/I_x} \\
\delta_s &= \delta \frac{M_{mid,xy}E/I_{xy}}{M_{mid,x}E/I_x}
\end{align*}
\]

where \( I_y \) is the moment of inertia of the total frame, \( I_x \) the moment of inertia of the column strip, and \( I_{xy} \) the moment of inertia of the middle slab strip.

2. Center deflection, \( \delta_{xy} = \frac{1}{6} L \) due to rotation at the left end while the right end is considered fixed, where \( \delta_c = \frac{M_{col,y}}{E/I_y} \) and \( \delta_s = \frac{M_{mid,xy}}{E/I_{xy}} \) is the flexural stiffness of equivalent column (moment per unit rotation).

3. Center deflection, \( \delta_{ww} = \frac{1}{6} L \) due to rotation at the right end while the left end is considered fixed, where \( \delta_c = \frac{M_{col,x}}{E/I_x} \). Hence

\[
\begin{align*}
\delta_{ww} = \delta_c + \delta_s + \delta_{y} + \delta_{xy} \\
\delta_{ww} = \delta_c + \delta_s + \delta_{xy}
\end{align*}
\]

(Use in Eqs. 11.19a and 11.19b the values of \( \delta_c, \delta_{ww}, \) and \( \delta_{xy} \) that correspond to the applicable span directions.) From Figure 11.24b and c, the total deflection is

\[
\delta = \delta_c + \delta_s + \delta_x
\]

11.8.2 Example 11.5: Central Deflection Calculations of a Slab Panel on Beams

A 7-in. (177.8-mm) slab of a five-panel by five-panel floor system spanning 26 ft in the E-W direction (7.92 m) and 20 ft in the N-S direction (6.10 m) is shown in Figure 11.25a. The panel is monolithically supported by beams 15 in. × 27 in. in the E-W direction (381 mm ×
Figure 11.28: Example on equivalent frame deflection evaluation.

686 mm) and 15 in. x 74 in. in the N-S direction (381 mm x 610 mm). The floor is subjected to a time-dependent deflection due to an equivalent uniform working load intensity \( w = 400 \text{ psf} \) (19 kPa). Material properties of the floor are:

- \( f' = 4000 \text{ psi} \) (27.6 MPa)
- \( f_e = 6000 \text{ psi} \) (414 MPa)
- \( E_e = 3.6 \times 10^9 \text{ psi} \) (24.6 \times 10^6 kPa)

Assume the following:
1. Not moment $M_y$ from adjacent spans (6b):

- **E-W**
  - Support 1: $20 \times 10^3$
  - Support 2: $5 \times 10^3$

- **N-S**
  - Support 1: $40 \times 10^3$
  - Support 4: $20 \times 10^3$

2. Equivalent column stiffness $K_{eq} = 400 E/L$, lb-in. per radian in both directions. Find the maximum central deflection of the panel due to the long-term loading and determine if its magnitude is acceptable if the floor supports sensitive equipment that can be damaged by large deflections.

3. Cracked moment of inertia:

   - **E-W**: $I_y = 45,500$ in.²
   - **N-S**: $I_y = 32,500$ in.²

**Solution**: Calculate the gross moments of inertia ($I_g$) of the sections in Figure 11.25: the total equivalent frame $E_y$ in part (b), the column strip beam $I_y$ in part (c), and the middle strip slab $I_y$ in part (d). These values are

<table>
<thead>
<tr>
<th></th>
<th>$I_y$</th>
<th>$I_g$</th>
<th>$I_y$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>E-W</strong></td>
<td>63,600</td>
<td>53,700</td>
<td>3430</td>
</tr>
<tr>
<td><strong>N-S</strong></td>
<td>47,000</td>
<td>40,000</td>
<td>4288</td>
</tr>
</tbody>
</table>

Next, calculate factors $a_y$, $a', \text{ and } a''$, as in Ex. 1.2. In both cases they are greater than 1.0. Hence, the factor moments coefficients (percent) obtained from the tables in Section 11.4.2 are as follows:

<table>
<thead>
<tr>
<th>Column Strip</th>
<th>Middle Strip</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+ and -)</td>
<td>(+ and -)</td>
</tr>
<tr>
<td><strong>E-W</strong></td>
<td>81.0</td>
</tr>
<tr>
<td><strong>N-S</strong></td>
<td>67.5</td>
</tr>
</tbody>
</table>

**E-W direction deflections (span = 25 ft)**

long-term $w_{ls} = 450$ psf

\[ h_2 = \frac{45 \times 20(25)^2}{384 \times 3.6 \times 10^6 \times 63,600} = 0.0691 \text{ in.} \]

\[ h_0 = 0.6691 \times 0.81 \times 63,600 = 0.0663 \text{ in.} \]

\[ h_1 = 6.0691 \times 0.19 \times 63,600 = 0.243 \text{ in.} \]

Rotation at end 1 is

\[ \theta_1 = \frac{M_1}{K_{eq}} = \frac{20 \times 10^3 \times 12}{400 \times 3.6 \times 10^6} = 1.67 \times 10^{-4} \text{ rad} \]

and rotation at end 2 is

\[ \theta_2 = \frac{M_1}{K_{eq}} = \frac{5 \times 10^3 \times 12}{400 \times 3.6 \times 10^6} = 0.42 \times 10^{-4} \text{ rad} \]

where $\theta$ is the rotation at one end if the other end is fixed.
8* = deflection adjustment due to rotation at supports 1 and 2 = \( \frac{8}{8} \)

\[ 8* = \left( 1.67 \times 0.42 \right) \times 10^{-4} \times 300 = 6.0078 \text{ in.} \]

Therefore,

Net \( d_1 = 0.0665 + 0.0078 = 0.0741 \) try 0.07 in.
Net \( d_0 = 0.243 + 0.0078 = 0.2508 \) try 0.25 in.

**N-S direction deflections (tensile > 20 ft)**

\[ \delta_6 = \frac{450 \times 25(300)}{384 \times 3.6 \times 10} = 0.0479 \text{ in.} \]
\[ \delta_5 = 0.0479 \times 0.675 = 0.032 \text{ in.} \]
\[ \delta_4 = 0.0479 \times 0.325 = 0.0158 \text{ in.} \]

rotation \( \theta = \frac{M_4}{K_w} = \frac{40 \times 10^3 \times 12}{400 \times 3.6 \times 10^6} = 3.3 \times 10^{-4} \text{ rad} \)

rotation \( \theta = \frac{M_4}{K_w} = \frac{20 \times 10^3 \times 12}{400 \times 3.5 \times 10^6} = 1.67 \times 10^{-4} \text{ rad} \)

\[ 8* = \frac{60}{8} \left( 3.3 \times 10^{-4} \times 240 \right) = 0.0149 \text{ in.} \]

Therefore,

Net \( d_0 = 0.03^{\circ} + 0.0149 = 0.0539 \) try 0.05 in.
Net \( d_0 = 0.171 + 0.0149 = 0.1859 \) try 0.19 in.
Total central deflection \( \delta = \delta_0 + \delta_5 + \delta_6 \)
\( \delta_{tot} = \delta_0 + \delta_5 + \delta_6 + 0.25 = 0.35 \text{ in.} \)
\( \delta_{tot} = \delta_0 + \delta_5 + 0.19 = 0.26 \text{ in.} \)

Hence the average deflection at the center of the interior panel is

\[ \frac{1}{2} \left( \delta_{tot} + \delta_{tot} \right) = 0.28 \text{ in. (7.1 mm)} \]

**Adjustment for cracked section:** Use Bramson's effective moment of inertia equation,

\[ I_e = \left( \frac{M_e}{M} \right) I + \left[ 1 - \left( \frac{M_e}{M} \right) \right] I_f \]

as discussed in Chapter 8. Calculation of ratio \( M_e/M \),

\[ M_e = \frac{I_e}{I} \]

where \( f_r \) = modulus of rupture of concrete
\( I_e = \) distance of center of gravity of section from outer tension fibers

E-W (240-in. flange width): \( v = 21.64 \text{ in.} \)
N-S (300-in. flange width): \( v = 19.26 \text{ in.} \)

\[ f_r = 7.5 \sqrt{v} = 7.5 \sqrt{400} = 474 \text{ psi} \]

Hence,
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\[
M_{x} \quad (E-W) = \frac{474 \times 63,600}{21.34} \times \frac{1}{12} = 1,17 \times 10^{6} \text{ ft-lb}
\]

\[
M_{x} \quad (N-S) = \frac{474 \times 47,000}{19.29} \times \frac{1}{12} = 0.97 \times 10^{6} \text{ ft-lb}
\]

interior panel \( M_{x} = \frac{wL^{2}}{12} = \frac{20 \times 450(25)^{2}}{12} = 3.52 \times 10^{6} \text{ ft-lb} \)

\[
= \frac{25 \times 450(10)^{2}}{12} = 2.81 \times 10^{6} \text{ ft-lb}
\]

Note that the moment factors \( K_{a} \) used to be on the safe side, although the actual moment coefficients for two-way action would have been smaller.

**E-W effective moment of inertia \( I_{e} \)**

\[
\frac{M_{x}}{M_{e}} = \frac{1.17 \times 10^{6}}{3.52 \times 10^{6}} = 0.332
\]

\[
\frac{M_{x}}{M_{e}} = 0.037
\]

\[
L_{e} = 0.037 \times 63,600 + (1 - 0.037)45,500 = 46.17 \text{ in.}^{4}
\]

**N-S effective moment of inertia \( I_{e} \)**

\[
\frac{M_{x}}{M_{e}} = \frac{0.97 \times 10^{6}}{2.81 \times 10^{6}} = 0.345
\]

\[
\frac{M_{x}}{M_{e}} = 0.061
\]

\[
L_{e} = 0.061 \times 47,000 + (1 - 0.061)32,500 = 33.095 \text{ in.}^{4}
\]

average \( I_{e} = \frac{1}{2} \left( \frac{63,600}{45,500} \right) = 1.49 \text{ in.}^{4} \)

adjusted neutral deflection for cracked section effects

\[
\frac{L}{L_{e}} = 1.40 \times 0.25 = 0.39 \text{ in. (9.9 mm)}
\]

Hence the long-term central deflection is acceptable.

11.9 CRACKING BEHAVIOR AND CRACK CONTROL
IN TWO-WAY-ACTION SLABS AND PLATES

11.9.1 Flexural Cracking Mechanism and Fracture Hypothesis

Flexural cracking behavior in concrete structural slabs under two-way action is significantly different from that in one-way members. Crack-control equations for beams underestimate the crack widths developed in two-way slabs and plates and do not tell the
designer how to space the reinforcement. Cracking in two-way slabs and plates is controlled primarily by the steel stress level and the spacing of the reinforcement in the two perpendicular directions. In addition, the clear concrete cover in two-way slabs and plates is nearly constant (1 in. (19 mm) for interior exposure), whereas it is a major variable in the crack-control equations for beams. The results from extensive tests on slabs and plates by Nawy et al. demonstrate this difference in behavior in a fracture hypothesis on crack development and propagation in two-way plate action. As seen in Figure 11.2b, stress concentration develops initially at the points of intersection of the reinforcement in the reinforcing bars and at the welded joints of the wire mesh, that is, at grid nodal points, thereby dynamically generating fracture lines along the paths of least resistance: \( A_1 A_2, A_1 A_3, A_2 B_2, \) and \( B_1 B_2. \) The resulting fracture pattern is a total repetitive cracking grid, provided that the spacing of the nodal points \( A_1, A_2, A_3, \) and \( B_1 \) is close enough to generate this preferred initial fracture mechanism of orthogonal cracks narrow in width as a fracture mechanism.

If the spacing of the reinforcing grid intersections is too large, the magnitude of the stress concentration and the energy absorbed per unit grid area is too low to generate cracks along the reinforcing wires or bars. As a result, the principal cracks follow diagonal yield-line cracking in the plain concrete field away from the reinforcing bars early in the loading history. These cracks are wide and few.

This hypothesis also leads to the conclusion that surface deformations of the individual reinforcing elements have little effect in arresting the generation of the cracks or controlling their type or width in a two-way-action slab or plate. In a similar manner, we may conclude that the scale effect on two-way-action cracking behavior is insignificant, since the cracking grid would be a reflection of the reinforcement grid if the preferred orthogonal narrow cracking widths develop. Therefore, to control cracking in two-way-action floors, the major parameter to be considered is the reinforcement spacing in two perpendicular directions. Concrete cover has only a minor effect, since it is usually a small, constant value of 0.75 in. (19 mm).

**Figure 11.24** Grid unit in two-way-action reinforcement.
For a constant area of steel determined for bending in one direction, that is, for energy absorption per unit slab area, the smaller the spacing of the transverse bars or wires, the smaller should be the diameter of the longitudinal bars. The reason is that less energy has to be absorbed by the individual longitudinal bars. If we consider that the magnitude of fracture is determined by the energy imposed per specific volume of reinforcement acting on a finite element of the slab, a proper choice of the reinforcement grid size and bar size can control cracking into preferred orthogonal grids.

It must be emphasized that this hypothesis is important for serviceability and reasonable overload conditions. In relating orthogonal cracks to yield-line cracks, the failure of a slab ultimately follows the generally accepted rigid-plastic yield-line criteria.

11.6.2 Crack Control Equation

The basic equation (Section 8.11) for relating crack width to strain in the reinforcement is

$$ w = \alpha e_s ^* $$

(11.21)

The effect of the tensile strain in the concrete between the cracks is neglected as insignificant. $e_s$ is the crack spacing, $s$, the unit strain in the reinforcement, and $\alpha$ and $\beta$ are constants. As a result of this fracture hypothesis, the mathematical model in Eq. 11.21, and the statistical analysis of the data of 90 slabs tested to failure, the following crack-control equation emerged:

$$ w = K \cdot f_r \cdot \sqrt{\frac{d_{01}b_t}{Q_i}} $$

(11.22)

where the quantity under the radical, $G_i = d_{01}b_t/Q_i$, is termed the grid index and can be transformed into

$$ G_i = \frac{d_{01}b_t}{d_o} $$

(11.23)

where $K$ = fracture coefficient, having a value of $K = 2.8 \times 10^3$ for uniformly loaded, restrained, two-way-action square slabs and plates. For concentrated loads or reactions or when the ratio of short to long span is less than 0.75, but larger than 0.5, a value of $K = 2.1 \times 10^3$ is applicable. For a span aspect ratio of 0.5, $K = 1.6 \times 10^3$. Units of coefficient $K$ are in in. $/lb$.

$\beta$ = ratio of the distance from the neutral axis to the tensile face of the slab to the distance from the neutral axis to the centroid of the reinforcement grid (to simplify the calculations use $\beta = 1.25$, although it varies between 1.20 and 1.35).

$f_r$ = actual average service load stress level, or 40% of the design yield strength,

$d_{01}$ = diameter of the reinforcement in direction 1 closest to the concrete outer fibers (in.)

$d_c$ = concrete cover to centroid of reinforcement (in.)

$s_1$ = spacing of the reinforcement in direction 1

$s_2$ = spacing of the reinforcement in perpendicular direction 2

$L_1$ = direction of the reinforcement closest to the outer concrete fibers; this is the direction for which crack control check is to be made.

$Q_i = \frac{\text{area of steel} \times \text{per foot width}}{12(d_{01} + 2s_1)}$

where $c_r$ is clear concrete cover measured from the tensile face of the concrete to the nearest edge of the reinforcing bar in direction 1.

$w$ = crack width at face of concrete caused by reversed load (in.)
11.9 Cracking Behavior and Crack Control in Two-way Slabs and Plates

Subscripts 1 and 2 pertain to the directions of reinforcement. Detailed values of the fracture coefficients for various boundary conditions are given in Table 11.10.

A graphical solution of Eq. 11.21 is given in Figure 11.27 for

\[ f_a = 60,000 \text{ psi (414 MPa)} \]
\[ f_r = 40\% \text{, } f_r = 20,000 \text{ psi (136 MPa)} \]

for rapid determination of the reinforcement size and spacing needed for crack control. Equation 11.21. in SI units is

\[ u_{\text{max}} (\text{mm}) = 0.145 \left( \frac{K}{G} \right) \sqrt{G} \]

where \( f_r = \text{MPa}, G = e_{\text{cr}}/d_e \times 8/m, \text{ and } u_{\text{cr}}, d_e, \text{ and } d_e \text{ are in millimeters.} \)

The grid index, \( G \), specifies the size and spacing of the bars in the two perpendicular directions of any concrete floor system, and \( u_{\text{max}} \) is the maximum tolerable crack width.

The crack control equations and guidelines presented are important not only for the criteria of corrosion in the reinforcement but also for deflection control. The reduction of the stiffness \( E_I \) of the two-way slab or plate due to orthogonal cracking when the limits of tolerable crack width in Table 8.5 are exceeded can lead to excessive deflection in both the short and long term. Deflection values several times those anticipated in the design, including deflection due to construction loading, can be reasonably controlled through proper control of the flexural crack width in the slab or plate. Proper selection of the reinforcement spacing \( s_s \) and \( s_d \) in both perpendicular directions, as discussed in this section, and not exceeding 12 in. centers in cover can maintain good seismic performance of a slab system under normal and reasonable overload conditions. The majority of codes generally follow the principles presented on the selection of reinforcement size and spacing in slabs and plates, and good engineering practice mandates such caution.

### Table 11.10 Fracture Coefficients for Slabs and Plates

<table>
<thead>
<tr>
<th>Loading Type</th>
<th>Slab Shape</th>
<th>Boundary Condition</th>
<th>Span Ratio, S/L</th>
<th>Fracture Coefficient, 10⁻² K</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Square</td>
<td>4 edges r</td>
<td>1.0</td>
<td>2.1</td>
</tr>
<tr>
<td>A</td>
<td>Square</td>
<td>4 edges r</td>
<td>1.0</td>
<td>2.1</td>
</tr>
<tr>
<td>B</td>
<td>Rectangular</td>
<td>4 edges r</td>
<td>0.5</td>
<td>1.6</td>
</tr>
<tr>
<td>B</td>
<td>Rectangular</td>
<td>3 edges r</td>
<td>0.7</td>
<td>2.2</td>
</tr>
<tr>
<td>B</td>
<td>Rectangular</td>
<td>1 edge h</td>
<td>0.7</td>
<td>2.3</td>
</tr>
<tr>
<td>B</td>
<td>Rectangular</td>
<td>2 edges h</td>
<td>0.7</td>
<td>2.7</td>
</tr>
<tr>
<td>B</td>
<td>Square</td>
<td>4 edges r</td>
<td>1.0</td>
<td>2.8</td>
</tr>
<tr>
<td>B</td>
<td>Square</td>
<td>3 edges r</td>
<td>1.0</td>
<td>2.9</td>
</tr>
<tr>
<td>B</td>
<td>Square</td>
<td>1 edge h</td>
<td>1.0</td>
<td>4.2</td>
</tr>
</tbody>
</table>

*Loading types: A, concentrated; K, uniformly distributed.

*Boundary conditions: r, rest couple; s, simply supported; h, hinged.

*Span ratios: 1, span short; 2, span long; 3, span very long.
Figure 11.27 Crack-control reinforcement distribution in two-way-action slabs and plates for all exposure conditions: $f_e = 60,000$ psi, $f_y = 24,000$ psi ($= 0.40 f_e$).

11.9.3 Example 11.6: Crack-Control Evaluation for Serviceability in an Interior Two-way Panel

Check the bar size and spacing used in an interior panel to determine if it satisfies serviceability through crack control. As shown in Figure 11.24, the panel has a width/length ratio $l/l_o = 1.0$, and its thickness is 5 in. (125 mm). The floor is subjected to normal weather conditions. Reinforcement used for flexure is No. 4 bars at 9 in. center to center in each direction. Given:

- $B = 1.25$
- $w_{min} = 0.016$ in.
- $f_e = 60$ ksi (414 MPa)
- $K = 3.8 \times 10^{-7}$ in/psi

Solution:

$$f_e = 0.40 f_y = 0.40(24,000) = 9,600 \text{ psi} = 66.0 \text{ MPa}$$
The fracture coefficient \( K \) for this panel aspect ratio is \( K = 2.8 \times 10^{-6} \). The maximum tolerable crack width for normal interior conditions is \( w_{\text{max}} = 0.016 \text{ in.} \) (0.41 mm). (Table 8-3).

\[
\begin{align*}
  w_{\text{max}} &= K G_i \sqrt{G_i} \\
  G_i &= \frac{t_2 d_1}{d_1} \times \frac{8}{\pi} \\
  0.016 &= 2.8 \times 10^{-6} \times 1.25 \times 24 \sqrt{G_i}
\end{align*}
\]

to give

\[
G_i = 383 \text{ in.}^2 = \frac{t_2 d_1}{d_1} \times \frac{8}{\pi}.
\]

If \( t_i = t_j \) for this square panel, cover \( d_i = 0.25 + 0.25 = 1.0 \) in. to the center of the first reinforcement layer and \( d_i = 0.5 \) in. = diameter of the No. 4 bar.

\[
383 = \frac{1.0 \times 8}{0.5} \times \frac{8}{\pi} \quad \text{giving} \quad s = 8.4 \text{ in. or 8.5 in. maximum}
\]

Hence the 8 in. center-to-center spacing specified for flexure is not satisfactory. Reduce the spacing of reinforcement to No. 4 bars at 9 in. (225 mm) center to center for crack control (12.7 mm diameter at 216 mm center to center).

**11.9.4 Example 11.7: Crack-control Evaluation for Serviceability in a Rectangular Panel Subjected to Severe Exposure Conditions**

Select the bar size and spacing necessary for crack control at the column intersection region of the 7-in.-thick slab shown in Figure 11.29 that is uniformly loaded. Select the bar size for two conditions:

**Condition A:** Floor is subjected to severe exposure of humidity and moist air.

**Condition B:** Floor sustains an aggressive chemical environment where the design working stress level in the reinforcement is limited to 15 ksi (103.4 MPa).

**Figure 11.29** Rectangular panel.
Given:

\[ \beta = 1.20 \]
\[ l/h = 0.8 \]
\[ f = 60 \text{ ksi (414 MPa)} \]

Solution: Condition A: Hydration and Moist Air

Tolerable \( w_{\text{max}} = 0.012 \text{ in. (0.3 mm)} \) (Table 8.4). Try No. 4 bars \( d = 0.5 \), \( d = 0.75 + 0.25 = 1.0 \text{ in.} \). Assume that \( z = a = 1 \) for the given panel. The aspect ratio, \( l/h = 0.8 \), \( K = 2.1 \times 10^{-5} \) for concentrated reactions at the column support (Table 11.10).

\[ 0.012 = 2.1 \times 10^{-5} \times 1.20 \times 0.4 \times 60 \sqrt{G} \]

to give \( G = 394 \text{ in.}^2 \). Therefore,

\[ 394 = \frac{t d}{d_{cl}} \times \frac{2}{\pi} \times \frac{z}{h} \times \frac{2}{\pi} \times \frac{8}{5} \]

\[ z = 8.8 \text{ in.} \]

Hence use No. 4 bars at 8 in. center to center each way for crack control.

Condition B: Aggressive chemical environment

Tolerable \( w_{\text{max}} = 0.007 \text{ in. (0.18 mm)} \) (Table 8.4); \( f = 15 \text{ ksi} \) to be used as a low stress level for sanitary or water-retaining structures instead of 0.4 \( f \). Try No. 5 bars \( d_{cl} = 0.625 \text{ in.} \).

\[ 0.007 = 2.1 \times 10^{-5} \times 1.20 \times 15.0 \sqrt{G} \]

to give a grid index \( G = 325 \text{ in.}^2 \).

\[ d_{cl} = 0.75 + 0.312 = 1.06 \text{ in.} \]

\[ G = 325 \times \frac{2}{\pi} \times \frac{2}{\pi} \times \frac{8}{5} \times \frac{8}{6.25} \]

\[ z = 8.9 \text{ in.} \]

Use No. 5 bars at 9-in. (229-mm) center-to-center spacing each way for crack control.

Reinforcement Summary

Condition A: No. 4 bars at 8 in. c-c (12.7-mm diameter at 216 mm c-c)

Condition B: No. 5 bars at 9 in. c-c (15.9-mm diameter at 229 mm c-c)

11.9.5 Example 11.8: SI Example on Crack Control in Two-way Slabs and Plates

Solve Ex. 11.6 using SI units.

Data

\[ f = 166 \text{ MPa} \]
\[ w_{\text{max}} \text{ (mm)} \]
\[ K = 3.0 \times 10^{-5}; \quad \beta = 1.25 \]
\[ d_{cl} \text{ (mm)} \]

Solution:

\[ w_{\text{max}} \text{ (mm)} = \frac{0.145 K f}{\sqrt{G}} \]

where

\[ G = \frac{4 t d_{cl}}{d_{cl}} \times \frac{8}{w} \]
11.10 Yield-Line Theory for Two-way Action Plates

\[
0.40 = 0.145 \times 2.8 \times 10^{-1} \times 1.23 \times 106 \sqrt{G_c} \\
\sqrt{G_c} = \frac{0.40 \times 10^6}{0.145 \times 2.8 \times 1.23 \times 106} = 495
\]

If \( s_1 = s_2 \) for this square panel,

\[
(475)^2 = \frac{e \times 25}{12.7} \times \frac{8}{\pi} \]

\[
s = \left[ \frac{(475) \times 12.7 \times \pi}{25 \times 8} \right]^{1/2} = 212 \text{ mm}
\]

Hence space the bars at 20 cm c-c even.

11.10 YIELD-LINE THEORY FOR TWO-WAY ACTION PLATES

A study of the hinge-field mechanism in a slab or plate at loads close to failure aids the engineering student in developing a feel for the two-way-action behavior of plates. Hinge fields are successions of hinge bands that are idealized by lines; hence the name yield-line theory by K. W. Johansen.

To do justice to this subject, an extensive discussion over several chapters or a whole textbook is necessary. The intention of this chapter is only to introduce the reader to the fundamentals of the yield-line theory and its application.

The yield-line theory is an upper-bound solution to the plate problem. This means that the predicted moment capacity of the slab has the highest expected value in comparison with test results. Additionally, the theory assumes a totally rigid-plastic behavior; that is, the plate stays planar at collapse, producing rigid plastic failure systems. Consequently, deflection is not accounted for, nor are the compressive membrane forces that will act in the plane of the slab or plate considered. The plates are assumed to be considerably underreinforced such that the maximum reinforcement percentage \( p \) does not exceed 1% of the section \( b h \).

Since the solutions are upper bound, the slab thickness obtained by this process is in many instances thinner than what is obtained by the other lower-bound solutions, such as the direct design method. Consequently, it is important to apply rigorously the serviceability requirements for deflection control and for crack control in conjunction with the use of the yield-line theory as given in Sections 11.8 and 11.9.

One distinct advantage in this theory is that solutions are possible for any shape of a plate, whereas most other approaches are applicable only to the rectangular shapes with rigorous computations for boundary effects. The engineer can, with ease, find the moment capacity for a triangular, trapezoidal, rectangular, circular, and any other conceivable shape provided that the failure mechanism is known or predictable. Since most failure patterns are presently identifiable, solutions can be readily obtained, as seen in Section 11.10.2.

11.10.1 Fundamental Concepts of Hinge-Field Failure

Mechanisms in Flexure

Under the action of a two-dimensional system of bending moments, yielding of a rigid-plastic plate occurs when the principle moments satisfy Johansen's square yield criterion, as shown in Figure 11.30. In this criterion, yielding is considered to have occurred when the numerically greater of the principal moments reaches the value of \( \pm M \) at the yield-line cracks. The directions of the principal curvature rates are considered to coincide with the curvatures of the principal moments. The idealized moment-curvature relationship is shown as the solid line in Figure 11.31.
Johansen's square yield criterion.

Line OA is considered almost vertical at point O and strain hardening is neglected.

If we consider the simplest case of a square slab with supports, with degree of fixity 
1 varying from $i = 0$ for simply supported to $i = 1.0$ for fully restrained on all four sides, the
failure mechanism would be as shown in Figure 11.32, when a uniformly distributed load
is applied.

Take the simply supported case (a). The yield-line moments along the yield lines
are the principal moments. Hence the twisting moments are zero in the yield lines and in
most cases the shearing forces are also zero. Consequently, only moment $M$ per unit
length of the yield line acts about the lines $AD$ and $BE$ in Figure 11.33. The total
moments can be represented by a vector in the direction of the yield line whose value is $M \times
length of the yield line$, that is, $M[\mu2 \cos \theta]$ in Figure 11.33e. The virtual work of the yield
moments of the shaded triangular segment $ABO$ is the scalar product of the two moment
vectors $M[\mu2 \cos \theta]$ on fracture lines $AO$ and $BO$ and a rotation $\theta$. In other words, the in-
ternal work

$$E_i = \sum M\theta$$

If the displacement of the shaded segment at its center of gravity $c$ is $\delta$, the external work

$$E_x = \text{force} \times \text{displacement} = \int \left[ w_c \cdot dx \cdot dy \right] \delta$$

where $w_c$ is the intensity of external load per unit area. But $E_i = E_x$; hence

$$\sum M\theta = \int \left[ w_c \right] \cdot dx \cdot dy \delta$$

Applying Eq. 11.24 to the particular case under discussion gives us

![Figure 11.31: Moment-curvature relationship.](image)
Figure 11.32  Failure mechanism of a square slab: (a) $i = 0$; (b) $i = 0.5$; (c) $i = 1.0$.

$$M_\theta = Ma \frac{\Delta}{h/2}$$

since angle $\theta$ in Figure 11.33b is small, where $\theta = \Delta/(a/2)$.

Work per one triangular segment:

$$E_s = M_\theta = 2M\Delta$$

$$F_{xy} = \frac{\pi a^2 \theta}{4} \cdot \frac{\Delta}{3}$$

Figure 11.33  Vector moments on slab segment at failure.
Chapter 11  Design of Two-Way Slabs and Plates

Photo 11.5  Testing setup of four-panel prestressed concrete floor. (Tests by Neely et al.)

Photo 11.6  Yield-line pattern at failure at column reaction and panel boundaries of a two-way composite floor. (Tests by Neely, Chakraborti, et al.)
where deflection at center of gravity of the triangle = \( 0.55 \). Therefore,

\[
(2M\Delta) = 4 \left( \frac{wL^4}{12} \Delta \right)
\]

Or

\[
\text{unit } M = \frac{wL^4}{24} \Delta
\]

(11.25)

If the square slab was fully fixed on all four sides, \( E_I = 4(4M\Delta) \) since fracture lines develop around not only the diagonals but also the four edges, as shown in Figure 11.32c. Hence

\[
\text{unit } M = \frac{wL^4}{48} \Delta
\]

(11.26)

It is to be noted that a lower-bound solution as proposed by Mansfield's failure pattern in Figure 11.32c gives a value \( M = wL^4/42.85 \). Hence, for a uniformly loaded square slab with load intensity \( w \) per unit area and degree of support \( i \) on all sides,

\[
wL^4 = M [24(1 - i)]
\]

(11.27)

The general equation for the yield-line moment capacity of a rectangular isotropic slab on beams and having dimensions \( a \times b \) as shown in Figure 11.34, with side \( a \) being the shorter dimension, is

\[
\text{unit } M \left( \text{ft} \cdot \text{lb} \right) = \frac{wL^4}{24} \Delta \left( 3 + \frac{a}{b} \right) - a \Delta b
\]

(11.28)

where

\[
a = \frac{2a}{\sqrt{1 + \epsilon_1 + \sqrt{1 + \epsilon_4}}}
\]

\[
b = \frac{2b}{\sqrt{1 + \epsilon_1 + \sqrt{1 + \epsilon_4}}}
\]

\( i = \) degree of restraint depending on stiffness ratios discussed in Section 11.2

Note that Eq. 11.28 reduces to the simplified form of Eq. 11.26 or 11.27 for the case of a square slab restrained on all four sides \( (i = 1.0) \).

---

**Figure 11.34** Rectangular slab. (Note the sequence of side numbers.)
11.10.1.1 Affine slabs. Slabs that are reinforced differently in the two perpendicular directions are called orthotropic slabs or plates. The moment in the $x$ direction equals $M_x$ and the moment in the $y$ direction equals $M_y$. The moment in the $z$ direction equals $M_z$, where $\mu$ is a measure of the degree of orthotropy or the ratio

\[
\frac{M_x}{M_y} = \frac{\langle A_x \rangle}{\langle A_y \rangle}.
\]

To simplify the analysis, the slab should be converted to an affine (isotropic) slab where the strength and reinforcement area in both the $x$ and $y$ directions are the same. Such conversion can be made as follows:

1. Divide the linear dimension by $\sqrt{\mu}$ in the $x$ direction of the positive moment for a slab to be reinforced for a moment $M$ in both directions using the same unit load intensity $w$, per unit area.
2. In the case of concentrated loads or total loads, also divide such loads by $\sqrt{\mu}$.
3. In the case of line loads, the line load has to be divided by $\sqrt{\mu}$, where $\theta$ is the angle between the line load and the $M$ direction.

If the slab is to be analyzed as an affine slab with the moment $\mu M$ in both directions, the dimension in the $x$ direction would have to be multiplied by $\sqrt{\mu}$. In either case, the result would of course have to be the same (see Eq. 11.9).

11.10.2 Failure Mechanisms and Moment Capacities of Slabs of Various Shapes Subjected to Distributed or Concentrated Loads

The preceding concise introduction to the virtual-work method of yield-line moments evaluation should facilitate good understanding of the mathematical procedures of most standard rectangular shapes subjected to uniform loading. More complicated slab shapes and other types of symmetrical and non-symmetrical loading require additional and more detailed knowledge of the subject as discussed in the introduction. Also, the assumed
failure shape and minimization energy principles can give values for particular cases that can differ slightly from one author to another depending on the mathematical assumptions made with respect to the failure shape.

The following summary of failure patterns and the respective moment capacities in terms of load, many of them due to Mansfield (Ref. 11.13) should give the reader adequate coverage in a capsule of solutions to most cases expected in today's and tomorrow’s structures.

1. Point load to corner of rectangular cantilever plate:

![Diagram of a rectangular cantilever plate with a point load at the corner.](image)

- Case (a) \( P = 2M \)
- Case (b) \( P = \frac{2}{3} \rho \) (M)

2. Square plate centrally loaded having boundaries simply supported against both downward and upward movements:

![Diagram of a square plate centrally loaded.](image)

\( P = 8M \)
3. Regular n-sided plate with simply supported edges and centrally loaded \((n > 4)\):

\[ F = \text{with } \sin \left( \frac{\pi}{n} \right) \]

4. Square plates centrally loaded, having boundaries simply supported against downward movement but free for upward movement:

\[ F = 0.5M \]

5. Circular centrally loaded plate simply supported along the edges:

\[ I = 2M \]

6. Circular plate with fully restrained edges and centrally loaded by point load \(P\):

7. Point load \(P\) applied anywhere in arbitrarily shaped plate fully restrained on all boundaries:

\[ F = 0.6M \]
8. Equilateral triangular plate with simply supported edges and centrally loaded by
point load \( P \):

\[
P = 4(MS \cos \frac{\pi}{3} + 12k - 24k) \\
P_{max} = 4(MS) \text{ for } l = a/4
\]

9. Acute-angled triangular plate on simply supported edges loaded with point load \( P \)
at the center of the inscribed circle:

\[P = MS + 4l\]

10. Oblique-angled triangular plate with simply supported edges and load \( P \) at the cen-
ter of the inscribed circle:

\[P = MS + 2l + 2 \cos \frac{\pi}{3}l, \text{ where } l \text{ is in radians}
\]

As \( k \) approaches \( \infty \), the plate degenerates into case 11

11. A long strip simply supported along edges and load with point \( P \) midway between
edges:

\[P = M(4 - 2l)\]

12. Simply supported strip with equal loads \( P \) between edges:

(a) Loads \( P \) sufficiently far apart; hence no mutual interaction between the two loads
\[P = M(4 + 2l)\]

(b) Loads \( P \) sufficiently close
\[P = M(4 + 2l(1 + \frac{b}{w}))\]

Limiting spacing \( b \) between the two loads is \( b_{max} = (1 + 1/2w)l\)
13. Strip simply supported with unequal loads $P$ and $kP$ midway between the edges, where $k < 1.0$ and the loads are sufficiently apart:

$$P \cdot B \left( \frac{1}{2} + \frac{k}{2} \left[ 2 + \frac{k}{1} \right] \right)$$

where $k$ is less than 1.

14. Uniformly loaded square slab with degree of finity $v$ varying between zero and 1.0:

(a) $v = 0$ and no upward movement
$$w_{x^2} = 5AB$$

(b) $v = 0$ and free upward movement
$$w_{x^2} = 22.5AB$$
$$k_{xx} = 1/2 \ (\text{tan}^{-1}(3))$$

(c) $v = 0.5$ (partial restraint)
$$w_{x^2} = 34.75AB$$

(d) $v = 1.0$ (full restraint)
$$w_{x^2} = 44AB \ (\text{upper bound})$$
$$w_{x^2} = 42.5AB \ (\text{lower bound})$$
15. Equilateral triangular plate ($\alpha = 60^\circ$) uniformly loaded:

\[ w_{max} = 30.85 \text{ kips} \]

16. Rectangular slab uniformly loaded with unit load of intensity $w_u$ supported on all four sides with degree of fixity $i$ varying from zero to 1.0 (note sequence of numbers assigned to panel sides):

Yield lines on tension side

Yield lines on compression side

\[ M = \frac{w_u a^2}{24} \left[ \frac{2}{\sqrt{1 + \frac{4}{n^2}}} - \frac{4}{n^2} \right] \]

where

\[ n = \frac{2p}{\sqrt{1 + \frac{4}{n^2}}} + \frac{\sqrt{1 + \frac{4}{n^2}}}{2} \]

General note: Load $P$ is assumed in the foregoing expression to act at a point. To adjust for the fact that $P$ acts on a finite area, assume that it acts over a circular area of radius $p$. For a slab fully restrained on all boundaries, the hinge field would be bound by a circle touching the slab boundary (circle radius $= r$). In such a case,

\[ M + M' = \frac{P}{2\pi} \left( 1 - \frac{2p}{3r} \right) \]

where $M = \text{positive unit moment}$

$M' = \text{negative unit moment}$

Reaction of columns supporting flat plates can be similarly considered for analyzing the flexural local capacity of the plate in the column area. For rectangular supports, an approximation to equivalent circular support can be made in the use of Eq. 11.29.
11.10.3 Example 11.9: Rectangular Slab Yield-line Design

The reinforced concrete slab shown in Figure 11.35 is 14 ft 6 in. x 24 ft in plan (4.42 m x 7.32 m). It carries an external factored ultimate uniform load $w_u = 220$ psf (10.3 kPa), including its self-weight. It is simply supported on one long edge and the adjacent short edge is built in on the opposite edge. Let the reinforcement spacing the short direction be twice the reinforcement spacing the long direction. Also assume the reinforcement on the built-in edges to be equal to the strong reinforcement. Design the slab structure for flexure, including the reinforcement needed and its spacing, using the yield-line theory. Given:

- $f'_c = 4000$ psi (27.6 MPa), normal-weight concrete
- $f'_y = 60,000$ psi (414 MPa)

Solution: \( \mu \) is ratio of reinforcement in the strong direction to the weak direction = 2. From Eq. 11.20, the expression for the unit moment in an affine rectangular slab is

\[
M_x = \frac{\eta f'_c}{24} \left( \sqrt{3 + \frac{a^2}{b^2} - \frac{a}{b}} \right)^2
\]

Change to affine slab converting the span dimension:

\[
as = 14.5 \times \frac{1}{\sqrt{2}} = 14.5 \times \frac{1}{1.414} = 10.25 \text{ ft}
\]

\[
a_s = \frac{2a}{\sqrt{a + b} + \sqrt{a + b}} = \frac{2 \times 10.25}{\sqrt{10.25} + \sqrt{10.25}} = \frac{20.50}{2 \times 1.80} = 5.81
\]

\[
b_s = \frac{2b}{\sqrt{a + b} + \sqrt{a + b}} = \frac{2 \times 24.0}{\sqrt{24.0} + \sqrt{24.0}} = \frac{48.0}{2 \times 4.90} = 4.80
\]

\[
a = 8.492 \quad 0.427
\]

\[
b = 19.884
\]
11.10 Yield-Line Theory for Two-way Action Plates

\[ w_0 = \frac{w_{0,1}}{0.9} = 244 \text{ psf} \]

\[ M_0 = \frac{w_{0,1} h^2}{24} \left[ \sqrt{3} + \left(0.427\right)^2 - 0.427 \right]^2 \]

\[ = \frac{w_{0,1} \times 72.11}{24} \left[ 1.841 - 1.350 = 5.52 \times 11 = 1550 \text{ lb} \right] \]

\[ \mu M_0 = 2 \times 1350 = 2700 \text{ lb ft or 24k f/ft} \]

Assume that \( d = 4 \text{ in.} \) (or \( s = 5 \text{ in.} \)). \( M_0 = \frac{w_{0,1} h}{1 - 0.5 \eta} \), where \( w = \frac{f'_{c}}{f'} \) (see Chapter 5).

\[ z_{0,0} = (0.5)^{0.6} \times 60,000 \left( 1 - 0.5 \eta \right) \left( \frac{f'_{c}}{f'} \right) \]

to get \( \eta = 0.00389 = 0.389\% \) in the short direction.

Reinforcement \( A_0 \) on 12-in. (305-mm) strip \( = 0.0389 \times 4 \times 12 = 0.139 \text{ in.} \) (1/2 in).

This steel area is less than required for the limit state in tension \( \eta = 0.00389 \) hence O.K. No. 3 bars at 9 in. center to center = 0.139 ft. Hence use No. 3 bars at 9 in. center to center in the short direction and on the tension top face of the fixed supports (9.33-mm bars at 34 in. center to center).

Maximum allowable spacing \( s = 2 \times 5 = 10 \text{ in.} \) (254 mm)

Percentage of reinforcement in the long direction gives \( p = 0.00143 = 0.143\% \). Use No. 3 bars at 10 in. center to center for the long direction.

Alternate affine slab in the perpendicular direction

Multiply the \( M \) direction by \( \sqrt{2} \) to \( \mu M \) in both directions. Affine \( b = 24.0 \sqrt{2} = 33.9 \text{ ft} \).

\[ a_0 = \frac{2 \times 14.5}{2.414} = 12.0 \text{ ft} \]

\[ b_0 = \frac{2 \times 33.9}{2.414} = 28.1 \text{ ft} \]

\[ \frac{a_0}{b_0} = \frac{12.0}{28.1} = 0.427 \]

(Same as in the preceding solution)

Hence

\[ M_0 = \frac{w_{0,1} (12.0)^2}{24} \left[ \sqrt{3} + \left(0.427\right)^2 - 0.427 \right]^2 \]

\[ = 6,000 (1.839) = 2700 \text{ lb} \] (as before)

A check of the thickness for minimum deflection and crack-control requirements would have to be made before the design is complete.

11.10-4 Example 11.10: Moment Capacity and Yield-Line Design of a Triangular Balcony Slab

The balcony floor in Figure 11.16 is triangular in shape and is supported at the two perpendicular sides and carries a factored uniform line load of intensity \( p = 400 \text{ lb per linear foot} \) (5.34 kN/m) acting on the triangle hypotenuse. The reinforcement in the short direction is three times the reinforcement in the long direction (\( a = 3 \)).

Analyze the floor moment capacity by the yield-line theory and design the thickness and reinforcement necessary in both directions if the length of the two perpendicular supports is 16 ft (4.88 m) and it subtends an angle of \( 30^\circ \) with the hypotenuse. The shorter side is \( 9.24 \text{ ft} \) (2.82 m). Given:

\( f'_c = 4000 \text{ psi (27.6 kPa)} \) normal-weight concrete

\( f'_c = 6000 \text{ psi (41.5 kPa)} \)

Assume that the self-weight of the plate can be neglected in the solution on the assumption that it is small compared to the live load.
Figure 11.36 Balcony floor plate: (a) floor geometry, (b) affine slab.

Solution:

\[ A_B = 16 \tan 30^\circ = 9.24 \text{ ft} \]
\[ A_D = \sqrt{10^2 + 9.24^2} = 13.88 \text{ ft} \]

The yield line at failure is expected to be line \( BC \) subtending an angle \( \phi \) with side \( DD \). Its components in the \( x \) and \( y \) directions are \( c \cos \phi \) and \( c \sin \phi \), respectively.

Assume that point \( C \) deforms by a magnitude \( \Delta \), causing the center of gravity of the load on segment \( BC \) at its cover line to deflect \( 1.5\Delta \). Summing vertically, the work in the \( x \) and \( y \) directions in Fig. 11.36 gives:

\[ F_x = 3M \cos \phi - \frac{3g}{2} = 9.24p \Delta \]
\[ F_y = 18.49p \times \frac{3}{2} = 27.73p \Delta \]

Since \( E_t = p \Delta \),

\[ M = \frac{0.324 \cos \phi + 0.108 \tan \phi}{3} \]
\[ d(gx) = -0.324 \cos^2 \phi + 0.104 \sec^2 \phi = 0 \]
\[ \sec^2 \phi = 3.87 \text{ or } \tan \phi = 1.732 \quad \phi_{sat} = 61^\circ \]

Hence,

\[ p = M \left( \frac{0.334 \times \frac{1}{1.732} + 0.108 \times 1.732}{3} \right) = 0.373M \]

Affine slab solution

\( A'B \) in Fig. 11.36b becomes \( 9.24 \sqrt{3} = 5.335 \text{ ft} \). Free edge \( A'D \) becomes 16.666 ft.

\[ p = \text{affine linear load} = \frac{p}{\sqrt{\cos \phi + \sin \phi}} \]

where \( \theta \) = angle between the line load and 6 direction, or
\[ \rho' = \frac{\rho}{\sqrt{3 \cos 30^\circ + \sin 30^\circ}} = 0.632 \rho \]

\[ E_t = \Delta M \cot \phi + \Delta M \tan \phi \]

\[ E_x = \rho' \times 10.80 \times \frac{A}{2} = 0.443 \rho' \Delta \]

But \( E_x = E_s \); therefore,

\[ \frac{\rho'}{M} = 0.1186 \cot \phi + 0.1186 \tan \phi \]

\[ \theta_{xx} = 45^\circ \tan \phi = 1.3 \]

Therefore,

\[ \rho' = 2 \times 0.1186M = 0.237M \]

or

\[ \rho = \frac{0.237}{0.612} M = 0.375M \] (as before)

**Design of Reinforcement**

400 = 0.375M to give \( M = 1067 \) lb or \( aM = 3 \times 1067 = 3201 \) lb or ft·lb/ft. Assume that \( h = 3 \) in (\( d = 4 \) in. = 100 mm).

\[ d_{eff} = d(1 - 0.5h) \]

or

\[ 3200 = 4d_{eff} \times 60,000 \left( 1 - 0.5 \frac{h}{d} \frac{60,000}{4000} \right) \]

to give \( \rho = 0.0034 \). The required \( A_e = 0.0054 \times 4 \times 12 = 0.162 \) in.\(^2\). Use No. 3 bars at 8 in. center to center in the short direction \( = 0.165 \) in.\(^2\) (9.53-mm diameter at 203-mm center to center) and No. 3 bars at 2h = 10 in. center to center in the long direction to satisfy moment requirements.

**SELECTED REFERENCES**

11.1 ACI Committee 318, _Building Code Requirements for Structural Concrete_ ACI 318-05 and Commentary (ACI 318R-45), American Concrete Institute, Farmington Hills, MI, 2005, 444 pp.


PROBLEMS FOR SOLUTION

11.1 An end panel of a floor system supported by beams on all sides carries a uniform service live load \( w_1 = 75 \, \text{psf} \) and an external dead load \( w_2 = 20 \, \text{psf} \) in addition to its self-weight. The center-line dimensions of the panel are 18 ft \( \times 20ft \) (the dimension of the discontinuous side is 18 ft). Design the panel and the size and spacing of the reinforcement using the ACI Code direct design method. Given:
11.2 Solve Problem 11.1 by the direct design method and by the equivalent-frame method of the ACI Code, assuming that the floor is a flat plate system with no edge beams.

11.3 Determine the size and spacing of reinforcement in the N-S direction of the floor slab in Ex. 11.3 for both the columns and the middle strips.

11.4 Solve for the factored moments for the floor slab on beams in Ex. 11.1 using the coefficients in Figure 11.23.

11.5 An interior flat-plate panel is supported on columns spaced 18 ft × 20 ft. The panel dimensions, loading, and material properties are the same as those in Problem 11.1. Design the panel and size and spacing of the main as well as shear-moment transfer reinforcement by the ACI Code direct design method.

11.6 Calculate the time-dependent deflection at the center of the panel in (a) Problem 11.1, (b) Problem 11.5. Check also if the panels satisfy the serviceability requirements for deflection control and crack control for aggressive environments. Assume that $A_{w}/A_{c} = 350$ in. per radian for part (a) and $A = 221$ in. per radian for part (b).

11.7 Use the yield-line theory to evaluate the slab thickness needed in the column zone of the flat plate in Problem 11.3 for flexure, assuming that the hinge load would have a radius of 24 in.

11.8 An isotropically reinforced long strip is simply supported on the edges. A concentrated load $P$ acts on the minor axis of the slab midway between the long edges. Prove that the magnitude of the column load is $P = MY = 2W$.

11.9 A slab 2 ft × 3.5 ft in. carries an external factored ultimate load of 200 lb per square foot, including its own weight. It is simply supported on one long edge and the adjacent short edge and built in on the opposite edges. Let the reinforcement spanning the short way be three times the reinforcement spanning the long way. Also assume the reinforcement on the built-in edges to be equal to the strong reinforcement. Design the slab structures and the reinforcement needed and its spacing using the yield-line theory. Given:

- $f' = 4000$ psi, normal-weight concrete
- $f = 60,000$ psi

11.10 Calculate the maximum crack width in a two-way interior panel of a reinforced concrete floor system. The slab thickness is 8 in. (203 mm) and the panel size is 20 ft × 20 ft (6.09 m × 6.09 m). Also design the size and spacing of the reinforcement necessary for crack control, assuming that (a) the floor is exposed to normal environments; (b) the floor is part of a parking garage. Given $f' = 6000$ psi (414 MPa).
FOOTINGS

12 INTRODUCTION

Cumulative floor loads of a superstructure are supported by foundation substructures in direct contact with the soil. The function of the foundation is to transmit safely the high concentrated column and/or wall reactions or lateral loads from earth-retaining walls to the ground without causing unsafe differential settlement of the supported structural system or soil failure.

If the supporting foundations are not adequately proportioned, one part of a structure can settle more than an adjacent part. Various members of such a system become overstressed at the column-beam joints due to uneven settlement of the supports leading to large deformations. The additional bending and torsional moments in excess of the resisting capacity of the members can lead to excessive cracking due to yielding of the reinforcement and ultimately to failure.

If the total structure undergoes even settlement, little or no overstress occurs. Such behavior is observed when the foundation is excessively rigid and the supporting soil highly yielding such that a structure behaves similar to a floating body that can sink or tilt without breaking. Numerous examples of such structures can be found in such locations.

Photo 12.1  One Shell Plaza, New Orleans. (Courtesy of Portland Cement Association.)
as Mexico City with buildings on mat foundations or rigid supports that sank several feet over the years due to the high consolidation of the supporting soil. Examples of other famous cases of very slow and relatively uneven consolidation process can be cited. Gradual loss of stability of a structure undergoing tilting with time, like the leaning Tower of Pisa, is an example of foundation problems resulting from uneven bearing support. Layouts of structural supports vary widely and soil conditions differ from site to site and within a site. As a result, the type of foundation to be selected has to be governed by these factors and by optimal cost considerations. In summary, the structural engineer has to acquire the maximum economically feasible soil data on the site before embarking on a study of the various possible alternatives for site layout.

Basic knowledge of soil mechanics and foundation engineering is assumed in presenting the topic of design of footings in this chapter. Background knowledge of the methodology of determining the resistance of cohesive and noncohesive soils is necessary to select the appropriate bearing capacity value for the particular site and the particular foundation system under consideration.

The bearing capacity of soils is usually determined by borings, test pits, or other soil investigations. If these are not available for the preliminary design, representative values as the footing level can normally be used from Table 12.1.

### Table 12.1 Presumptive Bearing Capacity (ton/ft²)

<table>
<thead>
<tr>
<th>Type of Soil</th>
<th>Bearing Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Massive crystalline bedrock, such as granite, diorite, gneiss, and trap rock</td>
<td>100</td>
</tr>
<tr>
<td>Foliated rocks, such as schist or slate</td>
<td>40</td>
</tr>
<tr>
<td>Sedimentary rocks, such as hard shales, sandstones, limestones, and silts</td>
<td>15</td>
</tr>
<tr>
<td>Gravel and gravel-sand mixtures (SW and GP soils)</td>
<td></td>
</tr>
<tr>
<td>Densely compacted</td>
<td>5</td>
</tr>
<tr>
<td>Medium compacted</td>
<td>4</td>
</tr>
<tr>
<td>Loose, not compacted</td>
<td>3</td>
</tr>
<tr>
<td>Sands and gravelly sands, well graded (SW soil)</td>
<td></td>
</tr>
<tr>
<td>Densely compacted</td>
<td>31</td>
</tr>
<tr>
<td>Medium compacted</td>
<td>3</td>
</tr>
<tr>
<td>Loose, not compacted</td>
<td>21</td>
</tr>
<tr>
<td>Sands and gravelly sands, poorly graded (SP soil)</td>
<td></td>
</tr>
<tr>
<td>Densely compacted</td>
<td>3</td>
</tr>
<tr>
<td>Medium compacted</td>
<td>2k</td>
</tr>
<tr>
<td>Loose, not compacted</td>
<td>1k</td>
</tr>
<tr>
<td>Silty gravel and gravel-sand-silt mixtures (GM soil)</td>
<td></td>
</tr>
<tr>
<td>Densely compacted</td>
<td>2k</td>
</tr>
<tr>
<td>Medium compacted</td>
<td>2</td>
</tr>
<tr>
<td>Loose, not compacted</td>
<td>1k</td>
</tr>
<tr>
<td>Silty sand and silt-sand mixtures (SM soil)</td>
<td>2</td>
</tr>
<tr>
<td>Clayey gravel, gravel-sand-clay mixtures, clayey sands, sand-clay mixtures</td>
<td></td>
</tr>
<tr>
<td>(GC and SC soils)</td>
<td>2</td>
</tr>
<tr>
<td>Inorganic silts, and fine sands; silty or clayey fine sands and clayey silts</td>
<td></td>
</tr>
<tr>
<td>with slight plasticity; inorganic clays of low to medium plasticity; gravelly clays; sandy clays; silty clays, lean clays (ML and CL soils)</td>
<td>1</td>
</tr>
<tr>
<td>Inorganic clays of high plasticity; fine clays, silts, or other fine silts</td>
<td></td>
</tr>
</tbody>
</table>
12.2 TYPES OF FOUNDATIONS

There are basically six types of foundation substructures, as shown in Figure 12.1. The foundation area must be adequate to carry the column loads, the footing weight, and any overburden weight within the permissible soil pressure.

Figure 12.1 Types of foundations: (a) wall footing; (b) isolated footing; (c) combined footing; (d) strip footing; (e) pile foundation; (f) raft foundation.
12.3 SHEAR AND FLEXURAL BEHAVIOR OF FOOTINGS

To simplify foundation design, footings are assumed to be rigid and the supporting soil layers elastic. Consequently, uniform or uniformly varying soil distribution can be assumed. The net soil pressure is used in the calculation of bending moments and shear by subtracting the footing weight intensity and the surcharge from the total soil pressure. If a column footing is considered as an inverted floor segment where the intensity of net soil pressure acts along the column width, the bending and shear forces in the footing can be calculated using the flexural theory of beams. The flexural behavior of footings is governed by their depth, shape, and location relative to the loads they transmit. The shear forces in footings are critical for designing against overturning and sliding, especially in unstable soil conditions. The flexural behavior is important to resist the bending moments resulting from the distributed loads and concentrated loads at the edges of footings.
pressure is considered to be acting as a column-supported cantilever slab, the slab would be subjected to both bending and shear in a similar manner to a floor slab subjected to gravity loads.

When heavy concentrated loads are involved, it has been found that shear rather than flexure controls most foundation designs. The mechanism of shear failure in footing slabs is similar to that in supported floor slabs. However, the shear capacity is considerably higher than that of beams, as will be discussed in the next section. Since the footing in most cases bends in double curvature, shear and bending about both principal axes of the footing plan have to be considered.

The state of stress at any element in the footing is due primarily to the combined effects of shear, flexure, and axial compression. Consequently, a basic understanding of the fundamental behavior of the footing slab and the cracking mechanism involved is essential. It enables developing a background feeling for the underlying hypothesis used in the analysis and design requirements of footings both in shear and in flexure.

12.3.1 Failure Mechanism

The inclined shear cracks develop in essentially the same manner as in beams, stabilizing at approximately 65% of the ultimate load and extending rapidly toward the neutral axis. Thereafter, the cracks propagate slowly toward the compression zone such that a very shallow depth in compression remains at failure.

The inclined cracks always form close to the concentrated load or column reaction in two-way slabs or footings, as seen in Figure 12.2a. This is due partly to the heavy concentration of bending moments in the region close to the column face, forming a truncated pyramid at the foot of the column region. The column as a perimetrically punch through the slab in this failure form if the slab is not adequately designed to resist shear failure (also called diagonal tension or punching shear). The action of the confining surrounding punched slab on the column base interface punching through the slab in Figure 12.2b can be represented by the resisting shear forces $V_I$ and $V_o$, the compressive forces, $C_I$ and $C_o$, and the tensile forces $T_I$ and $T_o$, in addition to the internal crack and membrane action of the slab.

Figure 12.2c shows an infinitesimal element taken from the compression zone above the inclined crack. The element is subjected to the following four stress components: (1) vertical water stress $v$, (2) direct compressive stress $f$, (3) vertical compressive stress $f_o$, and (4) lateral compressive stress $f_e$.
The vertical shearing stress $\tau_v$ is the result of the total shear that has to be entirely transmitted by the compression zone above the inclined crack. The direct compressive stress $f_c$, which varies along the length of the critical section, results from the bending moments. The vertical compressive force $f_c$ is due to the heavy concentrated column load. It has a major influence on increasing the shear capacity of the slab, as demonstrated in Ref. 12.2 for pressure on an infinite semi-infinite solid loaded at the surface. The horizontal compressive stress $f_h$ is the result of the bending moment about an axis perpendicular to the critical section. It contributes further to the increase in the compressive strength of the concrete as a result of the triaxial state of stress. Consequently, the existence of the multiaxial forces and stresses in Figure 12.2b explains why the shear capacity of a slab subjected to concentrated loads is considerably higher than that of a beam.

In addition, the inclined crack occurring close to the critical section in two-way slabs and footings due to the high moment concentration justifies considering the critical section to be at a distance of $d/2$ from the face of the column in slabs and footings, while in beams and one-way slabs and footings, the ACI Code specifies the critical section at a distance of $d/4$ from the face of the column. The nominal shearing stress at failure varies between $4\sqrt{f_c}$ and $9\sqrt{f_c}$ for the footing slab, whereas it does not exceed $2\sqrt{f_c}$ to $4\sqrt{f_c}$ in beams. The Code, however, allows a maximum nominal resisting shear strength of plain concrete not to exceed $\tau_v = 4\sqrt{f_c}$ for the unsupported two-way slab or the footing and $\tau_v = 2\sqrt{f_c}$ for beams and one-way-action footings. For plain concrete footings cast
against soil, the effective thickness used in computing stresses is taken as the overall thickness minus 3 in. The overall thickness should not be less than 8 in.

12.3.2 Loads and Reactions

Based on the foregoing discussion, it is essential to make the correct assumptions for evaluating all the combined forces acting on the foundation. The footing slab has to be proportioned to sustain all the applied factored loads and induced reactions, which include axial loads, shears, and moments to be resisted at the base of the footing. After the permissible soil pressure is determined from the available site data and the principles of soil mechanics and the local codes, the footing area size is computed on the basis of the unfactored (service) loads, such as dead, live, wind, or earthquake loads in whatever combination governs the design. The minimum eccentricity requirement for column slenderness considerations is neglected in the design of footings or pile caps and only the computed end moments that exist at the base of a column are considered to have been transferred to the footing. In cases where eccentric loads or moments exist due to any loading combinations, the extreme soil pressure resulting from such loading conditions has to be within such permissible bearing values such as those in Table 12.1 or as determined by actual soil tests. Once the size of a footing or pile cap for a single pile or a group of piles is determined, the design of the footing geometry becomes possible using the principles and methodologies presented in the preceding chapters for shear and flexure design. The external service loads and moments used to determine the size of the foundation area are converted to their ultimate factored values using the appropriate load factors and strength reduction factors for determining the nominal resisting values to be used in the analysis and proportioning the size and reinforcement distribution in the footing.

12.4 SOIL BEARING PRESSURE AT BASE OF FOOTINGS

The distribution of soil bearing pressure on the footing depends on the manner in which the column or wall loads are transmitted to the footing slab and the degree of rigidity of the footing. The soil under the footing is assumed to be a homogeneous elastic material, and the footing is assumed to be rigid as a most common type of foundation. Consequently, the soil bearing pressure can be considered uniformly distributed if the reaction load acts through the axis of the footing slab area. If the load is not axial or symmetrically applied, the soil pressure distribution becomes trapezoidal due to the combined effects of axial load and bending.

12.4.1 Eccentric Load Effect on Footings

As indicated in Section 12.2, exterior column footings and combined footings can be subjected to eccentric loading. When the eccentric moment is very large, tensile stress on one side of the footing can result, since the bending stress distribution depends on the magnitude of the eccentricity. It is always advisable to proportion the area of these footings such that the load falls within the middle kern, as shown in Figures 12.3 and 12.4. In such a case, the location of the load is in the middle third of the footing dimension in each direction, thereby avoiding tension in the soil that can theoretically occur prior to stress redistribution.

1. Eccentricity case \( e < L/6 \) (Figure 12.3a). In this case, the direct stress \( P_w \), is larger than the bending stress \( k \), the stress
Figure 12.3 Eccentrically loaded footings.

\[ P_{\text{max}} = \frac{P}{A_{f}} = \frac{P_{\text{e}}}{A_{f}} \]  
\[ P_{\text{max}} = \frac{P}{A_{f}} = \frac{P_{\text{e}}}{A_{f}} \]  

1. Eccentricity case \( e_1 \leq L/6 \) (Fig. 12.3b):

- Direct stress: \( \sigma = \frac{P}{A_{f}} = \frac{P_{\text{e}}}{A_{f}} \)  
- Bending stress: \( M_{f} = P_{\text{e}} \times \frac{c}{I} \)  
- \( c = \frac{L}{2} \) (see equation)  
- \( \frac{L}{2} = \frac{L}{6} + \frac{L}{6} = \frac{L}{3} \) (see equation)

Figure 12.4 'Skeletal loading of footing.'
where \( s \) and \( L \) are the width and the length of the footing, respectively. In order to find the limiting case where no tension exists on the footing, the direct stress \( P/A_f \) has to be equivalent to the bending stress so that

\[
\frac{P}{A_f} - \frac{P c}{I} = 0
\]  

(12.24)

Substituting for \( P/A_f \) and \( c/I \) from Eqs. 12.2a and \( z \) into Eq. 12.2d,

\[
\frac{P}{sL} - \frac{P c}{z} = 0 \quad \text{or} \quad c = \frac{L}{6}
\]

Consequently, the eccentric load has to act within the middle third of the footing dimension to avoid tension on the soil.

3. Eccentricity case \( e_r > L/2 \) (Figure 12.3c). As the load acts outside the middle third, tensile stress results at the left side of the footing, as shown in Figure 12.3c. If the maximum bearing pressure \( p_{max} \) due to load \( P \) does not exceed the allowable bearing capacity of the soil, no split is expected at the left end of the footing, and the center of gravity of the triangular bearing stress distribution coincides with the point of action of load \( P \) in Figure 12.3c.

The distance from the load \( P \) to the top of footing is \( r = (L/2) - e_r \), where \( e_r \) is the distance of the centroid of the stress triangle from the base of the triangle. Therefore, the width of the triangle is \( 3r = 3[(L/2) - e_r] \). Hence, the maximum compressive bearing stress is

\[
p_{max} = \frac{P}{sL} = \frac{2P}{3s[(L/2) - e_r]}
\]  

(12.3a)

4. Eccentricity about two axes, bi-axial loading (Fig. 12.4). In the case where a concentrated load has an eccentricity in two directions (both within their respective ken points), the stresses are

\[
p_{max} = \frac{P}{A_f} = \frac{P c_1}{I_1} = \frac{P c_2}{I_2}
\]  

(12.3b)

12.4.2 Example 12.1: Concentrically Loaded Footings

A column support transmits axially a total service load of 400,000 lb (1779 kN) to a square footing at the front line (3 ft below grade), as shown in Figure 12.5. The front line is the sub-grade soil level below which the groundwater does not freeze throughout the year. Test bo-
ings indicate a densely compacted gravel-sand soil. Determine the required area of the foot-
ing and the net soil pressure intensity \( p_n \) to which it is subjected. Given:

- unit weight of soil \( \gamma = 155 \text{ lb/ft}^3 \) (2.1 \text{ kN/m}^3)
- footing slab thickness \( e = 2 \text{ ft (0.61 m)} \)

Solution: Since the footing is concentrically loaded, the soil bearing pressure is considered uniformly distributed assuming the footing is rigid. From the soil test borings and Table 12.1, the presumptive bearing capacity of the soil is 5 tons/ft\(^2\) at the level of the footing, that is, 10,000 lb/ft\(^2\) (47.8 kPa). Assume that the average weight of the soil and concrete above the footing is 135 psf. Since the top of the footing has to be below the frost line (minimum 3 ft below grade), the net allowable pressure is

\[
p_n = 10,000 - (5 \times 135 + 100 \text{ psf for surcharge paving}) = 9225 \text{ psf}.
\]

minimum area of footing \( A_f = \frac{400,000}{9,225} = 43.36 \text{ ft}^2 \)

Use square footing 6 ft x 6 ft inches. (2.03 m x 2.03 m):

\[
A_f = 44.44 \text{ ft}^2 (4.13 \text{ m}^2) > 43.36 \text{ ft}^2
\]

12.4.3 Example 12.2: Eccentrically Loaded Footings

A reinforced concrete footing supports a 14 in x 14 in column reaction \( P = 400,000 \text{ lb (1779 kN)} \) at the frost line (3 ft below grade). The load acts at an eccentricity \( e = 0.4 \text{ ft}, e_f = 1.3 \text{ ft}, \) and \( e = 2.8 \text{ ft}. \) Select the necessary area of footing assuming that it is rigid and has a thickness \( h = 2 \text{ ft}. \) Soil test borings indicate that the bearing area is composed of layers of shale and clay to a considerable depth below the foundation. Use a unit weight \( \gamma = 140 \text{ lb/ft}^3 \).

Solution: From Table 12.1, assume an allowable bearing capacity \( p_n = 6.5 \text{ tons/ft}^2 \) (13,000 lb/ft\(^2\)) at the footing base level.

Eccentricity \( e = 0.4 \text{ ft}. \)

By trial and adjustment, assume a footing 5 ft x 9 ft (1.52 m x 2.74 m), \( A_f = 45 \text{ ft}^2 \). Assume that the footing base is 6 ft below grade and that a slab on grade surcharge weighs 120 psf. Assume that the average weight of the soil and footing \( k = 140 \text{ psf}. \)

net allowable bearing pressure \( p_n = 10,000 - (6 \times 140 + 120) = 12,040 \text{ lb/ft}^2 (576.5 \text{ kPa}) \)

Stress due to the service eccentric column load is

\[
\rho = \frac{P}{A_f} - \frac{e\cdot e}{h} = \frac{400,000}{45} - \left( \frac{6 \times 0.4}{2} \right) = 8889 \text{ psf} = 11,259 \text{ lb/ft}^2 (C) \text{ and 6519 lb/ft}^2 (C) < 12,040 \text{ lb/ft}^2 (C)
\]

The distribution of the bearing pressure is as shown in Figure 12.5a; therefore, O.K.

Eccentricity \( e_f = 1.3 \text{ ft}. \)

By trial and adjustment, assume a footing 6 ft x 10 ft (1.83 m x 3.05 m), \( A_f = 60 \text{ ft}^2 \) (5.57 m\(^2\)). The actual service load-bearing pressure is

\[
\rho = \frac{400,000}{60} - \frac{60 \times 0.4}{2} = 6667 \text{ psf} = 8467 \text{ lb/ft}^2 (C) \text{ and 6519 lb/ft}^2 (C)
\]

< 12,040 lb/ft\(^2\) O.K.
Figure 12.6 Bearing area and bearing stress distribution in Ex. 12.2.

Notice in comparing the two cases that as the moment increases leading to larger eccentricities, the minimum bearing pressure decreases, as seen from Figures 12.6a and b.

**Eccentricity $e = 2.2$ ft**

By trial and adjustment, try a footing $7 \times 11$ ft ($2.13 \times 3.35$ m). $A = 77$ ft$^2$ ($7.15$ m$^2$).

$p = \frac{400,000}{77.0} = \frac{400,000 \times 2.2 \times 6}{77.0^2} = 5195 \pm 6234 \text{ lb/ft}^2$ (C) and $= 1039 \text{ lb/ft}^2$ (T)

Check by Eq. 12.3a for $e > L/6 > 11.09 > 1.83$ ft:
12.5 Design Considerations in Flexure

The maximum external moment on any section of a footing is determined on the basis of computing the factored moment of the forces acting on the entire area of footing on one side of a vertical plane assumed to pass through the footing. This plane is taken at the following locations:

1. At the face of column, pedestal, or wall for an isolated footing, as in Figure 12.7a
2. Halfway between the middle and edge of wall for footing supporting a masonry wall, as in Figure 12.7b
3. Halfway between face of column and edge of seal base for footings supporting a column with steel base plates

12.5.1 Reinforcement Distribution

In one-way footings and in two-way square footings, the flexural reinforcement should be uniformly distributed across the entire width of the footing. This recommendation is conservative, particularly if the soil bearing pressure is not uniform. However, no meaningful saving can be accomplished if refinement is made in the bending moment assumptions.

![Diagram](image)

Figure 12.7 Critical planes in flexure: (a) concrete column; (b) masonry wall.
in two-way rectangular footings supporting one column, the bending moment in
the short direction is taken as equivalent to the bending moment in the long direction.
The distribution of reinforcement differs in the long and short directions. The effective
depth is assumed without meaningful loss of accuracy to be equal in both the short and
long directions, although it differs slightly because of the two-layer reinforcing mats. The
following is the recommended reinforcement distribution:

1. Reinforcement in the long direction is to be uniformly distributed across the entire
   width of the footing.
2. For reinforcement in the short direction, a central band of width equal to the width
   of footing in the short direction shall contain a major portion of the reinforcement
   total area (as in Eq. 12.4) uniformly distributed along the band width:

   \[
   \frac{\text{reinforcement in band width}}{\text{total reinforcement in short direction}, A_i} = \frac{2}{\beta + 1} \quad (12.4)
   \]

   where \( \beta \) is the ratio of long to short side of footing. The remainder of the reinforce-
   ment required in the short direction is uniformly distributed outside the center
   band of the footing.

In all cases, the depth of the footing above the reinforcement has to be at least 6 in.
(152 mm) for footings on soil and at least 12 in. (305 mm) for footings on piles (footings
on piles must always be reinforced). A practical depth for column footings should not be
less than 9 in. (229 mm).

12.6 DESIGN CONSIDERATIONS IN SHEAR

As discussed in Section 12.3.1, the behavior of footings in shear is not different from that
of beams and supported slabs. Consequently, the same principles and expressions as
those used in Chapter 6 on shear and diagonal tension are applicable to the shear design
of foundations. The shear strength of slabs and footings in the vicinity of column reac-
tions is governed by the more severe of the following two conditions.

12.6.1 Beam Action

The critical section for shear in slabs and footings is assumed to extend in a plane across
the entire width and located at a distance \( d \) from the face of the concentrated load or re-
action area. In this case, if only shear and flexure act, the nominal shear strength of the
section is

\[
V_c = 2\sqrt{f_c} b_i d \quad (12.5)
\]

where \( b_i \) is the footing width. \( V_c \) must always be larger than the nominal shear force
\( V_n = V_c / k \) unless shear reinforcement is provided.

12.6.2 Two-way Action

The plane of the critical section perpendicular to the plane of the slab is assumed to be so
located that it has a minimum perimeter \( b_p \). This critical section need not be closer than
\( d / 2 \) to the perimeter of the concentrated load or reaction area. The fundamental shear
failure mechanism in two-way action as presented in Section 12.3.1 demonstrates that the
critical section occurs at a distance \( d / 2 \) from the face of the support and not at \( d \) as in
beam action. Maximum allowable nominal shear strength is the smallest of
(i) \[ V_c = \left( 2 + \frac{d}{b} \right) \sqrt{f_{c}'} b_d \]  
(12.6a)

(ii) \[ V_c = \left( \frac{a_d}{b_p} + 2 \right) \sqrt{f_{c}'} b_d \]  
(12.6b)

(iii) \[ V_c = 4 \sqrt{f_{c}'} b_d \]  
(12.6c)

where \( b_p \) = long side \( c \)/short side \( c \) of the concentrated load or reaction area

\[ b_p = \text{perimeter of the critical section, that is, the length of the idealized failure plane} \]

\[ a = 40 \text{ for interior columns, 30 for edge columns, and 20 for corner columns.} \]

Figure 12.8 gives the relationship of the column side ratio \( b_p \) to the shear strength \( V_c \) of the footing. \( V_c \) must always be larger than the nominal shear force \( V_s = V_f b_d \) unless shear reinforcement is provided.

In cases of both one- and two-way action, if shear reinforcement consisting of bars or wires is used,

\[ V_c = V_c + V_s = 6 \sqrt{f_{c}'} b_d \]  
(12.7)

where \( V_s = 2 \sqrt{f_{c}'} b_d \) and \( V_s \) is based on the shear reinforcement size and spacing as described in Chapter 6, unless shear heads made from steel I or channel shapes are used.

It is worthwhile to keep in mind that in most footing slabs, as in most supported superstructure slabs or plates, the use of shear reinforcement is not popular, due to practical considerations and the difficulty of holding the shear reinforcement in position.

12.6.3 Force and Moment Transfer at Column Base

The forces and moments at the base of a column or wall are transferred to the footing by bearing on the concrete and by reinforcement, dowels, and mechanical connectors. Such reinforcement can transmit the compressive forces that exceed the concrete bearing strength of the footing or the supported column as well as any tensile force across the interface.

The permissible bearing stress on the actual loaded area of the column base or footing top area of contact is

\[ f_p = 4f_c^{0.5} f_{c} \]  
where \( f_c = 0.7 \)  
(12.8a)

or

\[ f_p = 0.6f_{c} \]  
(12.8b)

![Image of Figure 12.8: Shear strength in footings.](image)
Hence the permissible bearing stress on the column can normally be considered 0.60 \( f'_{c} \) for the column concrete. The compressive force that exceeds that developed by the permissible bearing stress at the base of the column or at the top of the footing has to be carried by dowels or extended longitudinal bars.

As the footing supporting surface is wider on all sides than the loaded area, the code allows the design bearing strength on the loaded area to be multiplied by \( \sqrt{A_2/A_1} \), but the value of \( \sqrt{A_2/A_1} \) cannot exceed 2.0. \( A_1 \) is the loaded area and \( A_2 \) is the maximum area of the supporting surface that is geometrically similar and concentric with the loaded area.

A minimum area of reinforcement of 0.005\% \( , \) but not less than four bars \( \) has to be provided across the interface of the column and the footing even when the concrete bearing strength is not exceeded. \( A_3 \) \( (\text{in}^2) \) being the gross area of the column cross section.

Lateral forces due to horizontal normal loads, wind, or earthquake can be resisted by shear friction reinforcement, as described in Section 6.10.

12.7 OPERATIONAL PROCEDURE FOR THE DESIGN OF FOOTINGS

The following sequence of steps can be used for the selection and geometrical proportioning of the size and reinforcement spacing in footings:

1. Determine the allowable bearing capacity of the soil based on site boring test data and soil investigations.
2. Determine the service loads and bending moments acting at the base of the columns supporting the superstructure. Select the controlling service load and moment combinations.
3. Calculate the required area of the footing by dividing the total controlling service load by the selected allowable bearing capacity of the soil if the load is concentric or by also taking into account the controlling bending stress if combined load and bending moments exist.
4. Calculate the factored loads and moments for the controlling loading condition and find the required nominal resisting values by dividing the factored loads and moments by the applicable strength reduction factors \( \phi \).
5. By trial and adjustment, determine the required effective depth \( d \) of the section that has adequate punching shear capacity at a distance \( d/2 \) from the support face for one-way action and at a distance \( d \) for two-way action such that \( V' = 2 \sqrt{f'_{c} \cdot b \cdot d} \) for one-way action and \( V' = \min \) of values from Eqs. 12.5 for two-way action, where \( b \) is the footing width for one-way action and \( b_0 \) is the perimeter of the failure plane in two-way action. Use an average value of \( d \), since there are two reinforcing mats in the footing. If the footing is rectangular, check the beam shear capacity in each direction on planes \( x \) a distance \( d \) from the face of the column support.
6. Calculate the factored moment of resistance \( M' \) on a plane at the face of the column support due to the controlling factored loads from that plane to the extremity of the footing. Find \( M' = M_0/0.60 \). Select a total reinforcement area \( A_4 \) based on \( M' \) and the applicable effective depth.
7. Determine the size and spacing of the flexural reinforcement in the long and short directions:

(a) Distribute the steel uniformly across the width of the footing in the long direction.
(b) Determine the portion \( A_5 \) of the total steel area \( A_4 \) determined in step 6 for the short direction to be uniformly distributed over the central band.
12.7 Operational Procedure for the Design of Footings

\[ A_{ri} = \frac{2}{B + 1} A_i \]

Distribute uniformly the remainder of the reinforcement \((A_i - A_{ri})\) outside the center band of the footing. Verify that the area of steel in each principal direction of the footing plan exceeds the minimum value required for temperature and shrinkage: \(A_i = 0.008d_i f'c\) for sections reinforced with grade 60 steel and 0.009d_i f'c with grade 40 steel.

8. Check the development length and anchorage available to verify that bond requirements are satisfied (see Chapter 10).

9. Check the bearing stresses on the column and the footing at their area of contact such that the bearing strength \(P_{ba}\) for both is larger than the nominal value of column reaction \(P_c = \frac{P}{f'c} = 0.70\). For footing bearing \(P_{ba} = \sqrt{A_{f}/A_i} (0.85f'c\alpha)\), \(\sqrt{A_{f}/A_i}\) not to exceed 2.0.

10. Determine the number and size of the dowel bars that transfer the column load to the footing slab.

Figure 12.9 presents a flowchart for the sequence of calculation operations.

12.7.1 SI Footing Design Expressions

\[ E_i = w_i^{0.4} 0.043 \sqrt{f'_c} \text{ MPa} \]

\[ f_i = 0.7 \sqrt{f'_c} \]

\[ K_{re} = \frac{A_{re} f_{re}}{200f'c} \]

where \(f_i\) is in MPa

\[ \ell_s = \frac{d_i}{16} \left[ \frac{15f'c(\alpha + \lambda)}{6 + K_{re} \ell_s} \right] \]

If \(\ell_s = d_i\), \(\alpha = 1.0\) and \(f_i = 27.6\) MPa,

- bars ≤ No. 20 M, \(s \geq 2d_i\): \(\ell_s = 38d_i\)
- bars ≤ No. 20 M, \(s < 2d_i\): \(\ell_s = 57d_i\)

- bars ≤ No. 25 M, \(s > 2d_i\): \(\ell_s = 48d_i\)
- bars ≤ No. 25 M, \(s > 2d_i\): \(\ell_s = 72d_i\)

Shear in beam action

\[ V_i = 2 \sqrt{f'_c} b_i d_i \]

Shear in two-way action

The smallest of

\[ V_i = \left(2 - \frac{4}{B}\right) \sqrt{f'_c} b_i d_i / 12 \]
\[ V_i = \left(\frac{\alpha d_i}{b_i} + 2\right) \sqrt{f'_c} b_i d_i / 12 \]
\[ V_i = 4 \sqrt{f'_c} b_i d_i / 12 \]

\(\alpha = 40\) for interior columns, \(\alpha = 30\) for edge columns, and \(\alpha = 20\) for corner columns.
Figure 12.9 Flowchart for footing design.
12.8 EXAMPLES OF FOOTING DESIGN

12.8.1 Example 12.3: Design of Two-way Isolated Footing

Design the footing thickness and reinforcement distribution for the isolated square footing in Ex. 12.1 if the total service load \( P = 400,000 \) lb comprises 230,000 lb (1023 kN) dead load and 170,000 lb (756 kN) live load. Given:

- \( f_c' = f_{c} = 3000 \text{ psi (20.7 MPa)} \) normal weight concrete (footing)
- \( f_y' = f_y = 5500 \text{ psi (37.6 MPa)} \) in column
- \( f_y = 60,000 \text{ psi (413.7 MPa)} \)

**Solution:**

**Factored load intensity** (step 4)

Data from Ex. 12.1:

- Column size = 14 in. x 14 in. (356 mm x 356 mm)
- Footing area = 6 ft 8 in. x 6 ft 8 in. (2.03 m x 2.03 m)
- \( A_f = 44.49 \text{ ft}^2 \)

Assumed footing slab thickness \( h = 2 \text{ ft} \)

Factored load \( U = 1.2 \times 230,000 + 1.6 \times 170,000 = 548,000 \) lb

**Shear capacity** (step 5)

Assume that the thickness of the footing slab = 2 ft. The average depth \( d = h - 3 \text{ in.} \)

**Beam action (at \( d \) from support face):** The area to be considered for factored shear \( V_x \) is shown as \( ABCD \) in Figure 12.10.

- Factored \( V_x = 12.31 \left( \frac{9 \text{ ft 8 in.}}{2} \right) \left( -\frac{14}{2 \times 12} - \frac{20}{2 \times 12} \right) = 88,020 \text{ lb} \)

- Required \( V_x = \frac{V_x}{6} = \frac{88,020}{0.75} = 118,060 \text{ lb} \)

- \( b_n = 6 \text{ ft 8 in.} = 80 \text{ in.} (2.03 \text{ m}) \)

- Available \( V_x = 2 \sqrt{2} b_n d = 2 \sqrt{3000} \times 80 \times 20 = 175,271 \text{ lb} \)

**Two-way action (at \( \frac{d}{2} \) from support face):** The area to be considered for factored shear \( V_x \) is equal to the total area of footing less area \( EFGH \) of the failure zone.

- Factored \( V_x = 12.31 \left( \frac{44.49}{12} \right) \left( \frac{14 + 20}{12} \right) = 449,105 \text{ lb} \)

- Required \( V_x = \frac{V_x}{6} = \frac{449,105}{0.75} = 598,807 \text{ lb (2663 kN)} \)

- \( b_n = \) perimeter of failure zone \( EFGH = (14 + 20) = 34 \text{ in.} \)

- \( \beta = \frac{14}{34} = 0.42 \)

The available nominal shear \( V_x \) from Eqs. 12.6 is the smallest of

\[ V_x = \left( 2 + \frac{4}{\beta} \right) \sqrt{2} b_n d \leq 4 \sqrt{2} b_n d \]
Figure 12.10 Details of footing in Ex. 12.3.

or

\[ V_r = \left( 2 + \frac{a}{b_s} \right) \sqrt{3000 \times 136 \times 20} = 293,882 \text{ lb} \]

\[ V_r = \left( \frac{6.27}{0.83} \times 2 \right) \sqrt{3000 \times 136 \times 20} = 1,174,317 \text{ lb} \]

where \( a \) = 40 used for interior column value

\[ V_r = 6 \sqrt{3000 \times 136 \times 20} = 595,922 \text{ lb} \]

Since available \( V_r \) = 595,922 lb. = required \( V_r \) = 595,807 lb. Therefore, \( d = 20 \text{ in.} \) is adequate for shear.

**Bending moment capacity (ساکس ۶ و ۷)***

The critical section is at the face of the column.

- Moment arm \( = \frac{6 \times 8 \text{ in.}}{2} = \frac{1}{2} \times 12 = 2 \text{ ft in.} \)
- Factored moment \( M = 12.317 \times 8.67 \times \frac{1}{2} \times 12 = 3,727.55 \text{ in.-lb} \)
12.8 Examples of Footing Design

\[ M_c = \frac{M_s}{d} = \frac{3,727.755}{1.01} = 3,703.63 \text{ in. lb} \]

\[ (468 \text{ kN-m}) \]

\[ M_s = A_f \left( d - \frac{d}{2} \right) \]

Assume that \((d - 0.03) = 0.9d\). Use average \(d = 20\) in.

\[ 4,141,950 = A_f \times 60,000 \times 0.9 \times 20 \]

or

\[ A_f = \frac{4,141,950}{60,000 \times 0.9 \times 20} = 3.84 \text{ in.}^2 / \text{90-in. band} \]

\[ a = A_f f_y = 3.84 \times 60,000 \]

\[ 0.85 \times 3000 \times 80 = 1.13 \text{ in.} \]

\[ 4,141,950 = A_f \times 60,000 \left( \frac{20.0 - 1.13}{2} \right) \]

\[ A_f = 3.55 \text{ in.}^2 \]

\[ \rho = \frac{A_f}{bd} = \frac{3.55}{80 \times 50} = 0.0022 \]

Minimum allowable shrinkage steel

\[ \rho_{min} = 0.0018 < \rho \quad \text{O.K.} \]

Use 12 No. 5 bars \((A_f = 3.66 \text{ in.}^2)\) each way spaced \(s = 64\) in. (165 mm) center to center.

Development of reinforcement (step 8)

The critical section for development length determination is the same as the critical section in flexure, that is, at the face of the column. From Table 10.2, \(l_b = 34\) in. for No. 5 bars (bottom bar).

\[ \frac{d}{l_b} = \sqrt{s \times d} \]

\[ s = 6.25 \text{ in.} > 2d \]

Use \(l_b = 24\) in. The projection length of each bar beyond the column face is

\[ \frac{1}{2} (6.8 \text{ in.} - 14\text{ in.}) - 3\text{ in. cover} = 30\text{ in.} > 24\text{ in.} \quad \text{O.K.} \]

Force transfer at interface of column and footing (step 9)

Column \(f_c = 5500\) psi. Factored \(P_c = 548,000\) lb.

(a) Bearing strength on column using Eq. 12.8b:

\[ \phi P_{ub} = 0.70 \times 0.85 A_f \]

or

\[ \phi P_{ub} = 0.65 A_f = 0.60 \times 500 \times 14 \times 14 \]

\[ = 648,000 \text{ lb} > 548,000 \quad \text{O.K.} \]

From step 9 of the design operational procedure on bearing strength on footing concrete,

\[ \sqrt{\frac{A_f}{A_t}} = \sqrt{\frac{6.6 \times 8}{14 \times 144}} = 5.714 > 2.0 \quad \text{use 2.0} \]

\[ \phi P_{ub} = 2.0 \phi A_f = 2.0 \times 0.60 A_f = 2.0 \times 0.60 \times 3000 \times 14 \times 14 = 705,600 \text{ lb} \]

\[ > 548,000 \quad \text{O.K.} \]
Dowel bars between column and footing (step 10)

Even though the bearing strength at the interface between the column and the footing slab is adequate to transfer the factored \( P \), a minimum area of reinforcement is necessary across the interface. The minimum \( A_s = 0.025 (14 \times 14) = 0.98 \text{ in}^2 \), but not less than four bars. Use four No. 5 bars as dowels (\( A = 1.22 \text{ in}^2 \)).

Development of dowel reinforcement in compression. From Eqs. 10.7 a and b for No. 5 bars and Section 10.3.5,

\[
L_a = \frac{0.025 f_y}{\sqrt{f_c}}
\]

and \( L_a \geq 0.003 f_y / f_c \), where \( f_y \) is the dowel bar diameter. Within column,

\[
L_a = \frac{0.02 \times 0.625 \times 60,000}{\sqrt{5500}} = 10.11 \text{ in.}
\]

Available length for development above the footing reinforcement assuming column bars size to be the same as the dowel bars size:

\[
L_a = 0.02 \times 0.625 \times 60,000 = 13.69 \text{ in.}
\]

Available length for development above the footing reinforcement assuming column bars size to be the same as the dowel bars size:

\[
l_a = 24 - 3 \text{ (cover)} - 2 \times 0.625 \text{ (footing bars)} - 0.625 \text{ (dowels)} = 19.13 > 13.69 \text{ in.} \quad \text{O.K.}
\]

12.8.2 Example 12.4: Design of Two-way Rectangular Isolated Footing

Determine the size and distribution of the bending reinforcement of an isolated rectangular footing subjected to a concentrated eccentric factory column load \( P_c = 680,000 \text{ lb} \) (3025 kN) and having an area 10 ft \( \times \) 15 ft (3.05 m \( \times \) 4.57 m). Given:

- \( f_c = 2000 \text{ psi} \) (13.8 MPa), footing
- \( f_y = 60,000 \text{ psi} \) (413.7 MPa)
- column size = 14 in. \( \times \) 15 in.

Solution: factored load intensity \( q_c = \frac{680,000}{10 \times 15} = 4533 \text{ lb/ft}^2 \)

Shear capacity (step 5)

Through trial and adjustment, assume that the footing slab is 2 ft 4 in. thick.

Beam action (distance \( d \) from column face): Average effective depth \( = 2 \text{ ft 4 in.} - 3 \text{ in. (cover)} = 1 \text{ in. (diameter of bars in first layer) J} = 24 \text{ in.}

From Figure 12.11. Length \( CO \) subjected to bearing intensity \( q_c \), in one-way beam action:

\[
\frac{15 \text{ ft}}{2} \times \frac{18 \text{ in.}}{2} = \frac{24 \text{ in.}}{12} = 4 \text{ ft 9 in.} \quad = 57 \text{ in.}
\]

factored \( V_c = 4533 \times 10 \text{ ft} \times 4 \text{ ft} 9 \text{ in.} = 213,316 \text{ lb}

required \( V_c \) \[ V_c = \frac{V_c}{d} = \frac{213,316}{0.75} = 284,419 \text{ lb}

available \( V_c = 2 \sqrt{f_y b} d = 2 \sqrt{60,000 \times 120 \times 24}\n
= 315,485 \text{ lb} > 284,419 \text{ O.K.}

O.K.}
Figure 12.1.1 Beam action and two-way action planes in Ex. 12.4.

Notice that the shorter side length was used for \( b_1 \), to give the lower available \( V_c \) value.

Two-way action (at distance \( d/2 \) from column face): loaded area outside the failure zone \( L M N P \) in Figure 12.11

\[
\begin{align*}
V_c &= 15 \times 10 - (c - d)(c - d) \\
&= 150 - \frac{(18 + 24)(14 + 24)}{144} \\
&= 138.92 \text{ ft}^2 \\
\text{factored } V_c &= 4333 \times 138.92 = 620.724 \text{ lb} \\
\text{required } V_c &= \frac{620.724}{0.75} = 894.352 \text{ lb (37.5 kN)} \\
\text{perimeter of shear failure plane } &b_1 = 2[c - d] + [c - d] \\
&= 2[18 + 24] + (14 + 24) = 160 \text{ in.}
\end{align*}
\]
From Eqs. 12.6,

\[ V_r = \left(2 + \frac{4}{h_d} \right) \sqrt{f_{yd}} b_d d \approx 4 \sqrt{f_{yd}} b_d d \]

\[ \beta = \frac{22}{12} = 1.83 \]

or

\[ V_r = \left(2 + \frac{4}{1.83} \right) \sqrt{3000 \times 160 \times 24} = 1.074,851 \text{ lb} \]

\[ V_r = \left(\frac{d \cdot d}{b_d} \right) \sqrt{f_{yd}} b_d d = \left(\frac{40 \times 24}{160} + 2 \right) \sqrt{3000 \times 160 \times 24} = 1,682,604 \text{ lb} \]

and

\[ V_r = 4 \sqrt{f_{yd}} b_d d = 4 \sqrt{3000 \times 160 \times 24} = 841,302 \text{ lb} \]

Design of two-way reinforcement

The critical section for bending is at the face of the column. The controlling moment arm is in the long direction:

\[ \frac{15}{2} \times \frac{18}{12} = 6.75 \text{ ft} (2.06 \text{ m}) \]

Factored moment, \( M_r = \frac{4533 \times 10^{6.75^2}}{2} \)

\[ = 1,022,674 \text{ ft}-\text{lb} = 12,392,689 \text{ in} \cdot \text{lb} (1400 \text{ kN} \cdot \text{m}) \]

\[ M_r = \frac{12,392,689}{9} = 1,376,965 \text{ in} \cdot \text{lb} (155 \text{ kN} \cdot \text{m}) \]

Assume that \( (d - d') = 0.9d \)

\[ M_o = A_f \left( \frac{d - d'}{2} \right) \text{ or } 13,768,988 = A_f \times 60,000 \times 0.9 \times 24 \]

\[ A_f = \frac{13,768,988}{60,000 \times 0.9 \times 24} = 10.62 \text{ in}^2 / 10 \text{-ft-wide strip} \]

Check:

\[ \sigma = \frac{A_f (d - d')}{0.85 b_d d} = \frac{10.62 \times 60,000}{0.85 \times 3000 \times 12} = 2.08 \text{ in.} \]

\[ 13,768,988 = A_f \times 60,000 \left( \frac{24 - 2.08}{2} \right) \]

\[ A_f = \frac{13,768,988}{60,000 \times 22} = 1.00 \text{ in}^2 / 10 \text{ ft/width} \]

Try No. 8 bars, \( A_f = 0.79 \text{ in}^2 / \text{per bar} \)

number of bars in the long direction = \( \frac{10.62}{0.79} = 13.68 \]

Use 14 bars.
Reinforcement in the short direction

The bond width = \( s = 10.0 \) (Fig. 12.11). From Eq. (12.4),

\[
\beta = \frac{15}{10} = 1.5
\]

\[
\frac{A_d}{A_s} = \frac{2}{s - 1} \quad \text{or} \quad \frac{A_d}{A_s} = \frac{2}{1.5} = 1.33
\]

Therefore,

\[
A_d = \frac{2 \times 10.0}{2.5} = 8.0 \text{ in.}^2
\]

to be placed in the central 10.0 in. wide band and the balance (10.0 - 8.0 = 2.0 in.\(^2\)) to be placed in the remainder of the footing. Use twelve No. 8 bars in the central band = 9.48 in.\(^2\) and two No. 8 bars at each side of the head, as in Figure 12.12. To complete the design, a check of the development length, bearing stress at the column-foothing interface, and dowel action has to be made, as in Ex. 12.3.

12.8.3 Example 12.8: Proportioning of a Combined Footing

A combined footing has the layout shown in Figure 12.13. Column \( A \) at the property line is subjected to a total service axial load \( P_a = 200,000 \text{ lb (889.6 kN)} \), and the internal column \( B \) is subjected to a total service load \( P_b = 350,000 \text{ lb (1556.8 kN)} \). The live load is 35% of the total load.

![Figure 12.12 Footing reinforcement details of Ex. 12.4.](image-url)
load. The bearing capacity of the soil at the level of the footing base is 4000 lb/ft² (192.3 kPa), and the average value of the soil and footing unit weight \( \gamma = 120 \) psf (1822 kg/m³). A surcharge of 100 lb/ft² results from the slab on grade. Proportion the footing size and select the necessary size and distribution of the footing slab reinforcement and verify the development length required. Given:

\[
\begin{align*}
    f_c &= 3000 \text{ psi (20.7 MPa)} \\
    f_y &= 60,000 \text{ psi (413.7 MPa)}
\end{align*}
\]

base of footing at 7 ft below grade

Solution: total columns load \( P = 200,000 + 350,000 = 550,000 \) lb (2446.4 kN) net allowable soil capacity \( p_a = p_a = 120(7) \) psf (height to base of footing) \( = 100 \) psf

minimum footing area \( A_f = \frac{P}{p_a} = \frac{550,000}{100} = 5500 \text{ ft}^2 \)

Center of gravity of column loads from the property line:

\[
    x = \frac{200,000 \times 0.5 + 350,000 \times 20.5}{350,000} = 13.23 \text{ ft}
\]

length of footing \( L = 2 \times 13.23 = 26.46 \) ft

Use \( L = 27 \) ft.

width of footing \( s = \frac{17.8}{27.0} = 0.66 \) ft

Use \( s = 6 \) ft 6 in. as shown in Figure 12.13.

Factored shear and moments

- **column 1:**
  \[
  \begin{align*}
  P_1 &= 0.65 \times 200,000 = 130,000 \text{ lb} \\
  P_2 &= 200,000 - 130,000 = 70,000 \text{ lb} \\
  P_3 &= 1.5 \times 70,000 = 105,000 \text{ lb} \\
  P_4 &= 227,500 \text{ lb} \\
  P_5 &= 122,500 \text{ lb} \\
  P_6 &= 1.5 \times 227,500 + 1.6 \times 122,500 = 469,000 \text{ lb}
  \end{align*}
  \]

The net factored soil bearing pressure for footing structural design is
Figure 12.14  Shear diagram of footing in Ex. 12.5.

\[ \eta = \frac{P}{A_f} = \frac{268,000 + 469,000}{6.5 \times 20} = \frac{737,000}{127} = 5.83 \text{ ft/lb}^2 \]

Assume that the column loads are acting through their axes.

- Factored bearing pressure per foot width = \( \eta \times S = 5.83 \times 6.5 = 37.9 \text{ lb/ft} \)
- \( V_x \) at center line of column \( L = 268,000 - 27,300 \times \frac{6}{12} = 254,350 \text{ lb} \)
- \( V_y \) at center line of column \( R = 469,000 - 27,300 \times \frac{6}{12} = 451,350 \text{ lb} \)

The maximum moment is at the point C of zero shear in Figure 12.14 \((y=0)\) from the center of the left column \( L \):

\[ x = \frac{254,350 \text{ lb}}{27,300 \text{ lb/ft}} = 9.32 \text{ ft} \]

Taking a free-body diagram to the left of a section through \( C \), the factored moment at point \( C \) is

\[ M_x = \frac{w_y L}{2} - P_x x \]

\( M_x \) from left side = \( 27,300 \times \frac{(9.32 + 6.5) \times 2}{2} = 268,000 \times 9.32 \)

\[ = -1,181,458 \text{ ft-lb} = -14,177,496 \text{ in.-lb} \]  (Fig. 12.15)

\( M_x \) from right side = \( 27,300 \times \frac{(27.0 - 9.32) \times 2}{2} = 469,000 \times (20.0 - 9.32) \)

\[ = 980,090 \text{ ft-lb} = -11,761,080 \text{ in.-lb} \text{ less than } -14,177,496 \text{ in.-lb} \]

Hence \( M_x \) from the left side controls. Note that \( M_x \) from the right side differs from \( M_x \) from the left side because the footing length of 27 ft is used instead of the computed length of 26.46 ft and because \( x \) is rounded off. Therefore, the load is not exactly uniform due to the small eccentricity.

Figure 12.15  Free-body diagram.
Design of the footing in the longitudinal direction

(a) Shear. The combined footing is considered as a beam in the shear computations. Hence the critical section is at a distance \( d \) from the face of the support. Controlling \( V_s \) at the column center line

\[
V_s = \frac{291.5}{0.75} = 388.73 \text{ lb}
\]

Assume that the total footing thickness is \( 3 \) ft (0.92 m). The effective footing depth \( d = 32 \) in. for minimum steel cover \( c = 2 \) in. For the controlling column, the equivalent rectangular column size \( \sqrt{b_d h_d} = 13.29 \) in.

\[
\text{required } V_s = 388.73 - \frac{(13.29/2 + d)}{12} \times \frac{27,300}{2} = 271,456 \text{ lb (1208 kN)}
\]

\[
V_s = 2\sqrt{b_d h_d} = d = 2 \sqrt{300} \times 6.5 \times 12 \times 32
\]

\[
= 273,423 \text{ lb (1216 kN)} > 271,456 \text{ OK.}
\]

(b) Moment and reinforcement in the longitudinal direction (step 4). The distribution of shear and moment in the longitudinal direction is shown in Figure 12.16. The critical section for moment is taken at the face of the columns.

controlling moment \( M_c = \frac{M}{d} = \frac{14,117.46}{9} = 1,575,773 \text{ in.-lb (1854 kNm)} \)

\[
M_c = A_f \left( d - \frac{a}{2} \right)
\]

Assume that \( a = 0.6d \)

\[
15,752,773 = A_f \times 60,000 \times 0.9 \times 32
\]

or

\[
A_f = \frac{15,752,773}{60,000 \times 0.9 \times 32} = 8.11 \text{ in.}^2
\]

\[
\alpha = \frac{A_f h}{A_f h} = \frac{9.32 \times 60,000}{60,000} = 6.5 \text{ ft}
\]

\[
15,752,773 = A_f \times 60,000 \left( 32 - \frac{3.06}{2} \right)
\]

\[
A_f = 8.8 \text{ in.}^2 \text{ (20 bars, 1.91-mm diameter)}
\]

Use 20 No. 6 bars at the top for the middle span.

\( A_f = 8.8 \text{ in.}^2 \) (20 bars, 19.1-mm diameter)

Design of footing in the transverse direction

Both columns are treated as isolated columns. The width of the band should not be larger than the width of the column plus half the effective depth \( d \) on each side of the column. This assumption is on the safe side since the actual bending stress distribution is highly indeterminate. It is, however, possible to assume that the flasural reinforcement in the transverse direction can raise the shear punching capacity within the \( d/2 \) zone from the face of the rectangular left column \( L \) and the equivalent rectangular right column \( R \). Figure 12.17 shows the transverse band widths for both columns \( L \) and \( R \) determined on the basis of this discussion.

\[
\text{band width } h = \frac{b_y}{2} + \frac{32}{2} = 28 \text{ in.} = 2.33 \text{ ft}
\]

The rectangular column size equivalent to the circular interior 15-in.-diameter column = 13.29 in.
Figure 12.16 Longitudinal shear and moment distribution: (a) elevation; (b) shear; (c) moment.

- Band width $b_w = 13.52 + 2 \left( \frac{32}{2} \right) = 45.3$ in. $= 3.77$ ft

- Column transverse band reinforcement

  $\text{moment arm} = \frac{6.5 \times 6}{2} - \frac{18}{3 \times 12} = 2.58$ ft $= 30.0$ in.

The net factored bearing pressure in the transverse direction is

\[ q_s = \frac{768,000}{6.5} = 41,231 \text{ lb/ft} \]

\[ M_s = \frac{q_s}{2} = \frac{41,231 \times (2.5)^2}{2} = 129,847 \text{ ft-lb} = 1,546,164 \text{ in.-lb} \]

\[ M_s = \frac{M_s}{q_s} = \frac{1,546,164}{41,231} = 37.96 \text{ in.-lb} \]

\[ M_s = A_0 f_y \left( d - \frac{d}{2} \right) \]

\[ M_s = 1,717,964 \times 0.9 \times 32 \]

\[ A_0 = 1.0 \text{ in.}^2 \]
Figure 12.17 Footing transverse band widths.

\[ w = \frac{A_L f_c}{0.85/ \bar{f}_c} = \frac{1.9 \times 60,000}{0.85 \times 3000 \times 20} = 0.81 \text{ in.} \]

\[ 1717.96 \times 10^3 = A_i \times 60,000 \left( \frac{32 - 0.544}{2} \right) \]

\[ A_i = 0.91 \text{ in.}^2 \text{ (787 mm}^2) \]

\[ \text{min. } A_i = 0.80 \times 10^3, d = 0.0016 \times 28 \times 32 = 1.62 \text{ in.}^2 > 0.91 \text{ in.}^2 \]

\[ A_i = 0.82 \text{ in.}^2 \text{ actual} \]

Use six No. 5 bars, \( A_i = 1.86 \text{ in.}^2 \) (six bars, 15.9-mm diameter) equally spaced in the band, which is to be centered under the column.

Column K transverse band reinforcement

Equivalent square column size = 13.3 in. \( \times \) 13.3 in.

\[ \text{moment arm } = \frac{6 \text{ ft 6 in.}}{2} = 3.3 \text{ ft} = 39.3 \text{ in.} \]

Net factored bearing pressure in the transverse direction is

\[ f_s = \frac{469,000}{6.53} = 72,154 \text{ lb/ft} \]

\[ M_s = \frac{f_s}{2} = 72,154 \left( \frac{2.69}{2} \right) = 361,057 \text{ ft-lb} = 3,132,684 \text{ in.-lb} \]

\[ M_e = M_s / \Phi = 3,132,684 / 0.90 = 3,480,760 \text{ in.-lb} (393 \text{ kN-m}) \]

\[ M_e = A_i \left( d - \frac{d_i}{2} \right) \text{ assume that } d = \frac{d_i}{2} = 0.90 \]
or

\[
3,480.760 = A_c \times 60,000 \times 0.9 \times 32
\]

\[
A_c = \frac{3480.760}{60000 \times 0.9 \times 32} = 0.985\text{in}^2
\]

\[
a = \frac{A_c}{A_s} = \frac{0.985}{500} = 0.00197\text{in}
\]

\[
3,480.760 = A_x \times 60,000 \left(12 \times \frac{1.004}{2}\right)
\]

\[
A_x = \frac{3480.760}{60000 \times 12 \times 0.502} = 1.84\text{in}^2
\]

\[
r = \frac{1.84}{1186\text{mm}^2} = 0.0013 < \rho_{cr}
\]

where \(\rho_{cr} = 0.0018\) (shrinkage temperature reinforcement).

Minimum \(A_x = 0.0018 \times 45.3 \times 32 = 2.61\text{in}^2\)

Use nine No. 5 bars, \(A = 2.79\text{in}^2\) (tensile bars, 15.9-mm diameter) equally spaced.

Development length check for top bars in tension

\[
f_t = 3000\text{psi (20.7 MPa)}
\]

Top reinforcement more than 12-in. concrete below bars, \(a = 1.3\). From Eq. 10.6,

\[
f_t = \frac{3}{f_d} \frac{f_p}{\sqrt{a_d}}
\]

(a) **Longitudinal top bars.** Assume transverse reinforcing index \(K_s = 0.50\). Spacing \(c = 4.91\text{in.}\) \(d/d_s = 4.91/0.75 = 6.52\text{in.}\) use 2.5. No. 6 bar \(d_s = 0.75\text{in.}\) (19.1 mm).

\[
\phi_t = 1.7, \quad \phi_s = 1.0, \quad \alpha_t = 0.8, \quad \lambda = 1.0, \quad K_s = 0
\]

\[
\xi = 0.75 \times \frac{\frac{3}{40} \times 60000 \times \left(1.3 \times 1.0 \times 0.8 \times 1.0\right)}{2.5} = 25.7\text{in}
\]

\[
> \text{min } \xi = 12\text{in}
\]

(b) **Transverse bottom steel.** No. 5 bars, \(d_s = 0.625\)

\[
\phi_t = 1.0, \text{bottom steel}, \quad \phi_s = 1.0, \quad \alpha_t = 0.8, \quad \lambda = 1.0, \quad K_s = 0
\]

\[
c = 3.8\text{in.} \quad c/d_s = 3.8/0.625 = 6.1 > 2.5, \quad \text{use 2.5}
\]

\[
\xi = 0.625 \times \frac{3}{40} \times 60000 \times \left(1.0 \times 1.0 \times 0.8 \times 1.0\right) = 16.4\text{in}
\]

\[
\lambda = \frac{A_{\text{required}}}{A_{\text{provided}}}
\]

- Column \(L\) steel modifier: \(\lambda = \frac{1.62}{1.88} = 0.87\)
- Column \(R\) steel modifier: \(\lambda = \frac{2.00}{2.70} = 0.74\)

- Modified \(\xi = 16.4 \times 0.94 = 15.4\text{in.}\)

Available development length \(-32.35\text{in.}\) > 15.4 in. O.K.

Therefore, adopt reinforcement as in Figure 12.12. Check for dowel steel from the columns to the footing slab.
12.9 STRUCTURAL DESIGN OF OTHER TYPES OF FOUNDATIONS

From the discussion and the examples given in the foregoing sections, it is clear that the design of foundation substructures follows all the hypotheses and procedures used in proportioning the superstructures once the intensity and distribution of the soil bearing pressure is determined. If a cluster of piles supports a very heavy reaction through a pile cap, the analysis reduces to determining the punching load for each pile and determining the corresponding thickness of the cap. A determination of the center of gravity of the resultant of all pile forces has to be made if the system is subjected to bending in addition to axial load in order to choose the appropriate pile cap layout.

When raft foundations are necessary in poor soil conditions and deep excavations, the design of such a substructure is not too different from the design of any heavily loaded floor system. Once the soil pressure distribution is determined, the design becomes that of an inverted floor supported by deep beams longitudinally and transversely.

Variations are to be expected in the described foundation types, particularly in cases of specialized or unique structures. Through an understanding of the basic principles presented, the student and the designer should have no difficulty in utilizing the soil data developed by the geotechnical engineer in selecting and proportioning the appropriate foundation substructure.

SELECTED REFERENCES

PROBLEMS FOR SOLUTION

12.1. Design a reinforced concrete, square, isolated footing to support an axial column service live load \( P_s = 300,000 \text{ lb} (1334 \text{ kN}) \) and service dead load \( P_d = 625,000 \text{ lb} (2780 \text{ kN}) \). The size of the column is 10 in. × 10 in. (0.25 m × 0.25 m). The soil and concrete have a modulus of elasticity of 20,000 kN/m² and 20,000 MPa, respectively. Assume that the footing will be 1 ft below grade. Given:

- average weight of soil and concrete above the footing, \( \gamma = 130 \text{ psf} (6.4 \text{ kN/m}^2) \)
- footing depth, \( f = 3 \text{ ft} (1000 \text{ psf}) \)
- column depth, \( f_c = 4000 \text{ psf} (27.6 \text{ MPa}) \)
- footing depth, \( f = 60,000 \text{ psf} (413.7 \text{ MPa}) \)
- surcharge, \( s = 100 \text{ psf} (5.7 \text{ kPa}) \)

12.2. Design a reinforced concrete wall footing for (a) a 10-in. (0.25-m) reinforced concrete wall and (b) a 12-in. (0.30-m) masonry wall. The intensity of service live load \( W_s = 20,000 \text{ lb/ft} (292.9 \text{ kN/m}) \) and a service live load \( W_d = 12,000 \text{ lb/ft} (266 \text{ kN/m}) \) of wall length. Assume an evenly distributed soil bearing pressure and that the average soil bearing capacity is 3 tons/ft² (57.6 kN/m²). The frost line is assumed to be 2 ft below grade. Given:

- average weight of soil and footing above base, \( \gamma = 125 \text{ psf} (19.6 \text{ kN/m}^2) \)
- footing depth, \( f = 3000 \text{ psf} (20.7 \text{ MPa}) \)
- column depth, \( f_c = 5000 \text{ psf} (34.5 \text{ MPa}) \)
- footing depth, \( f = 60,000 \text{ psf} (413.7 \text{ MPa}) \)

12.3. A combined footing is subjected to an extension 16 in. × 16 in. (0.4 m × 0.4 m) columns shoring the property line carrying a total service load \( P_s = 300,000 \text{ lb} (1334 \text{ kN}) \) and an interior column 20 in. × 20 in. (0.5 m × 0.5 m) carrying a total factored load \( P_d = 400,000 \text{ lb} (1179 \text{ kN}) \). The live load is 30% of the total load. The center-line distance between the two columns is 22–2/3 ft (6.81 m). Design the appropriate reinforced concrete footing on a soil weighing 155 psf (21.2 kN/m²). The bearing capacity of the soil at the level of the footing base is 6000 lb/ft² (87.6 kN/m²). The frost line is assumed to be 3–3/4 ft (1.12 m) below grade. Assume a surcharge of 125 psf (1842 kN/m²) at grade level. Given:

- footing depth, \( f = 3500 \text{ psf} (21.4 \text{ MPa}) \)
- column depth, \( f_c = 5500 \text{ psf} (37.4 \text{ MPa}) \)
- footing depth, \( f = 60,000 \text{ psf} (413.7 \text{ MPa}) \)

12.4. Redesign the isolated reinforced concrete footing in Problem 12.1 if the load is applied at an eccentricity \( e = 0.5 \text{ ft} (0.15 \text{ m}) \) and (b) \( e = 1.8 \text{ ft} (0.55 \text{ m}) \).

12.5. Redesign the combined reinforced concrete footing in Problem 12.3 if the center-line distance between the two columns is 15–5/8 ft (4.6 m).
CONTINUOUS REINFORCED
CONCRETE STRUCTURES

13.1 INTRODUCTION

Proceeding chapters have covered the design and analysis of individual elements and isolated reinforced concrete sections. However, except for prestressed construction, concrete structural systems are constructed in continuous monolithic pours with the reinforcement well developed through adjoining spans and overlapping columns. Consequently, support sections and joints transmit flexural moments, thereby rendering a structure statically indeterminate in view of the continuity of the components. The three equations of equilibrium of forces and moments are no longer sufficient to solve for the unknown moments and reactions, and equations of geometry have to be developed to eliminate the indeterminacy.

Equations of geometry consider the deformation of the structure under load or stress, because the deformed shape of individual elements depends not only on load but also on the rotations and slopes of the element at its ends. The magnitudes of the slope and rotation depend on the rigidity (or flexibility) of the joint, that is, the relative rigidities of the adjoining members, which normally comprise horizontal beam elements and vertical column elements.

Photo 13.1 New York Hall of Science, Queens, New York. (Courtesy of Ammann & Whitney.)
13.1 Introduction

The equations of geometry ensure the compatibility of the deflections with the geometry of the structure, hence the name geometry conditions or compatibility conditions. An example of such a condition is that no deflection takes place at the intermediate support of a continuous beam, and the rotation is the same on both sides of that support. When the two adjacent spans are equal and similarly loaded, the manner of load application on one span affects the manner of deformation of the adjacent spans. It is essential to alternate live-load patterns on adjacent spans to give the maximum and minimum (reverse) moments and the resulting stresses in the various parts of the structural system.

Several methods of analysis of statically indeterminate structures have been developed over the years. Some of them are more exact than others, and some include approximations that facilitate relatively quick solutions when computer utilization is not readily possible or justified. The classical methods are based primarily on a physical understanding of the structural deformational behavior and are particularly important in interpreting the response of the system to the type and sense of applied load. The availability of computers transformed this need for understanding physical behavior to formulating a problem so that it can be understood by the computer through matrix formulation of the computations. In this manner, it becomes possible to keep track of a larger number of calculations and hence to be able to analyze more complex structural systems.

It is important for historical purposes to list the major methods of structural analysis. The common basic concept in these methods is either the force method with consistent deformation, for which the redundant forces are the primary unknowns, or the displacement method, for which displacements are the primary unknowns. The former is also termed the flexibility method and the latter is termed the stiffness method.

Emile Clapeyron's three moments theorem (1857) is a "force method" in which bending moments at supports are considered as the redundants to be evaluated by solving simultaneous equations whose number is equivalent to the number of indeterminacies. Carlo Casalino's energy method, embodying his second theorem of differential coefficients of internal work (1658), postulates that the partial differential of the internal work of an elastic system as a function of one of the external forces gives the relative displacement at the point of application of that force. Both force methods, while very powerful at the time, have the limitation of applicability to few spans with essentially unyielding supports and the necessity for exceedingly tedious computational effort. Another application of the force method is the elastic curve and the analogous column. In this method, the redundant are chosen as a point called the elastic center, involving computations similar to those for stresses in a column subjected to combined bending and direct stress.

The slope deflection concept is an example of the displacement method, where deflections and slopes are taken as the primary unknowns. It was developed independently by Axel Bendixen in Germany (1914) and George Manney in the United States (1915) and is the precursor of the moment distribution method of Hardy Cross (1929). It is worthwhile noting that it was originally developed because of the need to consider the secondary effects, that is, the internal bending moments and shears resulting from rotational restraints that develop in the bolted or welded joints.

The availability of computers for speedy solution of complex problems has decreased interest in use of the classical methods discussed thus far. The approaches here are more in the direction of formulating problems in such a manner that the computer can keep track of large quantities of numbers. Such formulating or bookkeeping can be achieved by use of matrix methods. Matrix formulations can be used for the "force method-method of consistent deformation" solutions through the use of the flexibility matrix. The displacement method can be applied through the use of the stiffness matrix. The unknown joint displacements are obtained by solving an equal number of simultaneous equations in matrix form. It should be noted at the outset that the matrix displacement method using stiffness matrices is the most powerful of the various methods of
13.2 LONGHAN DISPLACEMENT METHODS

13.2.1 Slope Deflection Method

This displacement method involves writing two equations for each span of a continuous structure, one at each end, expressing the end moments as the sum of the following four contributions:

1. The restraining moment resulting from an assumed fixed-end condition of the loaded span
2. The moment due to the rotation of the tangent to the elastic curve at one end of the span
3. The moment due to the rotation of the tangent to the elastic curve at the other end of the span
4. The moment resulting from the translation of one end of the span with respect to the other

The equations have to be set to conform to the requirements of equilibrium and compatibility at each joint of a continuous beam or a frame.

Consequently, a large set of simultaneous linear algebraic equations results for a total structural system, with displacements as the unknowns. For a structure with several spans or a high-rise building, the computational effort needed to solve the equations could be staggering and the probability of errors great. Use of this method is limited today, because other faster methods are available. Example 13.2 with two redundancies is given as an illustration.

13.2.2 Moment Distribution Method

This method is a numerical application of the displacement method in which the desired moments, shears, or stresses are obtained by a method of successive approximations suitable for longhand computations. The method lends itself to simple physical interpretation. Hence it can be used for quick approximate as well as exact solutions, depending on the number of successive cycles chosen. It is essentially an iterative solution of the slope deflection equations and has been used extensively since its development by Hardy Cross in the United States in 1919. The student and the designer are assumed to be well acquainted with this hand-computational method, and the reader is referred to the various texts on the subject of structural analysis given in the list of references to supplement the examples presented in this chapter using moment distribution.

13.3 FORCE METHOD OF ANALYSIS

13.3.1 The Force Method and the Flexibility Matrix

In this method, earlier forces or moments can be used as redundants. Moments will be used in this book as redundants. They are more convenient than forces in the analysis, particularly at the limit state of failure, as shown in A. L. L. Baker's method of "imposed rotations" presented in Section 13.7.2. Hence a somewhat more detailed discussion with an analysis example is presented here to fit the sequence of analysis methods. It is also
assumed that the reader has a background from other courses in structural analysis in both the force method and the matrix force method such that this discussion will serve only as a refresher on the topic.

In this method, cuts or hinges are inserted at suitable points in an indeterminate frame or continuous beam. As many supporting reactions or moments as necessary are removed until the structure becomes statically determinate, thereby facilitating the analysis. If the total strain energy $U$ with respect to the elastic redundant moment $X_i$ at any hinge $i$ is made equal to the elastic rotation at the hinge, that is,

$$\frac{\partial U}{\partial X_i} = -\theta_i$$

and if $\delta_0$ is assumed to represent the relative rotation of the $i$th hinge due to a moment at the $j$th hinge, then $\delta_0 = \delta_j$ from Maxwell's reciprocal theorem. The coefficients $\delta_0$ are called influence coefficients because they represent the displacement or rotation at a particular section due to a unit moment at another section, that is, $\delta_0 = -\theta_0$.

From the principle of virtual work,

$$\delta_0 = \sum \int_{A_i} \frac{M_j M_i}{EI} \, ds \quad (13.2)$$

The right side of Eq. 13.2 represents the integration of the products of the areas of the $M_j$ diagrams and the ordinates of the $M_i$ diagrams at the centroids of the $M_i$ diagrams along the horizontal distances along the span. In other words,

$$\delta_0 = \frac{A_i}{EI} \eta \quad (13.3)$$

where $A_i$ is the area under the primary $M_j$ bending moment diagram and $\eta$ is the ordinate of the $M_i$ moment diagram under the centroid of the $M_i$ diagram (Ref. 13.2). As an example, in Figure 13.1 the influence coefficient $\delta_0$ is obtained by superposing the moment

![Figure 13.1](image)

Figure 13.1 Influence coefficient determination from superposing $M_i$ and $X_i$: (a) primary structure moment; (b) redundant structure moment.
diagram $M_0$ of the primary structure on the moment diagram $X_i$ of the redundant structure created by the assumed developed hinge $i$.

$$A_i = \frac{2}{3} \frac{L}{l}$$

$\eta$ under the centroid of $M_0$ diagram $= c/2$.

$$\delta_{n} = \frac{1}{E I} \left( \frac{1}{2} \frac{L}{l} \right) \left( \frac{c}{2} \right) = -\frac{1}{3EI} \frac{c}{l}$$

$\delta_i$ is obtained by superposing the redundant structure $X_i$ on itself.

$$A_i = \frac{1}{2} \frac{L}{l}$$

$$\delta_{i} = \frac{1}{E I} \left( \frac{1}{2} \frac{L}{l} \times \frac{2}{3} c \right) = -\frac{1}{3EI} \frac{c}{l}$$

Table 13.1 gives the product integral values $\int M_i M_j ds$ for evaluating the influence coefficients $\delta_i$ for various combinations of primary and redundant moment diagrams. It can aid the designer in easily getting sets of equations 13.6 to follow.

Equation 13.2 can be rewritten as

$$\sum_{i=1}^{n} \frac{M_i M_j}{E I_i} ds = -\delta$$

(13.4)

Substituting $\delta_0$ and $\delta_0$ for $M_j$ in Eq. 13.4, the following expression is obtained:

### Table 13.1 Product Integral Values $\int M_i M_j ds$ for Various Moment Combinations

<table>
<thead>
<tr>
<th>$M_i$</th>
<th>$M_j$</th>
<th>$M_i M_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Diagram 1" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image2.png" alt="Diagram 2" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image4.png" alt="Diagram 4" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image5.png" alt="Diagram 5" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image6.png" alt="Diagram 6" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image7.png" alt="Diagram 7" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image8.png" alt="Diagram 8" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
<tr>
<td><img src="image9.png" alt="Diagram 9" /></td>
<td>$\frac{1}{2} Lc$</td>
<td>$\frac{1}{6} Lc^2$</td>
</tr>
</tbody>
</table>

\[ \frac{1}{2} Lc (a + c) \]
\[ \delta_0 + \sum_{i=1}^{n} \delta_i X_i = -\delta_0 \]  \hspace{1cm} (13.5)

where \( \delta_0 \) is the net elastic rotation.

Hence, to solve for a structure having \( n \) degrees of indeterminacy, it should satisfy the following conditions:

\[ \delta_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots + \delta_n X_n = -\delta_0 = 0 \]
\[ \delta_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots + \delta_n X_n = -\delta_0 = 0 \]
\[ \delta_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots + \delta_n X_n = -\delta_0 = 0 \]  \hspace{1cm} (13.6)

The number of equations in a set is equal to the number of redundancies. In matrix form, the structure flexibility matrix \( [B] \) can be defined for \( n \) loading conditions as:

\[ [B] = \begin{bmatrix}
\delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\
\delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{n1} & \delta_{n2} & \cdots & \delta_{nn}
\end{bmatrix} \begin{bmatrix}
X_{11} \\
X_{12} \\
\vdots \\
X_{1n}
\end{bmatrix} \hspace{1cm} (13.7a) \]

or in shorter form:

\[ [B][X] = [\delta] \hspace{1cm} (13.7b) \]

Solving for the unknown redundants by inversion of the \( [B] \) matrix,

\[ [X] = [B]^{-1} [\delta] \hspace{1cm} (13.8) \]

The parameters \( \delta_1, \ldots, \delta_n, X_1, \ldots, X_n, \delta_1', \ldots, \delta_n' \), can best be described in Figure 13.2 for a typical two-span continuous beam \( ABC \) with a hinge introduced at interior support \( B \).

![Figure 13.2 Reduction of indeterminate beam through introduction of hinge as redundant: (a) continuous beam elevation; (b) primary structure; (c) redundant structure.](image)
due to moment $X$, causing a rotation $\theta$, at this support. The beam is subjected to a single external loading condition of uniformly distributed load.

### 12.3.2 Example 13.1: Analysis of Two-span Continuous Beam by the Force and Matrix Methods

Consider the two-span continuous prismatic beam $ABC$ in Figure 13.3 having a fixed moment at the right end $C$. Solve for the moments at $B$ and $C$ due to a uniform load of intensity $w$ per unit length of span using (a) the force method (the method of consistent deformations) and (b) the matrix force method.

![Figure 13.3](image)

**Figure 13.3** Statically indeterminate beam in Ex. 13.1: (a) structure elevation; (b) primary structure ($M$); (c) redundant structure $X_1 = -1$; (d) redundant structure $X_2 = -1$; (e) final resisting moments.
Solution: (a) Method of consistent deformations. Since the beam is prismatic, the $EI$ values are not included for simplification.

\[ b_1 + X_b h_1 + X_c h_2 = 0 \quad (a) \]
\[ b_2 + X_b h_1 + X_d h_2 = 0 \quad (b) \]

From Eq. 13.3, $b_1 = (A_k / E) / \eta$; hence

\[ b_1 = 2 \left( \frac{w L^2}{E} \times L \times \frac{3}{2} \right) \left( \frac{1}{3} \right) = -\frac{w L^3}{12} \]
\[ b_2 = \left( \frac{w L^2}{E} \times L \times \frac{3}{2} \right) \left( \frac{1}{3} \right) = -\frac{w L^3}{12} \]
\[ b_{11} = 2 \left( \frac{1 \times L}{2} \right) \left( -\frac{1}{3} \right) = \frac{2L}{3} \]
\[ b_{12} = \left( \frac{1 \times L}{2} \right) \left( -\frac{2}{3} \right) = -\frac{L}{3} \]
\[ b_{22} = \left( \frac{1 \times L}{2} \right) \left( -\frac{2}{3} \right) = -\frac{L}{3} \]

Substituting the values of $b_1$ through $b_{22}$ in Eqs. (a) and (b),

\[ \frac{w L^3}{12} = \frac{2L}{3} X_1 + \frac{L}{6} X_2 = 0 \quad (c) \]
\[ -\frac{w L^3}{12} + \frac{L}{3} X_1 + \frac{L}{3} X_2 = 0 \quad (d) \]

Solving Eqs. (c) and (d) simultaneously gives

\[ X_1 = \frac{3}{20} w L^2 = 0.15w L^2 \]
\[ X_2 = \frac{2}{20} w L^2 = 0.0714w L^2 \]

Hence

\[ M_x = -0.15w L^2 \quad \text{and} \quad M_c = -0.0714w L^2 \]

The balance of moments at midspan and the support reactions can simply be found from statics.

(b) Matrix force method. The structure flexibility matrix $\{F\}$ from part (a) of the solution is

\[ \{F\} = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \]

The matrices $\{A\}$ and $\{b\}$ are, respectively,

\[ \{A\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]
\[ \{b\} = \begin{bmatrix} -\frac{w L^2}{12EI} \\ -\frac{w L^2}{24EI} \end{bmatrix} = -\frac{w L}{24EI} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]

The flexibility matrix is inverted to yield...
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\[
\begin{pmatrix}
F_1 \\
F_2
\end{pmatrix} = \begin{pmatrix}
\frac{6EI}{L} & 2 & -1 \\
-2 & 4
\end{pmatrix} \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} + \begin{pmatrix}
[\Delta] \\
-\Delta
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} = \begin{pmatrix}
\frac{6EI}{L} & 2 & -1 \\
-2 & 4
\end{pmatrix} \begin{pmatrix}
0 \\
0
\end{pmatrix} + \begin{pmatrix}
\frac{6\Delta L^2}{28} \\
\frac{2\Delta L^2}{28}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{3\Delta L^2}{28} \\
-\frac{\Delta L^2}{28}
\end{pmatrix}
\]

Hence

\[
M_x = -\frac{3wL^2}{28} = -0.1076wL^2
\]

\[
M_y = -\frac{2wL^2}{28} = -0.0714wL^2
\]

13.4 DISPLACEMENT METHOD OF ANALYSIS

13.4.1 Displacement Method and the Stiffness Matrix

The displacement method is analogous to the force (deformation) method except that the nodal displacements are considered as the unknowns instead of the redundant forces or moments. Essentially the slope deflection method, it can be considered the direct link to computer methods of structural analysis. Since the joint displacements represent the freedom to move or rotate, the term "degrees of freedom" represents the joint displacements as a measure of the kinematic degrees of indeterminacy.

A set of equilibrium equations equal to the number of unknown displacements has to be solved in order to determine these unknown displacements. The computational operation involves (1) computation of the force-displacement or moment-rotation relationships, that is, the stiffness; (2) setting up the equilibrium equations in order to determine the unknown displacements or rotations; and (4) calculation of the forces or moments by substituting the displacements or rotations computed in (3) in the force-displacement or moment-rotation relationship established in (1).

In matrix form, we must establish the kinematic degrees of freedom \( n \) to be used in the solution. Next, the static matrix \([A]\) and the deformation matrix \([D]\) have to be established using basic concepts, to be followed by a visual check to ensure that the matrix \([B] = [A]^T\); that is, \([B]\) is the transpose of \([A]\). The member stiffness matrix \([S]\) is then computed. The fixed-end moments \([M_e]\) are also computed, and the external forces (moments) matrix \([P]\) is established in which the elements of the \([P]\) matrix are the reversals of the forces (moments) acting on the member ends in the fixed condition.

Combining the equilibrium conditions as in Ref. 13.3,

\[
\begin{pmatrix}
\Phi_x \\
\Phi_y
\end{pmatrix}_{n \times 1} = \begin{pmatrix}
[A] & [D]
\end{pmatrix}_{n \times 2} \begin{pmatrix}
[M_e] \\
M_1
\end{pmatrix}_{2 \times 1}
\]

the moment-rotation relationships,

\[
\begin{pmatrix}
[M_e]_{n \times 2} \\
[M_1]
\end{pmatrix} = \begin{pmatrix}
[S]_{n \times 2} & [\Phi]\end{pmatrix}_{n \times 2}
\]

\[
\begin{pmatrix}
\Phi_x \\
\Phi_y
\end{pmatrix}_{n \times 1} = \begin{pmatrix}
[A] & [D]
\end{pmatrix}_{n \times 2} \begin{pmatrix}
[M_e] \\
M_1
\end{pmatrix}_{2 \times 1}
\]

(13.9)

(13.10)
and the compatibility conditions,
\[ (\theta)_{\text{ex}} = (\delta)_{\text{ext}} + (X)_{\text{det}} \]  
(13.12)

and using the inverse of the global stiffness
\[ [K] = [AS\text{A}^T]_{\text{sym}} \]  
(13.13)

where the inverse of the matrix is
\[ [K]^{-1} = [AS\text{A}^T]_{\text{sym}}^{-1} \]  

the following two equations of the joint displacement (rotation) matrix \([X]\) and internal force (moment) matrix \([M]\) respectively, are obtained:

\[ (X)_{\text{ex}} = [AS\text{A}^T]_{\text{sym}}^{-1} \{P\}_{\text{ext}} \]  
(13.14)

\[ (M)_{\text{ex}} = [AS\text{A}^T]_{\text{sym}}^{-1} \{P\}_{\text{ext}} \]  
(13.15)

The final end moments \((M)_{\text{ex}}\) are the sum of the moments for the joint fixed condition \((M)_{\text{f}}\) and joint rotation condition \((M)_{\text{rot}}\) of Eq. 13.14, so

\[ (M)_{\text{ex}} = (M)_{\text{f}} + (M)_{\text{rot}} \]  

13.4.3 Example 13.2: Analysis of Two-span Continuous Beam

by the Displacement and Matrix Displacement Methods

Solve Ex. 13.1 for the moments \((M)_{\text{f}}\) and \((M)_{\text{rot}}\) at the supports of the continuous beam using (a) the displacement method and (b) the matrix displacement method.

**Solution:** (a) Displacement method. This method is the longhand slope deflection method and hence is expected to be cumbersome if the number of redundants is large. As seen from Figure 13.3, the structure is statically indeterminate to the second degree; it has two redundants. The longhand solution would have been considerably quicker if the second displacement method of moment distribution could be used. Figure 13.4 shows the displacement (rotation) and the free-body diagrams of segments AB and BC of the structure.

The basic slope deflection equations are

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{4EI}{L} \theta_1 \]  
(13.15a)

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{2EI}{L} \theta_2 \]  
(13.15b)

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{4EI}{L} \theta_3 \]  
(13.15c)

Since the joints are not rotating, \(\Delta = 0\). Hence Eqs. 13.15a and b become

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{4EI}{L} \theta_1 \]  
(13.17a)

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{2EI}{L} \theta_2 \]  
(13.17b)

\[ (M)_{\text{f}} = (M)_{\text{rot}} = \frac{4EI}{L} \theta_3 \]  
(13.17c)

Writing the two joint equilibrium equations corresponding to two degrees of freedom, we have

\[ (M)_{\text{f}} = 0 \quad M_2 + M_3 = 0 \quad M_4 = 0 \]  
(13.18)

As a sign convention in Figure 13.4, \(M_1\), \(M_2\), \(M_3\), and \(M_4\) as unknowns are considered to act clockwise on member ends and counterclockwise on the joints.

1. **Fixed-end moments**

\[ M_1 = \frac{wL^3}{12} \quad M_2 = \frac{wL^3}{12} \]

\[ M_3 = \frac{wL^3}{12} \quad M_4 = \frac{wL^3}{12} \]
Figure 13.4 Slope deflection solution of Ex. 11.2: (a) continuous beam elevation; (b) fixed-end moments; (c) free-body diagrams with unknown moments; (d) free-body diagrams representing degrees of freedom \( f \_j \) and \( \theta \_j \) rotations.

(2) Slope deflection moments: From Eqs. 13.17a and b,

\[
M_j = \frac{w L^2}{12} + \frac{4EI}{L} \theta_j + \frac{2EI}{L} \theta_i \quad (13.19a)
\]

\[
M_j = \frac{w L^2}{12} + \frac{2EI}{L} \theta_j \quad (13.19b)
\]

\[
M_j = \frac{w L^2}{12} + \frac{2EI}{L} \theta_j + \frac{8EI}{L} \theta_i (j = 0) \quad (13.19c)
\]

\[
M_j = \frac{w L^2}{12} + \frac{2EI}{L} \theta_j + \frac{4EI}{L} (j = 0) \quad (13.19d)
\]

where \( \theta_i = 0 \) because support C is fixed.

(3) Joint equilibrium moment: Substituting the slope deflection moments of Eq. 13.19 into Eq. 13.14 yields.
13.4 Displacement Method of Analysis

\[ M_1 = \frac{wL^2}{12} + \frac{2EI}{L} \theta_a + \frac{2EI}{L} \theta_b = 0 \]  
(13.20a)

\[ M_2 + M_3 = \frac{2EI}{L} \theta_a - \frac{2EI}{L} \theta_b = 0 \]  
(13.20b)

\[ \tau_c = \frac{wL^2}{12} + \frac{2EI}{L} \theta_a = 0 \]  
(13.20c)

Solving for Eqs. 13.20a and b, we get:

\[ \theta_a = \frac{1}{42} wL^3 \]

\[ \theta_b = \frac{1}{42} wL^3 \]

(4) Final moments. Substituting the values of \( \theta_a \) and \( \theta_b \) into Eqs. 13.19, the following final moment values are obtained:

\[ M_1 = 0 \]

\[ M_2 = \frac{3}{28} wL^3 \]

\[ M_3 = \frac{3}{28} wL^3 \]

\[ M_4 = \frac{1}{28} wL^2 \]

Hence, \( M_1 = 0, M_2 = 0.107wL^3, \) and \( M_4 = 0.009wL^2 \).

(b) Stiffness matrix solution. Figure 13.5 gives the moments and forces on the joints of the continuous beam ABC identifying the matrix notations used in the solution.

(1) Static matrix \([A]\): From Figure 13.5a, the equilibrium of joints where \( P_1 = M_1 \) and \( P_2 = M_2 + M_1 \) gives static equilibrium in matrix form as follows:

\[
[A]_{2 \times 4} = \\
\begin{bmatrix}
    M \\
    P
\end{bmatrix} = \\
\begin{bmatrix}
    1 & 2 & 3 & 4 \\
    \frac{1}{10} & \frac{1}{10} & \frac{1}{10} & \frac{1}{10}
\end{bmatrix}
\]

(2) Deformation matrix \([B]\): This matrix relates the rotations (internal deformations) to joint displacements. From the geometry of Figure 13.5d, the deformation matrix is

\[
[B]_{3 \times 4} = \\
\begin{bmatrix}
    \theta \\
    0
\end{bmatrix} = \\
\begin{bmatrix}
    1 & 2 \\
    1 & 1 & 1 & 1 \\
    2 & 1 & 1 & 1 \\
    3 & 1 & 1 & 1 \\
    4 & 1 & 1 & 1
\end{bmatrix}
\]

where \( \theta_2 = X_1 \) and \( \theta_1 = X_2 \); when \( X_1 \neq 0 \) and \( X_2 = 0 \)

\( \theta_2 = X_2 \) when \( X_1 \neq 0 \) and \( X_2 = 0 \)

It should be emphasised that after establishing matrix \([B]\) independently from matrix \([A]\) a check is made that one is the transpose of the other.
Figure 13.5 Details of the matrix formulations in Ex. 13.2: (a) external P-X diagram; (b) internal M-E diagram; (c) fixed-end moments $M_{g}$; (d) compatibility of internal deformations (rotations) with joint displacement; (e) equilibrium moments at joints; (f) fixed-end moments at joints. Sign convention: Moment causing compression at bottom face is positive.

(3) Member stiffness matrix [5]: This matrix is obtained from the pair of Eqs. 13.17a and b and the slope deflection solutions given in Eqs. 13.19 of the previous solution, where

\[ M_{i} = \frac{4EI}{L} \theta_{i} \quad \frac{2EI}{L} \theta_{i} - \frac{wL^{2}}{12} \]

\[ M_{1} = \frac{2EI}{L} \theta_{1} \quad \frac{4EI}{L} \theta_{i} + \frac{wL^{2}}{12} \]

\[ M_{2} = \frac{4EI}{L} \theta_{2} - \frac{wL^{2}}{12} \]

\[ M_{3} = \frac{4EI}{L} \theta_{3} \quad \frac{wL^{2}}{12} \]
Hence

\[
[S]_{12} =
\begin{bmatrix}
1 & 2 & 3 & 4 \\
1 & \frac{4EI}{L} & \frac{2EI}{L} & L \\
2 & \frac{2EI}{L} & \frac{4EI}{L} & L \\
3 & \frac{4EI}{L} & L & \frac{2EI}{L} \\
4 & \frac{2EI}{L} & L & \frac{4EI}{L}
\end{bmatrix}
\]

(4) **External force (moment) matrix** \([P]\): From the fixed-end moments in Figure 13.5c and the rotational equilibrium of joints A and B in Figure 13.5f, the net reverse moments on the joints are

\[
P_1 = -\left(\frac{wL^3}{12}\right) + \frac{vL^4}{12} \\
P_2 = \left(\frac{wL^3}{12}\right) - \frac{vL^4}{12} = 0
\]

\[
(P)_{12} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 + \frac{wL^3}{12} \\ 2 \end{bmatrix}
\]

(5) **[SA] matrix** \(= [S]\): Transpose \([A]^T = [P]\); hence

\[
[A] = [S]^T =
\begin{bmatrix}
1 & 2 \\
1 & \frac{4EI}{L} & \frac{2EI}{L} & L \\
2 & \frac{2EI}{L} & \frac{4EI}{L} & L \\
3 & 0 & \frac{4EI}{L} & L \\
4 & 0 & \frac{2EI}{L} & L
\end{bmatrix}
\]

(6) **Global stiffness matrix** \([K]\): The complete stiffness matrix from Eq. 13.12

\[
[K] = [A]^T [A];
\]

\[
[K]_{12} =
\begin{bmatrix}
1 & 2 \\
1 & \frac{4EI}{L} & \frac{2EI}{L} & L \\
2 & \frac{2EI}{L} & \frac{4EI}{L} & L
\end{bmatrix}
\]
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Note that the global stiffness matrix \([K]\) is the same as the factors of the pair of equilibrium Eqs. 13.24a and b in the long-beam slope deflection solution of part (a) in this problem. (7) Inverse of the stiffness matrix \([K]^{-1}\):

\[
[K]^{-1} = \begin{bmatrix}
1 & 2 \\
1 & \frac{2L}{7EI} & -\frac{L}{14EI} \\
2 & -\frac{L}{14EI} & \frac{L}{7EI}
\end{bmatrix}
\]

(8) Joint displacement matrix \([X]\): The joint displacement matrix \([X]\) from Eq. 13.13 is the product of the pre-multiplier inverse stiffness matrix \([K]^{-1}\) and the post-multiplier external moment matrix \([P]\): \([X] = [K]^{-1} \cdot [P]\).

The matrix operation gives:

\[
X_1 = \frac{2L}{7EI} \times \frac{wL^3}{12} - 0 = \frac{wL^2}{43EI}
\]

\[
X_2 = -\frac{L}{14EI} \times \frac{wL^3}{12} = -\frac{wL^2}{4 \times 42EI}
\]

and in matrix format, the joint displacement matrix would thus be:

\[
[X]_{1,3} = \begin{bmatrix}
1 & \frac{wL^2}{43EI} \\
-\frac{wL^2}{4 \times 42EI}
\end{bmatrix}
\]

(9) Internal moment matrix \([M]\): From Eq. 13.14, the internal moment matrix is the product of the \([K][P]\) pre-multiplier matrix and the joint displacement \([X]\) post-multiplier matrix giving:

\[
M_1 = \frac{1}{12} wL^3
\]

\[
M_2 = \frac{1}{42} wL^3
\]

\[
M_3 = -\frac{1}{42} wL^3
\]

\[
M_4 = -\frac{1}{64} wL^3
\]

and in matrix form:

\[
[M]_{1,4} = \begin{bmatrix}
1 & \frac{1}{12} wL^3 \\
2 & \frac{1}{42} wL^3 \\
3 & -\frac{1}{42} wL^3 \\
4 & -\frac{1}{64} wL^3
\end{bmatrix}
\]
13.5 Approximate Analysis of Continuous Beams and Frames

The final end moments \( [M_e] \) from Eq. 13.15 in matrix form are the sum of the fixed-end moments \( [M_e] \) and the internal moments \([M_i]\). In matrix form, the final moment values are as follows:

\[
[M_{ie}]_{2\times1} = [M_{ie}]_{1\times1} + [M_{ie}]_{2\times1}
\]

or

\[
\begin{bmatrix}
\frac{1}{12} wL^3 \\
\frac{1}{12} wL^3 \\
\frac{1}{12} wL^3 \\
\frac{1}{12} wL^3 \\
\frac{1}{42} wL^3 \\
\frac{1}{84} wL^3
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
\frac{3}{28} wL^3 \\
\frac{3}{28} wL^3 \\
\frac{2}{28} wL^3
\end{bmatrix}
\]

Hence

\[
M_a = 0
\]
\[
M_b = \frac{3}{28} wL^3 = 0.107wL^3
\]
\[
M_c = \frac{2}{28} wL^3 = 0.072wL^3
\]

13.5 FINITE-ELEMENT METHODS AND COMPUTER USAGE

The finite-element method is an extension of the matrix displacement method in which the body or structure to be analyzed is modeled as an assembly of finite elements interconnected at specified nodal points. The difference between the two methods is the choice of the stiffness matrix, permitting the inclusion of different types of elements into the analysis. As a result, the solution is greatly facilitated through convergence using the computer. The accuracy of the stiffness matrices can be distinctly improved by the introduction of additional nodes along the length of the member or in the plane of a planar element, depending on the degree of accuracy needed.

Numerous canned computer programs for structural analysis are available using the matrix displacement method or the finite-element method. FORTRAN standard programming language is widely used in such problem-oriented programs as STRESS, discussed in Ref. 13.4; PSCST finite-element programs, presented in Ref. 13.6; ANSYS finite-element program for three-dimensional analysis (Ref. 12.7), the PCA ADOS program in BASIC language for the design of slabs and plate systems as part of continuous frames (Ref. 13.8); SAP 2000, STAAD Pro, ADAPT, and others. The reader would do well to become familiar with the available programs and acquire the background knowledge for the use of computer methods in the solution of highly indeterminate continuous structures and high-rise building frames.

It should be noted, however, that the designer must always execute computational checks on the computer output. Such checks can be accomplished through the use of the classical structural analysis methods on small subassemblies of elements or individual elements. Hence the preceding presentations in Sections 13.1 to 13.4 are a necessary refresher for conducting such checks for which handheld computers can often be adequate.
13.6 APPROXIMATE ANALYSIS OF CONTINUOUS BEAMS AND FRAMES

13.6.1 Idealization Principle

The use of computers has facilitated the rapid analysis of continuous structures with high
degrees of indeterminacy, giving relatively exact solutions. This advantage has come
through the digital computational process, applying matrix methods and finite-element
techniques and utilizing the revolutionary advances in the hardware and software capa-
bilities and speed of today's desktop computers. However, providing a detailed com-
putational analysis, the elastic properties of the members have to be assumed as an input
requirement. These include modulus of elasticity, cross-sectional area, cross-sectional
moment of inertia, and the height of members. All these parameters are needed for es-

tablishing preliminary stiffness values for the beams and columns. Also, the pattern of
load distribution that can give the worst loading conditions has to be set by the design en-
gineer. Hence approximate structural analysis has to be initially performed with the ap-
propriate idealizations prior to embarking on an “exact” solution using the computer. In
many instances for moderate-sized structures, the approximate solution is often sufficient
since the input of stiffness values into a computer involves idealizations and assumptions
based on engineering judgment and isolation of controlling segments of an indeterminate
structure to arrive at a preliminary solution.

Photo 13.2 Chicago Mercantile Exchange: a high-stress concrete unique van-
deur supports an office tower. (Courtesy of Robert B. Johnson, Allied Brinsh
and Co., Chicago.)
Taking the simple case of a fixed-end beam as in Figure 13.6a, the elastic curve of the beam changes slope at a distance of 0.211L from the fixed supports; thereby creating inflection points; points of zero moments at points C and D in the span. Consequently, AC and BD can be treated as cantilever beams, and segment CD can be considered a simply supported beam of span L/2 = 0.578L and solved by simple statics.

Frames can be treated in a similar manner through location of the inflection points. Figure 13.7 shows two portal frames, one with a hinged base and the other with a fixed base subjected to gravity loading. Note that the bending moment diagrams are consistently drawn on the tension side of the members. The bending moment at midspan of the horizontal top member would be the difference between the moment at B and C and the total static moment mL/8. If the same frame is subjected to horizontal wind forces, the inflection points and the resulting bending moments are as shown in Figure 13.8.

On the basis of the foregoing discussion for gravity and wind loads, a multistory frame can thus be idealized as shown in Figure 13.9a for gravity loading and in Figure 13.9b for wind loading, with the inflection points located at the sections where a change
in curvature direction takes place. Figure 13.9b is uniquely suitable for approximate analysis due to horizontal wind loads assumed concentrated at joints.

13.6.1.1 Portal Method of Wind Loading Frame Analysis. Figure 13.9(b) shows that the structure is divided into portals because moments are taken as zero at midspans of the horizontal members and at midheight of the vertical members, rendering the entire structure statically determinate. This method of analysis for wind on frames is termed the portal method. It is based on the following assumptions:

1. All wind loads are transferred to the joint.
2. Shear resisted by each exterior column is assumed to be one-half that resisted by each interior column.
Figure 13.8 Idealization of portal frame through location of inflection points.
wind loading: (a, c) frame elevation; (b, d) deflected shape; (e, f) bending moment diagrams.

3. The total horizontal shear in the columns of a given story is equal to the total lateral force above that story.

4. Inflection points, equivalent to hinges occur at midspan of beams and at column midheight except in basement levels. In that case, it can be assumed to occur at about one-fourth to one-third of the column height above the foundation.

These assumptions are based on the fact that the total shearing force due to wind at any floor level can be divided among the columns at that level in proportion to their stiffnesses, and the vertical reactions on the columns due to wind can be considered to be proportional to their distance from the center of the building. Such assumptions are true only if the beams are infinitely stiff relative to the columns. Yet they are nearly correct in most cases and do provide an adequate safety factor, considering that \( f \) values for
Figure 13.9 Idealization of continuous structures for approximate analysis: (a) gravity loading; hinges assumed at midspan from column support for preliminary analysis; (b) wind loading; (c) alternative idealization of multistory frame for gravity loading; (d) single-story multispans symmetrical portals; (e) portal idealization of structure in (d); (f) portal unit: ABC.
columns are constant between any two floors and the $I_i$ values for beams are in most cases constant throughout the frame. It must also be remembered that the ratio of stiffnesses of the beams to the columns does not significantly affect the value of the ultimate load because excessive stiffness of the columns is eliminated by plastic yield before failure.

Figure 13.9c shows the idealized portion of a high-rise building for approximate analyses where fixity of columns can be assumed at the $n + 1$ and $n - 1$ floors. The end moment values of the beam $AB-BC-CD$ are not drastically affected by this approximation, and the moment coefficients for continuous beams on knife-edge supports can be used in the approximate analysis. Figure 13.9d and e show an approximation procedure for a symmetrical one-story frame under gravity loading. Note that the replacement of the portal intermediate columns by fixed ends at column locations $C$ transformed the structure into simple and essentially single bay portals.

Figure 13.10 shows a portal subjected to wind load $P$ and having equal spans. Inflection points are at midheights and at midspans. It is seen that axial loads due to horizontal wind force $P$ occur at the exterior columns only, since the combined tension and compression due to the portal effect results in zero axial load in the interior columns. If the spans are unequal, wind loading $P$ would cause axial loads not only on the exterior columns but also on the columns between the unequal spans.

The general expression for axial load in the exterior columns of an $n$-bays frame with unequal spans is

$$\frac{P_i}{2m_i} \text{ and } \frac{P_i}{2m_i}$$

![Diagram](image-url)

**Figure 13.10** Three-bay, equal-span portal frame wind analysis: (a) horizontal shears and axial loads, (b) moments at joint B (inflection points at midheights and midspans).
The axial load in the first interior columns is \( P_{li} = \frac{P_{li}}{2} \), and in the second interior columns is \( P_{li} = \frac{P_{li}}{2} \).

As shown in Figure 15.6, column moments are determined by the column shear times one-half the column height. Consequently, for point B of the portal frame, the column moment becomes \( M_{li} = \frac{1}{2} P_{li} \) at the column level. This moment must be balanced by equal moments in \( BD \) and \( BC \) of a magnitude \( \frac{P_m}{2} \) without considering their relative stiffnesses. The shear in beams \( AB \) and \( BC \) is then determined by dividing the beam end moments by one-half the beam length. In this case, the end shears become \( \frac{P_{li}}{2} \), giving a value of \( \frac{P_{li}}{2} \) at each floor level. Summation is then made of the beam shears and column load values as we proceed from the top floor to the foundation floor level.

Example 15.1 illustrates the use of the portal frame method for the analysis of forces in an indeterminate frame due to wind loading.

It should be noted that the classical portal frame method is a conservative way of rapid analysis that can serve as a quick check of computer solutions. Canned computer programs such as PCA-Frame program, STAAD Professional, Sap 2000 and subsequent editions, and others are available for accurate analysis of indeterminate multi-story frames subjected to gravity loading plus wind and earthquake forces.

15.6.2 Indeterminate Frames and Portals

15.6.2.1 General Properties. Concrete frames are indeterminate structures consisting of horizontal and vertical or inclined members joined in such a manner that the connection can withstand the stresses and bending moments that act on it. The degree of indeterminacy depends on the number of spans, number of vertical members, and type of end reactions. Typical frame configurations are shown in Figure 15.11. If \( k \) is the number
of joints, b the number of members, r the number of reactions, and \( s \) the number of indeterminacies, the degree of indeterminacy is determined from the following inequalities:

\[
3n + r > 3b + r \\
3n + r = 3b + r \quad \text{statically determinate} \\
3n + r < 3b + r \quad \text{statically indeterminate}
\]

The degree of indeterminacy is

\[ s = 3b + r - 3n \]

where \( 3n \) equations of static equilibrium are always available and the total number of unknowns is \( 3b + r \).

As an example, the degree of indeterminacy of the frame in Figure 13.11a is

\[ s = 3 \times 3 + 2 \times 2 - 3 \times 4 = 1 \]

For the frame in Figure 13.11g,

\[ s = 3 \times 10 + 2 \times 3 - 3 \times 9 = 9 \]

For a frame to perform satisfactorily, the following conditions must be satisfied:

1. The design should be based on the most unfavorable moment and shear combinations. If moment reversal is possible due to reversal of live-load direction, the highest values of positive and negative bending moments have to be considered in the design.
2. Proper foundation support for horizontal thrust has to be provided. If the frame is designed as hinged, which is an expensive construction procedure, an actual hinge system has to be provided.

13.6.2.2. Forces and Moments in Portal Frames. The behavior of concrete frames before cracking can reasonably be considered elastic, as was done in the case of continuous beams at service-load and slight overload conditions. Consequently, well before the development of plastic hinging, the bending moment diagrams shown in Figures 13.12 and 13.13 can easily be used in the design of indeterminate reinforced concrete frames. It has to be assumed that the student or the design engineer is well versed in these procedures as a basic background, and only the minimum guidelines and simplifications are presented in this book.

13.6.2.3. Uniform Gravity Loading on Single-Bay Portal. Assuming that the moments of inertia \( I \) of the vertical columns and \( J \) of the horizontal beam of the portal in Figure 13.14a are not equal, the following values of the moments and shears can be deduced:

**End shear in beams**

\[
V_b = V_C = -\frac{1}{2} \frac{wI}{J}
\]

**Horizontal thrust**

\[
H = \frac{1}{h} C_1 wI
\]

where

\[
C_1 = \frac{1}{12 \left( \frac{2}{J} \frac{h}{b} \right)}
\]

Maximum negative moment at beam-column junction

\[
M_a = M_i = -hh_L = -C_1 wI^2
\]
Figure 13.12 Right-angle portal frame loaded with gravity load intensity \( w \) (T indicates tension fibers): (a) load intensity; (b) bending moment (hinged base frame); (c) bending moment (fixed base frame); (d) deformation of frame (b); (e) deformation of frame (c).

Figure 13.13 Right-angle portal frame loaded with wind load intensity \( w \) (T indicates tension fibers): (a) bending moment (hinged base frame); (b) bending moment (fixed base frame); (c) deformation of frame (a); (d) deformation of frame (b).
Figure 13.14 - Bending moment ordinates in single-bay frames: (a) uniform gravity loading, (b) concentrated gravity loading, (c) uniform horizontal pressure.

Maximum passive moment at midspan

\[ M_{\text{max}} = \frac{1}{8} Wh^2 - Kh = \left(\frac{1}{8} - C\right) Wh^2 \]  

(13.23a)

Bending moments at any point \( X \)

\[ M_x = \frac{1}{2} x(t - x)w - C \mu \rho \]  

(13.23b)

Where points of contrallexus from either corner of the portal are

\[ x_i = \frac{1}{2} \left( \sqrt{4(l - K^2)} - l \right) = C_i l \]  

(13.23c)

And

\[ C_i = \frac{1}{2} \left( 1 - \sqrt{1 - K^2} \right) \]  

(13.23d)
13.6.2.4. Concentrated Gravity Loading on Single-Bay Portal. Since the concentrated load $P$ does not have to act at midspan, non-symmetry of shear results. The end shears from Figure 13.3a are

$$V_0 = \left(1 - \frac{a}{l} \right) P \quad \text{and} \quad V_1 = \frac{a}{l} P \quad (13.24a)$$

**Horizontal thrust**

$$H = C_T \frac{a}{l} \left(1 - \frac{a}{l} \right) \left(1 - \frac{a}{l} \right) \frac{1}{h} \quad (13.24b)$$

where

$$C_T = \frac{1}{1 + \left( \frac{I_T}{h} \right)} \quad (13.24c)$$

**Bending moments at corners**

$$M_0 = M_C = -H_0 = -C_T \frac{a}{l} \left(1 - \frac{a}{l} \right) Pi \quad (13.24d)$$

**Bending moments at any point along BC; for $x < a$**

$$M_x = \left(1 - \frac{a}{l} \right) \frac{x}{l} \left(1 - \frac{a}{l} \right) C_T Pi \quad (13.24e)$$
For \( x > a \),
\[
M_x = \frac{q}{2} \left[ 1 - \frac{x}{L} - \left( 1 - \frac{a}{L} \right) \frac{C_3}{L} \right] \frac{PL}{L}
\]  
(13.24a)

Maximum positive moment at \( x = a \)
\[
M_{ma} = \frac{q}{2} \left( 1 - \frac{a}{L} \right) \frac{PL}{L} = (1 - C_3) \frac{q}{2} \left( 1 - \frac{a}{L} \right) \frac{PL}{L}
\]  
(13.24b)

Horizontal thrust for several compressed gravity loads
\[
H = \frac{1}{h} C_1 \left[ P_1 \frac{h}{L} \left( 1 - \frac{a}{L} \right) + P_2 \frac{h}{L} \left( 1 - \frac{a}{L} \right) + \cdots \right]
\]  
(13.24c)

or
\[
H = \frac{1}{h} C_1 \sum P_i \frac{h}{L} \left( 1 - \frac{a}{L} \right)
\]  
(13.24d)

13.6.2.5 Uniform Horizontal Pressure on Single-Bay Portal. From Fig. 13.14c,

Vertical reactions at supports
\[
R_A = -\frac{1}{2} ph \frac{h}{L} \quad \text{and} \quad R_D = +\frac{1}{2} ph \frac{h}{L}
\]  
(13.25a)

Horizontal reactions: For windward hinge, \( A \),
\[
H_A = \frac{11}{8} \frac{h}{L} \frac{h}{L} + 18 \left( \frac{P h}{2} \right) = C_4 ph
\]  
(13.25b)

where
\[
C_4 = \frac{11}{8} L \frac{h}{L} + 18 \left( \frac{P h}{2} \right) + 3
\]  
(13.25c)

For leeward hinge, \( D \),
\[
H_D = ph - H_A = (1 - C_4) ph
\]  
(13.25d)

Bending moments at any point \( y \) along the column height due to horizontal pressure, with \( y \) being measured from the bottom
\[
M_y = H_{y} \frac{y}{2} \left( \frac{h}{L} \right)^2
\]  
(13.25e)

Maximum moment at windward column
\[
M_{max} = \frac{1}{2} \left( \frac{11}{8} \frac{h}{L} \frac{h}{L} + 18 \left( \frac{P h}{2} \right) \right) \frac{1}{2} \left( \frac{1}{2} C_4 \right) \frac{P h^2}{L}
\]  
(13.25f)
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Point of maximum bending moment above support A

\[ y_A = \frac{1}{6} \left( \frac{11 L_1 h + 18}{L_1 h / l + 3} \right) h = C_A h \]  

(13.25g)

Bending moments in corners of portal

\[ M_A = H_A h - \frac{1}{2} ph^2 = \frac{1}{2} \frac{L_1 h}{L_1 h / l + 3} - \frac{1}{2} \frac{L_1 h}{L_1 h / l + 3} \]

\[ M_C = -H_A h = -(1 - C_C) ph^2 \]  

(13.25b)

(13.25c)

The constants \( C_1, C_2, C_3, \) and \( C_4 \) in Eqs. 13.23, 13.24, and 13.25 can be graphically presented as in Figure 13.17. Cased computer programs for the analysis of indeterminate
13.6 Approximate Analysis of Continuous Beams and Frames

beams and frames render the use of charts such as Figure 13.15 unnecessary except for a quick check of numerical values.

13.6.3 Example 13.3: Forces and Moments in a Warehouse Portal Frame

A warehouse structure is constructed of a single-bay portal frame hinged at the base. The frame has a clear span of 80 ft (24.4 m) and is built in 8-ft segments. It is subjected to a uniform gravity load intensity \( w_g = 360 \text{ plf (5.5 kN/m)} \) and a horizontal uniform wind pressure of intensity \( p_w = 65 \text{ plf (0.95 kN/m)} \) on the windward side and a surcharge of intensity \( p_s = 40 \text{ plf (0.58 kN/m)} \) at the leeward side, as shown in Figure 13.16. Compute the shears, moments, and reactions that would be needed for the design of the structure. Assume that \( L = 8 \text{ ft} \), and that the self-weight of the horizontal top member \( w_{c, w} = 600 \text{ plf (8.8 N/m)} \).

**Solution:** For horizontal beam BC, gravity loading

From Eq. 13.23c,

\[
C_s = \frac{1}{2 \left( \frac{1}{l} + 1 \right)}
\]

where

\[
l = \frac{L}{l} = \frac{36}{8} \times \frac{10}{60} = 0.05
\]

From Figure 13.15, \( C_s = 0.056 \). Assume a 2-in. concrete topping and 1-in. insulation and water proofing.

\[
\begin{align*}
\gamma_c & = \left( \frac{8}{12} \times 150 + 6 \right) \times 240 = 240 \text{ plf over 6-ft-wide strip} \\
w_c & = 1.2(200 + 240) + 1.6 \times 260 = 1584 \text{ plf (23.1 kN/m)}
\end{align*}
\]

From Eq. 13.24d,

\[
M_{e, w} = M_{e, c} = -w_c \times \gamma_c \times -w_c \times \gamma_c
\]

\[
= -7.79 \times 10^3 \text{ in.-lb (280 kNm)}
\]

From Eq. 13.33e,

Maximum \( M_e \) at midspan = \( \frac{1}{8} \times C_s \times \frac{w_c^2}{2} \)

\[
= \left( \frac{1}{8} - 0.056 \right) 1584(80)^3 \times 12
\]

\[
= +7.42 \times 10^5 \text{ in.-lb (859 kNm)}
\]

Columns vertical reactions \( R_A = R_C = \frac{w_c L}{2} = \frac{1584 \times 80}{2} = 63,360 \text{ lb (282 kN)} \)

\[
m_s = 240 \text{ plf}
\]

Figure 13.16 Warehouse frame elevation.
Column top moments and reactions: wind loading

Windward $p_1 = 45 \times 1.7 = 110.5$ plf
Leeward $p_2 = 40 \times 1.7 = 68.0$ plf

From Eqs. 13.25h and 13.25i,

\[ M_{w} = (C_k - 0.5)p_1 l^2 \]

\[ M_{c} = -(1 - C_k)p_2 l^2 \]

From before,

\[ k = \frac{l - h}{l} = 0.45 \text{ for } l = 1, \]

From Figure 13.15, $C_k = 0.73$.

Windward side moment $M_w$

\[ M_{w1} = (0.73 - 0.5)(10.5)(36)^2 \times 12 = 395,254 \text{ in.-lb} \]

\[ M_{w2} = (1 - 0.73)(68)(0.36)^2 \times 12 = 285,535 \text{ in.-lb} \]

Total $M_w = 395,254 + 285,535 = 680,789 \text{ in.-lb} (76.9 \text{ kNm})$

Leeward side moment $M_c$

\[ M_{c1} = -(1 - 0.73)(10.5)(36)^2 \times 12 = -463,994 \text{ in.-lb} \]

\[ M_{c2} = -(0.73)(68)(0.36)^2 \times 12 = -243,233 \text{ in.-lb} \]

Total $M_c = -463,994 - 243,233 = -707,227 \text{ in.-lb} (79.9 \text{ kNm})$

Controlling wind moment $M_{w,c} = 707,227 \text{ in.-lb}$ since wind can blow from either left or right.

By superposition of moments due to gravity on the moment due to wind,

Maximum $M_t = \max (M_c, M_t) = 7.99 \times 10^6 - 707,227 = 7.08 \times 10^6 \text{ in.-lb}$

Maximum midspan moment = $7.42 \times 10^6 \text{ in.-lb}$

Vertical support reactions: From Eq. 13.25a

\[ R_s = \frac{1}{2} p_1 h = \frac{(110.5 + 68.0)(26)^2}{2 \times 80} = -1446 \text{ lb} \]

\[ R_o = -R_s = 1446 \text{ lb (6.4 kN)} \]

Total $R_s$ due to gravity and wind load = $63,360 + 1446 = 64,806 \text{ lb (288 \text{ kN})}$. Hence design the vertical supports for a combined $p_1 = 64,806 \text{ lb}$ and $M_s = 7.08 \times 10^6 \text{ in.-lb}$. Note that the axial load value is so small compared to the moment magnitude that the design of the vertical supports would be governed by flexure. The design of the beam $BC$ and verticals $AB$ and $CD$ for flexure and shear would be accomplished in accordance with the discussions in Chapters 3 and 6. Check also for moment redistribution from support to midspan if necessary, as discussed in the ACI Code section on moment coefficients.

13.6.4 Loading

The first step in analyzing frames is determining the service loads and wind stresses, as required in the general building code under which the project is to be designed and constructed, such as ANSI standard A58-1 (Ref. 13.25). Dead load includes member
13.6 Approximate Analysis of Continuous Beams and Frames

self-weight, weight of fixed service equipment, such as electrical and plumbing, and the weight of built-in partitions, which is normally taken as 20 psf at service level. Live loads include loads due to movable objects and movable partitions. The uniformly distributed live loads range between 40 psf for residential use and 450 psf for heavy manufacturing and warehouse storage. Portions of buildings such as library stacks and film rooms require substantially heavier live loads. Live loads on roofs include maintenance equipment, snow loads, ponding of water, and landscaping where applicable. If concentrated live loads have to be included, they would more likely affect individual supporting members and do not generally need to be included in the frame analysis.

Live-load reduction for the design of beams, slabs, and columns is generally allowed in most building codes to account for the probability that the total floor area influencing the load on an individual element may not be fully loaded. For example, the influencing area for an interior column is the total area of the four surrounding bays, while for an edge column, the influencing area is the two adjacent bays, and for a corner column, it is one bay only.

The reduced live load \( L_r \), per square foot of floor area supported on columns, beams, and two-way slabs having an influence area of more than 400 ft² is

\[
L_r = L \left( 0.25 + \frac{15}{\sqrt{A_i}} \right)
\]

(13.26)

where \( L = \) unreduced service live load

\( A_i = \) influencing area

The reduced live load cannot be taken less than 50% for members supporting one floor or less than 40% of the unit live load \( L \).

13.6.5 ACI Moment Coefficients

13.6.5.1. Center-Line Moments. The ACI building code allows moment and shear coefficients for rapid analysis of standard buildings of usual types of spans, construction, and story heights. Table 13.2 represents these coefficients. The limitations are as follows:

1. Maximum allowable ratio of live to dead load is 2:1.
2. Maximum allowable span difference should be such that the larger of the two adjacent spans does not exceed the shorter by more than 20%.

It should be noted that the values in Table 13.2 are somewhat conservative, and economy can be achieved by more precise analysis of the multistory structural system. Figure 13.17 gives a graphical representation of the moment and shear coefficients of Table 13.2 for a typical three-span structure under various support conditions. Moment coefficients for continuous beams and alternative loadings where ACI moment coefficients cannot be used are given in Table 13.3.

13.6.5.2. Redistribution of Negative Moments in Continuous Non-Prestressed Flexural Members. Except where approximate values for moments are used, negative moments calculated by the elastic theory at supports of continuous flexural members on each be increased or decreased because of the redistribution of moment, with a decrease in the negative moment at the support, and a corresponding increase in the positive moment in the midspan.

The redistribution of negative moments can be made only when the section at which the moment is reduced is so designed that \( \epsilon_i = 0.0075 \leq 0.020 \).
Table 13.2  ACI Moment and Shear Coefficients\(^a\)

<table>
<thead>
<tr>
<th>Positive moment</th>
<th>Negative moment at exterior face of first interior support</th>
<th>Negative moment at other faces of interior supports</th>
<th>Negative moment at all supports (1) slabs with spans not exceeding 10 ft and (2) beams and girders where rise of sum of column stiffnesses to beam stiffness exceeds 8 at each end of the span</th>
<th>Negative moment at interior faces of exterior supports for members built integrally with their supports</th>
<th>WHERE THE SUPPORT IS A SPANDED BEAM OR GIRDER</th>
<th>WHERE THE SUPPORT IS A COLUMN</th>
<th>Shear in end members of first interior support</th>
<th>Shear at all other supports</th>
</tr>
</thead>
<tbody>
<tr>
<td>End span</td>
<td>If discontinuous end is unrestrained</td>
<td>1.0(w_i f_c)</td>
<td></td>
<td>(1.0w_i f_c)</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(1.12\frac{w_i l_i}{2})</td>
<td>(\frac{w_i l_i}{2})</td>
</tr>
<tr>
<td></td>
<td>If discontinuous end is integral with the support</td>
<td>(1.0w_i f_c)</td>
<td></td>
<td>(1.0w_i f_c)</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(1.12\frac{w_i l_i}{2})</td>
<td>(\frac{w_i l_i}{2})</td>
</tr>
<tr>
<td>Interior span</td>
<td></td>
<td>(1.0w_i f_c)</td>
<td></td>
<td>(1.0w_i f_c)</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(1.12\frac{w_i l_i}{2})</td>
<td>(\frac{w_i l_i}{2})</td>
</tr>
<tr>
<td>Negative moment at exterior face of first interior support</td>
<td>Two spans</td>
<td>1.0(w_i f_c)</td>
<td></td>
<td>1.0(w_i f_c)</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(1.12\frac{w_i l_i}{2})</td>
<td>(\frac{w_i l_i}{2})</td>
</tr>
<tr>
<td></td>
<td>More than two spans</td>
<td>(1.0w_i f_c)</td>
<td></td>
<td>1.0(w_i f_c)</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(\frac{1.0w_i f_c}{2})</td>
<td>(1.12\frac{w_i l_i}{2})</td>
<td>(\frac{w_i l_i}{2})</td>
</tr>
</tbody>
</table>

Sources: Ref. 13.13.

\(w_i\) = total factored load per unit length of beam or percent area of slab

\(L_i\) = shear span for positive moment and shear and the average of the two adjacent shear spans for negative moment

The code permits decreasing the negative moments at the supports for continuous members by not more than 1000\(e_1\)% with a maximum of 20%. The reason is that for ductile members, plastic hinge regions develop at points of maximum moment and cause a shift in the elastic moment diagram. The result is a reduction of the negative moment and a corresponding increase in the positive moment. The redistribution of the negative moment as permitted by the code can only be used when \(e_1\) is equal or greater than 0.0075 in./lin. at the section at which the moment is reduced. This redistribution is logically inapplicable to working stress design or to slab systems designed by the direct design method (DDM).

Figure 5.7 in Chapter 5 shows the permissible moment redistribution for minimum rotational capacity.

13.6.5.3. Effective Span Moments. The usual assumption in frame analysis postulates that the members are prismatic, having constant moment of inertia between center lines. In reality, a beam stops to have constant section at the face of the column support, and its moment of inertia is greatly increased as it approaches the column center line. To account for this discrepancy in the analysis, the moments and shears have to be adjusted by the change in moment values between the support center line and the support face, as shown in Figure 13.18. The adjusted moment value can be determined by mapping the area of the shear diagram between the column face and its center line. This area can be taken as \(\frac{1}{2} Val\) for the knife-edge support assumption of \(\frac{1}{2} Val\) for the finite support area as usually exists.
Figure 13.17 ACI moment and shear coefficients: (a) positive moments, (d) negative moments, (c) negative moments, slabs with span < 10 ft; (d) negative moments, slabs with spand columns 2 Kc/2 Kc > 3; (e) support shears, all cases.
Table 13.3 Bending Moments and Shear Diagrams for Continuous Beams

<table>
<thead>
<tr>
<th>Table 13.3 Bending Moments and Shear Diagrams for Continuous Beams</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1. CONTINUOUS BEAM – THREE EQUAL SPANS – ONE END SPAN UNLOADED</strong></td>
</tr>
<tr>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>Moment: 0.450 ft kips, Shear: 0.103 kips</td>
</tr>
</tbody>
</table>

| **2. CONTINUOUS BEAM – THREE EQUAL SPANS – END SPANS LOADED** |
| ![Diagram](image2) |
| Moment: 0.412 ft kips, Shear: 0.103 kips |

| **3. CONTINUOUS BEAM – THREE EQUAL SPANS – ALL SPANS LOADED** |
| ![Diagram](image3) |
| Moment: 0.412 ft kips, Shear: 0.103 kips |

| **4. CONTINUOUS BEAM – FOUR EQUAL SPANS – THIRD SPAN UNLOADED** |
| ![Diagram](image4) |
| Moment: 0.354 ft kips, Shear: 0.061 kips |

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Chapter 13 Continuous Reinforced Concrete Structures
### Table 13.3 Continued

5. **Continuous Beam—Four Equal Spans—Load First and Third Spans**

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
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<td>$R_3$</td>
<td>$R_4$</td>
<td>$S_1$</td>
</tr>
<tr>
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<td>0.4046</td>
<td>0.4546</td>
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</tr>
</tbody>
</table>

6. **Continuous Beam—Four Equal Spans—All Spans Loaded**

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_1$</td>
<td>$R_2$</td>
<td>$R_3$</td>
<td>$R_4$</td>
<td>$S_1$</td>
</tr>
<tr>
<td>0.4546</td>
<td>0.4046</td>
<td>0.4546</td>
<td>0.4046</td>
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</tbody>
</table>

7. **Continuous Beam—Two Equal Spans—Concentrated Load at Center of One Span**

<table>
<thead>
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<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
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<td>$R_1 + V_1$</td>
<td>$V_2$</td>
<td>$R_3 + V_3$</td>
<td>$V_4$</td>
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</table>

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{max}$ (at point of load)</td>
<td>$M_{max}$ (at point of load)</td>
<td>$M_{max}$ (at point of load)</td>
<td>$M_{max}$ (at point of load)</td>
</tr>
<tr>
<td>$11P$</td>
<td>$11P$</td>
<td>$11P$</td>
<td>$11P$</td>
</tr>
</tbody>
</table>

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### Diagrams

- **Continuous Beam—Four Equal Spans—Load First and Third Spans**
- **Continuous Beam—Four Equal Spans—All Spans Loaded**
- **Continuous Beam—Two Equal Spans—Concentrated Load at Center of One Span**
- **Continuous Beam—Two Equal Spans—Concentrated Load at Any Point**
Chapter 13  Continuous Reinforced Concrete Structures

13.6.6 Example 13.4: Analysis for Continuity by Moment Distribution

For continuous beams and one-way slabs not meeting the ACI moment coefficient conditions of Table 13.1 and for two-way action slabs and plates as part of a structural framing system, analysis by the other methods discussed in Sections 13.1 to 13.5 becomes necessary. The moment distribution method discussed in Section 13.2.2 will be used in the following illustrative example. Except in unique cases, building-frame and continuous beam moments computed by adjusting the fixed-end moments by two cycles of moment distribution are sufficiently accurate for rapid design purposes.

Problem statement. A flat plate floor of a multifloor framing system is shown in Figure 13.19. The end panels center-line dimensions are 17'6" x 30'-0" (5.33 m x 9.14 m), and the interior panel dimensions are 34'-0" x 20'-0" (10.36 m x 6.10 m). The floor heights, h, of intermediate floors are typically 8'-0" (2.44 m). Compute the factored controlling moments needed for the design of a typical floor panel to withstand service live load \( w_L = 40 \text{ psf} (1.92 \text{kN/m}^2) \) and a superimposed dead load \( w_D = 26 \text{ psf} \) due to partitions and flooring. Assume that the slab thickness, h, is 7 in. (17.8 cm) and that all panels are simultaneously loaded by the live load in the solution. Use the equivalent frame method for determining the stiffness of the members intersecting at the panel supports.

Solution: Equivalent frame characteristics

Take the equivalent frame in the N-S direction whose plane is shown in the shaded portion in Figure 13.19. The approximate stiffness of the column above and below the floor joint (moment per unit rotation) can be approximated by
Figure 13.19 Flat-flat apartment structure in Ex. 13.4.

\[ K_c = \frac{4E_b L_s}{L_c - 2b} \]  \hspace{1cm} (13.27)

where \( L_s = 7.00 = 105 \text{ in.} \)

Exterior column (14 in. × 12 in.) stiffness

\( b = 14 \text{ in.}, \quad I_c = 14(12)^2/12 = 1055 \text{ in.}^4 \)

Assume that

\[ \frac{E_{sl}}{E_{so}} = \frac{E_{so}}{E_{so}} = 1.0 \]

Use \( E_{so} = E_{so} = 1.0 \) in the calculations because \( E_{sl} \) drops out in the equation for \( K_c \).

Total \( K_c \) = \( \frac{4 \times 1 \times 2016}{105 - (2 \times 6.5)} \times 2 \) (for top and bottom columns)

= 175.3 in.\text{lin/ft} / E_{so}

Torsional constant \( C \) = \[ \frac{1}{3} \left( 1 - \frac{1}{6} \right) \left( \frac{5}{2} \right) \left( \frac{5}{2} \right) \]

= \left( 1 - \frac{1}{6} \right) \times \frac{5^2}{2} \times \frac{5^2}{3} = 724

Torsional stiffness of the slab at the column line is
\[ K_s = \sum \frac{9EJ_c}{k_d (L - c_i)^2} \]
\[ = \frac{9 \times 1 \times 724}{20 \times 12(1 - 14/(12 \times 20))^2} + \frac{9 \times 1 \times 724}{20 \times 12^2(1 - 14/(12 \times 20))^2} \]
\[ = 65 \text{ in.-lb/rad/}E, \]

The equivalent column stiffness is
\[ K_c = \left( \frac{1}{K_s} \right)^{1/2} = \left( \frac{1}{175.3} + \frac{1}{65} \right)^{1/2} = 47 \text{ in.-lb/rad/}E. \]

Interior column (14 in. x 20 in.) stiffnesses
\[ b = 14 \text{ in.}, \quad l = \frac{14(20)^3}{12} = 9333 \text{ in.}^3 \]

Total \( K_s = \frac{4 \times 1 \times 9333}{105 - 2 \times 6.5} = 812 \text{ in.-lb/rad/}E \)

\[ C = (1 - 0.65 \times 6.5/20) \times (6.5 \times 20/3) = 1456 \]

\[ K_s = \frac{9 \times 1456}{20 \times 12(1 - 14/(12 \times 30))^2} + \frac{9 \times 1456}{20 \times 12^2(1 - 14/(12 \times 20))^2} \]
\[ = 131 \text{ in.-lb/rad/}E, \]

\[ K_c = \left( \frac{1}{812} + \frac{1}{131} \right)^{1/2} = 115 \text{ in.-lb/rad/}E. \]

Slab stiffnesses: From Eq. 13.37,
\[ K_s = \frac{4EJ_s}{L_s} \]

where \( L_s = \) center-line span
\( c_s = \) column depth

Slab band width in E-W direction = 20/2 + 20/2 = 20 ft.

\[ I_s = 20 \times \frac{12(6.5)^3}{12} = 5493 \text{ in.}^4 \]

Slab at right of exterior column A:
\[ K_s = \frac{4 \times 1 \times 20(6.5)^3}{12 \times 17.5 - 12/2} = 108 \text{ in.-lb/rad/}E. \]

Slab at left of interior column B:
\[ K_s = \frac{4 \times 1 \times 20(6.5)^3}{12 \times 17.5 - 20/2} = 110 \text{ in.-lb/rad/}E. \]

Slab at right of interior column B:
\[ K_s = \frac{4 \times 1 \times 20(6.5)^3}{12 \times 24 - 20/2} = 110 \text{ in.-lb/rad/}E. \]

Slab distribution factor at points: \( DF = K_s/E, \) where \( \Sigma K = K_s + K_{	ext{slab}} + K_{	ext{slab}} \)

Outer joint A slab \( DF = \frac{110}{108} = 108 \)

Outer joint B slab \( DF = \frac{110}{79} = 108 \)

Inner joint C slab \( DF = \frac{110}{79} = 108 \)
13.8 Approximate Analysis of Continuous Beams and Frames

left joint B slab DF = \( \frac{110}{113 + 118 + 79} = 0.364 \)

right joint B slab DF = \( \frac{79}{113 + 118 + 79} = 0.262 \)

Two-cycle moment distributions:

slab self-weight \( w_s = \frac{7}{12} \times 150 = 88 \text{ psf} \)

\( w_s = 1.2(40 + 20) + 1.6 \times 40 = 201 \text{ psf} \)

Exterior spans AB and CD

fixed-end moment, FEM = \( \frac{w_s L^2}{12} \)

\( = \frac{201(17.5)^2}{12} \times 12 = 61556 = 61.56 \times 10^3 \text{ in.-lb/ft} \)

Interior span BC

fixed-end moment, FEM = \( \frac{201(24)^2}{12} \times 12 = 115776 = 115.78 \times 10^3 \text{ in.-lb/ft} \)

Running a moment distribution analysis as shown in Table 13.4, a carryover factor COF can be used for all spans. This assumption is justified because the effect of nonplanar actions would have negligible bearing on the fixed-end moments and carry-over factors. It also be assumed in multistory, multispand frames that the frame at a joint two spans away from the left joint (joint C) can be considered fixed in the distribution of moments. Hence, the slabs at interior supports B and C have to be designed for factored moments \( M_s = 97.5 \text{ in.-lb/ft} \) for spans BA and CD and \( M_s = 112,440 \text{ in.-lb/ft} \) for span BC. These moments can be adjusted for values at column face.

Proportioning of the reinforcement after similar evaluations for the E-W direction can be made similar to the discussion presented in Chapter 11, including proportioning for shear moment transfer at column junctions.

| Table 13.4 Two-cycle Moment of Distribution of Factored Moments \( M_s \) |
|---|---|---|---|---|
| | A | B | C | D |
| DF | 5.997 | 0.364 | 0.293 | 0.267 | 0.364 |
| COF | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| FEM (x 10^3) \( \text{in.-lb/ft} \) | -5156 | +81,500 | -115,780 | +115,780 | -61,560 |
| Distribution | +42,11 | +19,74 | +14,21 | -14,21 | +61,56 |
| CD | 0.87 | +21.46 | -7.14 | -8.22 | -3.76 |
| Final \( M_s \) \( (x 10^3 \text{ in.-lb/ft}) \) | -15.66 | +97.54 | -112.44 |
13.6.7 Example 13.5: Analysis of Wind Moments and Shears in a Multistory Frame by the Portal Method

The structure in Ex. 13.4 is four stories in height plus a basement, as shown in Figure 13.20. It is subjected to a wind intensity of 38 psf (1800 Pa). Assume all floor heights as well as the basement height to be the same: 8'-9" (2.67 m). Compute the moments and shears in the first- and second-story columns and the second-floor interior beams on lines A, B, C, or D. Assume that the building roof has a 2'-6" (76 cm) parapet wall.

Solution: External forces

Figure 13.20 shows the idealized structure and the inflection points for a portal frame analysis due to wind. The inflection points (virtual hinges) are at midheights of columns and at midspans of beams except for the basement, where the column hinges are one-third basement height above the foundation.

The wind forces are based on the tributary areas and are concentrated at the joints. Hence portal frame analysis is applicable.

half-floor height = 8'-9"/2 = 4.375 ft. Parapet height = 2'-0".
load factor = 1.6
wind force at roof joint = 1.6(4.375 + 2.020 ft x 38 psf) = 7752 lb
wind force at each floor joint = 1.6(4.375 + 20 x 38) = 10,640 lb

total shear in the second story is equivalent to the sum of the horizontal shear forces above that level = 7752 + 3 x 10,640 = 29,032 lb (129 kN)

total shear in the first story = 7752 + 3 x 10,640 = 39,672 lb (176 kN)

Second-story columns: shears and moments

shear in exterior columns, \( V_{p} = \frac{29,032}{6} = 4,839 \text{ lb} \)

shear in interior columns, \( V_{p} = 2 \times 4,839 = 9,678 \text{ lb} \)

moment in exterior columns, \( M_{p} = \frac{4,839 \times 8.75}{2} = 21,171 \text{ ft} \cdot \text{lb} \)

moment in interior columns, \( M_{p} = \frac{9,678 \times 8.75}{2} = 42,341 \text{ ft} \cdot \text{lb} \)

First-story columns: shears and moments

shear in exterior columns, \( V_{p} = \frac{39,672}{6} = 6,612 \text{ lb} \)

shear in interior columns, \( V_{p} = 2 \times 6,612 = 13,224 \text{ lb} \)

moment in exterior columns, \( M_{p} = \frac{6,612 \times 8.75}{2} = 28,926 \text{ ft} \cdot \text{lb} \)

moment in interior columns, \( M_{p} = \frac{13,224 \times 8.75}{2} = 57,855 \text{ ft} \cdot \text{lb} \)

Moments and shears in interior beams on lines A, B, C, and D

Beams in bay AB or CD

\[ M_{p} = M_{b} + M_{b} = 21,171 + 28,926 = 50,097 \text{ ft} \cdot \text{lb} / \text{bay} \]

\[ V_{p} = 50,097 / 17.5 \times 0.5 = 5726 \text{ lb} / \text{bay} \]
Figure 13.20 Portal frame wind analysis of structure in Ex. 13.4: (a) idealized structure frame with inflection points; (b) shear forces on second-story columns.

**Beams in Bay BC**

\[ M_{l} = (M_{1} + M_{2}) - M_{w} = (42.341 - 57.855) = -15.514 \text{ ft-kb} \]

\[ V_{l} = \frac{50.099}{5} = 10.019 \text{ lb/bay} \]

These shears and moments would have to be combined with the gravity load moments and shears computed by the frame analysis of Ex. 13.4. Axial loads on columns are computed similarly as discussed in Section 13.6.1. Since bay spans are not equal, not small axial loads on the interior columns due to wind loading have to be computed and added to those due to gravity loads and adjustments made in shears and moments. It should be noted that the numerical computations can become exceedingly cumbersome for tall buildings with a large
number of bays. In such cases, the use of computers in evaluating the axial forces, shear, and
moment becomes necessary.

The horizontal and vertical shear forces computed in this example for the second-floor
portals are shown in Figure 13.100. For wind blowing from left to right, the total axial tensile
force due to wind in column $A$ and the axial compressive force due to wind in column $D$ are
equal to the sum of all the vertical shears in the beam at all the floors and the roof in bays
$AB$ and $CD$, respectively. Note that the bending moment diagram in Figure 13.20b due to
wind is drawn on the tension side of the member.

**Note:** The portal method is an approximate and rapid method of analyzing the mome-
tnts and shears due to wind in multistory frames. Exact analysis taking into account the
$P$-δ effects on the columns can be achieved by the utilization of canned computer programs,
as explained in Section 9.13.

### 13.7 Limit Design (Analysis) of Indeterminate Beams and Frames

The discussions presented in this chapter thus far entail the elastic analysis of indetermi-
inate beam and frame systems with examples for "exact" solutions as well as by approxi-
mate methods based on the geometrical behavior of the structure. Also, redistribution
factors $p_i$ for continuity are introduced where permissible provided that adequate longi-
tudinal reinforcement is provided at the critical continuity zones to control the cracking
levels of such zones.

These procedures do not necessarily give the most efficient solution to a statically
indeterminate continuous beam or frame, since full redistribution at ultimate load is not
considered. As the applied load is gradually increased until the structure as a whole
reaches its ultimate capacity, the critical sections, such as the supports or corners of frames,
develop severe cracking, and rotations become so large that for practical purposes rota-
ting plastic hinges develop. If the number of plastic hinges that develops equals the
number of the indeterminacies, the structure becomes determinate, as full redistribution
of moments would have taken place throughout the structure. With the development of
an additional hinge, the structure becomes a mechanism resulting in a collapse.

Analysis of the structure at full moment redistribution is termed plastic or limit
analysis. Since concrete cracks severely at high overloads, it is possible for the designer to
impose the desirable locations of the plastic hinges by making the concrete member fail or
making it adequately strong at any section by decreasing or increasing the reinforcement
percentage without appreciably altering the stiffness of the member. This flexibility in
proportioning is not available in the plastic design of steel structures where the resulting
locations of the plastic hinges are obtained from mechanisms determined by upper and
lower-bound solutions. Details of the theory of imposed rotations by A. L. L. Baker are
well presented in Refs. 13.9, 13.10, and 13.11.

#### 13.7.1 Method of Imposed Rotations

The imposed locations of the plastic hinges coincide with the locations of the maximum
elastic moments for combined gravity loads and horizontal wind loads. These locations
occur at intermediate supports of continuous beams and beam-column corners of frames
as seen in the portal frame of Fig. 13.21. By superimposing Figs. 13.21a and b, the maxi-
num elastic moment occurs at corner $C$. As plastic moments are a magnification of the
elastic moments, the natural location for the development of a plastic hinge is corner $C$.
Since the structure is indeterminate to the first degree, only one hinge develops, resulting
in a basic frame $ABC$, which is the fundamental frame for the imposed hinges seen in Fig.
13.21c, numbered in the order in which they are expected to form. This structure has nine
indeterminacies; hence nine plastic hinges are formed. A tenth hinge reduces the struc-
Figure 13.21 Imposed plastic hinges in concrete frames: (a) gravity load elastic moment; (b) wind load intensity moments; (c) hinge 1 at C reducing frame to statically determinate; (d) basic plastic frame; (e) succession of plastic hinges in two span, two-level frame.

ture to a mechanism resulting in a collapse. Note that no plastic hinges are permitted to form at midspan of the horizontal members. The plastic moments corresponding to assumed hinges 1, 2, 3, ..., n are denoted by $X_1$, $X_2$, $X_3$, ..., $X_n$ and are assumed to remain constant throughout the progressive deformation of the structure.

Hence the derivative of the total strain energy $U$ with respect to the assumed plastic moments $X_i$ at any hinge $i$ is made equal to the plastic rotation at the hinge:

$$\frac{\partial U}{\partial X_i} = -6, \quad (13.28)$$

Equation 13.30 is similar to Eq. 13.1 except that plastic moment $X_i$ is used instead of the elastic moment $X_i$. If $h_{ij}$ is assumed to represent the relative rotation of the $i$th hinge due to a unit moment at $j$th hinge, $h_{ij} = h_{ji}$ from Maxwell's reciprocal theorem. The coefficients $h_{ij}$ are called influence coefficients because they represent the displacement or rotation at a particular section due to a unit moment at another section; that is, $h_{ii} = -6$. 


From the principle of virtual work,
\[ \delta_a = \sum \int_{s_a} \frac{M_a M_a}{E I} \delta s = - \delta_1 \] (13.30)
where \( \delta_1 \) has a finite value and equal to zero, as was the case in the elastic analysis represented by Eq. 13.4.

By substituting \( \delta_a \) and \( \delta_1 \) for \( M_a \) in Eq. 13.31, the following expression is obtained:

\[ \delta_0 + \sum \delta_0 = - \delta_1 \] (13.31)

Hence, the beam must develop \( n \) plastic hinges to reduce it to statically determinate.

\[ \delta_0 + \delta_0 X_1 + \delta_0 X_2 + \cdots + \delta_0 X_n = - \delta_1 \]
\[ \delta_0 + \delta_0 X_1 + \delta_0 X_2 + \cdots + \delta_0 X_n = - \delta_1 \]
\[ \delta_0 + \delta_0 X_1 + \delta_0 X_2 + \cdots + \delta_0 X_n = - \delta_1 \] (13.31)

The number of equations in a set is equal to the number of redundancies or indeterminacies. By trial and adjustment of the redundant plastic moments \( X_1, \ldots, X_n \) in the solution of the set of Eqs. 13.31 for controlled maximum allowable rotation of the largest rotating hinge \( \theta_1 \), the plastic moment at beam supports and column ends are obtained for the plastic design of the continuous structure. It has to be emphasized that the arbitrary plastic moment values \( X_1, X_2, X_3, \ldots, X_n \) are chosen in Eqs. 13.31 to result in plastic rotations \( \theta_1, \theta_2, \ldots, \theta_n \) that give full redistribution of moments throughout the structure.

As shown in Section 13.3.1, the influence coefficient \( \delta_a \) in Eqs. 13.31 is

\[ \delta_a = \frac{A}{E I} \delta \] (13.32)

where \( A \) is the area under the primary \( M \) bending moment diagram and \( \delta \) is the ordinate of the \( M \) moment diagram under the centroid of the \( M \) diagram (Ref. 13.9). Table 13.1 and the example accompanying it give solutions for products of integral values \( \delta_0 M_0 M_0 \) for various moment combinations \( E \delta \).
13.7 Limit Design (Analysis of Indeterminate Beams and Frames)

13.7.2 Example 13.6: Determination of Plastic Hinge Rotations
in Continuous Beams

Determine the required plastic hinge rotation in the four-span beam of Figure 13.22. The beam is subjected to a single span plastic moment \( M_p \), so that midspan moment - support moment = \( M_p \), before full rotation of the hinges and full moment redistribution takes place.

Solution: The structure is statically indeterminate to the third degree, so three hinges will develop at the plastic limit. Assume the maximum ordinate \( c \) of the redundant moment at hinge location to be unity. From Table 13.1 and Figure 13.23

\[
\begin{align*}
E \theta_{R1} &= -\frac{2}{3} M_p \quad E \theta_{R2} = \frac{2}{3} \\
E \theta_{R3} &= -\frac{1}{6} \\
E \theta_{R4} &= 0
\end{align*}
\]

From Eq. 13.33,

\[
-\theta_i = 8 \theta_1 + 8 \theta_2 + 8 \theta_3 + 8 \theta_1 \theta_2
\]

\[-E \theta_1 = -\frac{2}{3} M_p + 0.5 M_p \left( \frac{2}{3} \right) + 0.5 M_p \left( \frac{1}{6} \right) + 0 = -\frac{M_p}{4}
\]

Similarly, from Table 13.1 and Figure 13.23

\[
\begin{align*}
E \theta_{20} &= -\frac{2}{3} M_p \left( -\frac{1}{2} \right) + \frac{2}{3} M_p \left( -\frac{1}{3} \right) = -\frac{2}{3} M_p \\
E \theta_{21} &= \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) = \frac{1}{4} \\
E \theta_{22} &= 2 \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) = \frac{2}{3} \\
E \theta_{23} &= \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) = \frac{1}{6}
\end{align*}
\]

![Figure 13.22 Primary moments and plastic hinge rotations in Ex. 13.6.](image)
From Eq. (13.33),
\[ -M_y = b_2 M_d + 0.5 M_c \left( \frac{1}{2} \right) + 0.5 M_c \left( \frac{3}{4} \right) \]

\[ + 0.5 M_c \left( \frac{3}{2} \right) = - \frac{M_d}{b} \]

From symmetry, \( b_2 = 0 \). Therefore, the required plastic hinge rotations at the supports are:

\[ \theta_1 = \frac{M_d}{3EI} = \theta, \quad \text{and} \quad \theta_2 = \frac{M_d}{6EI} \]

Since \( \theta_2 \) is less than \( \theta_0 \), the first hinge to develop and the controlling one in the design is \( b_2 = M_d/6EI \). Note that the same procedure used in Eq. (13.1) can be used in the limit design of any continuous beam or multistory frame. It is also important to maintain the correct sign convention by drawing all moments at the section side of the member, as noted earlier in this chapter.

The preceding discussion gives the basic imposed rotations approach embodied in A. L. Baker's theory. Other modified approaches have been proposed by Colm (Ref. 13.31), Sawyer (Ref. 13.22), and Parfong (Ref. 13.23). Colm's method is based on the requirement of limit equilibrium and serviceability with subsequent check of rotation compatibility. Sawyer's method is based on simultaneous requirement of limit equilibrium, elastic compatibility with subsequent check of serviceability requirement.

Parfong's method is based on assigning ultimate moments for various loading patterns on the continuous span that would satisfy serviceability and limit equilibrium for the worst case. The sections are reinforced in such a manner that the ultimate moment strength for each span are equal to or greater than the product of the maximum ultimate moment \( M_u \) in the span when the end is free to rotate by a moment coefficient \( k \), for various boundary conditions as listed in Table 13.3.
13.7.3 Rotational Capacity of Plastic Hinges

Rotation is the total change in slope along the short plasticity length concentrated at the hinge zone. It can also be described as the angle of discontinuity between the plastic parts of the member on either side of the plastic hinge. There are two types of hinges, as shown in Fig. 13.24: tensile hinges and compressive hinges. So that the first hinge that develops in the structure, usually the critical hinge, can rotate without rupture until the nth hinge develops, the concrete section at this hinge has to be made durable enough through section core confinement to be able to sustain the necessary rotation. This is equally applicable to both tension and compression hinges, where confinement of the concrete core is obtained through concentration of closed stirrups at the supports and column ends. Fig. 13.24 shows typical tensile and compressive hinges.
Table 13.5  Beam Moment Coefficients for Assigned Moments

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Moment Type</th>
<th>Beam Loaded by One Concentrated Load at Midspan</th>
<th>All Other Beams</th>
</tr>
</thead>
<tbody>
<tr>
<td>Span with ends restrained</td>
<td>Negative</td>
<td>0.37</td>
<td>0.50</td>
</tr>
<tr>
<td>Span with one end restrained</td>
<td>Positive</td>
<td>0.42</td>
<td>0.22</td>
</tr>
<tr>
<td>Span with one end restrained</td>
<td>Negative</td>
<td>0.56</td>
<td>0.75</td>
</tr>
<tr>
<td>Span with one end restrained</td>
<td>Positive</td>
<td>0.50</td>
<td>0.46</td>
</tr>
</tbody>
</table>

The plasticity length \( l_p \) determines the extent of the severe cracking and the rotation magnitude of the hinge. Therefore, it is important to limit the magnitude of \( l_p \) through use of closely spaced ties or closed stirrups. In this manner, the strain capacity of the concrete at the confined section can be significantly raised, as demonstrated experimentally by several investigators, including the author's work in Refs. 13.17, 13.18, and 13.19. Several empirical expressions have been developed: 3ukor (Ref. 13.0), Cortley (Ref. 13.16), Navv and Potsendy (Ref. 13.19), Sawyer (Ref. 13.22), Mattock (13.24), and others.

The following simplified expressions for plasticity length \( l_p \) and concrete strain \( \varepsilon_c \) (Ref. 13.24) are presented:

\[
l_p = 0.5d + 0.5Z
\]

\[
\varepsilon_c = 0.003 + 0.02 \frac{l_p}{Z} + 0.2p
\]

Figure 13.24  Plasticity zones \( l_p \) in plastic hinges: (a) tensile hinge; (b) compressive hinge.
where \( d \) = effective depth of beam, in.
\( b \) = beam width, in.
\( Z \) = distance from the critical section to the point of contraflexure
\( \rho_p \) = ratio of volume of confining binder steel (including the compression steel) to the volume of the concrete core
\( l_p \) = half the plasticity length on each side of the center line of the plastic hinge

Equation 13.33 can be more conservative for high values of \( \rho_p \).
Additionally, such a \( \rho_p \) should be chosen that the necessary confinement is achieved as detailed in Ref. 13.26. The maximum spacing of the confining hoops should not exceed the smaller of the following: \( d/4 \) or 8 times the diameter of the smallest longitudinal bars, 24 times the diameter of the hoop bars, or 12 in. in beams and 4 in. in columns.

Once the concrete strain \( \varepsilon_c \) is determined, the angle of rotation of the plastic hinge is readily determined from the expression

\[
\psi_p = \left( \frac{\varepsilon_c}{\varepsilon_p} - \frac{\varepsilon_p}{klf} \right) l_p
\]

where \( c \) = neutral axis depth at the limit state at failure
\( \varepsilon_c \) = strain in the concrete at the extreme compression fibers when the yield curvature is reached
\( klf \) = neutral axis depth corresponding to \( \varepsilon_p \)
\( \varepsilon_p \) = concrete compressive strain at the end of the inelastic range or at the limit state at failure

The strain \( \varepsilon_p \) can usually be taken at the load level where the strain in the tension reinforcement reaches the yield strain \( \varepsilon_y = f_y/E_c \). The strain \( \varepsilon_p \) can be taken as 0.001 in./in. or higher, depending on whether the tension steel yields before the concrete crushes at the extreme compression fibers in cases of over-reinforced beams as is the case in some prestressed beams. If concrete crushes first, the value of \( \varepsilon_p \) would have to be higher than 0.001 in./in. A limit of allowable \( \varepsilon_p = 1.0\% \) is recommended in determining the maximum allowable plastic rotation \( \theta_p \), although strains of confined concrete as high as 1.5% could be obtained, as shown in the author's work (Ref. 13.19). A typical comparison of plastic
rotations obtained by several authors for various degrees of confinement is shown in Figure 13.26.

It should be stated that the discussions presented are equally applicable to reinforced and prestressed concrete indeterminate structures at the plastic loading range, where full redistribution of moment has taken place. As the load reaches the limit state at failure, the flexural behavior of the prestressed concrete elements is expected to resemble closely that of reinforced concrete elements.

Discussion and design examples on the confinement of members by the ACI and UBC Codes and the International Building Code, IBC 2000, for resisting seismic loading are presented in detail in Chapter 15. Additional discussion in the case of prestressed concrete structures in seismic zones is given in Ref. 13.26.

13.7.4 Example 13.7: Calculation of Available Rotational Capacity

Determine the required and the available rotational capacities of the critical plastic hinges in the continuous beam of Ex. 13.6 for both confined and unconfined concrete. Given:

\[ M_r = \frac{1}{2} M_e = 400 \text{ft}^2 \]
\[ c = 0.28 d \]
\[ kd = 0.375d \]
\[ \epsilon_{tu} = \begin{cases} 0.001 \text{ in./in. at end of the elastic range (unconfined)} \\ 0.004 \text{ in./in. at end of the inelastic range for unconfined sections} \end{cases} \]
\[ \epsilon_{tu} = 0.01 \text{ in./in. (confined)} \]
\[ E, I_c = 15,000,000 \text{ psi in.}^2 \text{ft} \]
\[
\frac{Z}{d} = 5.5
\]
\[
f'_l = 3000 \text{ psi}
\]
\[
f'_t = 60,000 \text{ psi for the mild steel}
\]

Also calculate the maximum allowable span-to-depth ratio \( \frac{b}{d} \) for the beam if full redistribution of moments is to occur at the limit state at failure.

**Solution:**

\[M_b = 2 \times 4000 \text{ft}^3 = 8000 \text{ft}^3\]

From Eq. 13.6,

- \[
\text{required } f_l = \left( \frac{M_b}{L} \right) \left( \frac{8000 \text{ft}^3}{4 \times 150,000 \text{ft}^3/\text{d}} \right) = \frac{1}{750} \text{ rad}
\]
- \[
\text{required } f_t = \left( \frac{M_b}{L} \right) \left( \frac{8000 \text{ft}^3}{6 \times 150,000 \text{ft}^3/\text{d}} \right) = 1/1125 \text{ rad}
\]

From Eq. 13.33,

\[l_e = 0.5d = 0.0525d = 0.5d + 0.05 \times 3.5d = 0.775d
\]

Total plasticity length on both sides of the hinge center line = \( 2 \times 0.775d = 1.55d \).

(a) **Unconfined sections:** From Eq. 13.35,

- available \( f_l = \left( \frac{5}{c} \right) \left( \frac{5}{10} \right) = \left( \frac{0.004}{0.25} \right) = 0.001 \) rad
- available \( f_t = \left( \frac{5}{c} \right) \left( \frac{5}{10} \right) = \left( \frac{0.375}{0.25} \right) = 1.5d = 0.018 \) rad

For full moment redistribution,

\[
\frac{1}{750} \leq 0.018 \quad \text{and} \quad \frac{1}{1125} \leq 0.018
\]

or

\[
\frac{1}{d} = 13.5 \quad \text{and} \quad \frac{1}{d} = 20.3
\]

(b) **Confined sections:**

- maximum allowance \( f_l = 0.01 \) in./in.
- available \( f_l = \left( \frac{5}{10} \right) = \left( \frac{0.004}{0.25} \right) = 0.001 \) rad
- available \( f_t = \left( \frac{5}{10} \right) = \left( \frac{0.375}{0.25} \right) = 1.5d = 0.051 \) rad

For full moment redistribution,

\[
\frac{1}{750} \leq 0.051 \quad \text{and} \quad \frac{1}{1125} \leq 0.051
\]

or

\[
\frac{1}{d} = 38.3 \quad \text{and} \quad \frac{1}{d} = 77.4
\]

It can be seen from comparing the results of the unconfined sections in case (a) to the confined sections in case (b) that confinement of the concrete at the plastic hinge zone permits more slender sections for full plasticity and hence a more economical indeterminate structural system.

**13.7.5 Example 13.8: Check for Plastic Rotation Serviceability**

If closed stirrup hinders are used in Ex. 13.7 with header ratio \( p_h = 0.025 \) and \( \theta \) = 35 with \( c \) at failure = 0.25d, verify whether the continuous beam satisfies rotation serviceability criteria.

Given \( b = 2d \).
Solution: \( \varepsilon_f = 5.3 \)

Hence

\[
\frac{b}{t} = \frac{1}{11}
\]

available \( \varepsilon_a = 0.003 + 0.2 \times \frac{b}{t} = 0.2 \times \frac{1}{11} \)

\( = 0.003 \times 0.2 = 0.0006 \text{ in./in.} \)

Maximum allowable strain to be utilized as \( \varepsilon_c = 0.01 \text{ in./in.} \). For \( \varepsilon_c = 0.01 \), the corresponding available plastic rotation is

\[
\theta_p = \frac{0.01}{0.002} = 5 \text{ rad}
\]

required \( \theta_0 = \frac{1}{750} = 0.0006 \text{ rad} \)

required \( \theta_0 = \frac{1}{1125} = 0.0009 \text{ rad} \)

Available, \( \theta_p = 0.009 \text{ rad} \) requires \( \theta = 0.009 \text{ rad} \). Thus, the beam satisfies serviceability criteria for plastic rotation.

The foregoing discussion for the limit design of reinforced and prestressed concrete indeterminate beams and frames permits the design engineer to provide ductile connections at beam-column supports and generate full moment redistribution throughout the structure, resulting in full utilization of the strength of the structural system. Also, continuity to withstand seismic loading can be effectively utilized through the appropriate adjustment of the beam-column zones utilizing the procedures presented in this section, as discussed in Chapter 15 on design for seismic loading.

SELECTED REFERENCES


Problems for Solution


13.13. ACI Committee 318, Building Code Requirements for Structural Concrete (ACI 318-89) and Commentary (ACI 318 R-89), American Concrete Institute, Farmington Hills, MI, 2005, pp. 444.


PROBLEMS FOR SOLUTION

13.1. A continuous reinforced concrete beam is shown in Figure 13.26. Find the moments and reactions at supports B, C, and D using the following methods of analysis: (a) flexibility matrix: (b) stiffness matrix: (c) slope deflection; (d) moment distribution. Assume that \( I_{AB} = I_{BC} = 2 I_{CD} \).
13.2. Solve Ex. 13.4 for service live-load intensity of 60 psf (2.97 MPa). Use a height between floors \( h = 12\text{ ft-6\,in.} (3.81\,m) \). Use in your solution the appropriate slab thickness as required by moment, shears, and deflection. Given:

\[
f_e = 5000 \text{ psi, nominal weight (34.5 MPa)}
\]

\[
f_s = 60,000 \text{ psi (413.7 MPa)}
\]

13.3. Find the moments and shears caused by a wind intensity of 24 psf acting on the structural system in Problem 13.2 using the portal method of wind analysis. Assume that the wind load factor is 1.7.

13.4. Assume that the continuous structure in Problem 13.4 is subjected to a constant intensity of dead load \( w_d = 800 \text{ psf (11.7 kN/m)} \) and live load \( w_w = 2000 \text{ psf (92.2 kN/m)} \) across all spans and the cantilever segment \( AB \). Analyze the structure by the limit theory of imposed rotations, and design the section for a maximum rotation capability for limit strain in confined concrete of value \( f_{c} = 0.01 \text{ in./in.} \). Design the confining reinforcement to achieve such ductility.

13.5. A two-bay warehouse portal frame has the dimensions shown in Figure 13.27. Analyze the structure for limit moments by the method of imposed rotations assuming that the portal is hinged at the foundation. Design the portal beam and column elements for service gravity live load of \( w_g = 40 \text{ psf (1915 Pa)} \) and wind and suction load concentrated at the upper joints having a service-level intensity \( w_w = 125 \text{ psf of height.} \) Use a degree of confinement that permits a limit strain \( f_{c} = 0.01 \text{ in./in.} \). Select the confining steel. Use Ref. 13.10 expressions for two-bay portal frame redundants:

\[
\begin{align*}
\bar{X}_1 &= 0.16m + 0.5M \\
\bar{X}_1 &= 0.175m + 0.4M \\
\bar{X}_2 &= 0.1m - 0.01M
\end{align*}
\]

\[w_w = 40 \text{ psf}\]

\[w_g = 40 \text{ psf}\]

\[l_e = 40\,\text{in.}
\]

\[l_e = 40\,\text{in.}
\]

\[w_w = 125 \text{ psf of height.}
\]

\[w_w = 125 \text{ psf of height.}
\]

where \( m \) = wind moment and \( M \) = gravity load moment. Assume:

\[
f_e = 5900 \text{ psi, nominal weight (34.5 MPa)}
\]

\[
f_s = 60,000 \text{ psi (413.7 MPa)}
\]

The portals are spaced 20 ft on centers.
INTRODUCTION TO PRESTRESSED CONCRETE

14.1 BASIC CONCEPTS OF PRESTRESSING

Reinforced concrete is weak in tension but strong in compression. In order to maximize utilization of its material properties, an internally or externally compressive force \( P \) is induced on the structural element through the use of stressed high-strength prestressing wires or tendons prior to loading. As a result, the concrete section is generally stressed only in compression under service and sometimes overload conditions. Such a system of construction is termed as prestressed concrete.

The prestressing force \( P \) that satisfies the particular conditions of geometry and loading of a given element (Figure 14.1) is determined from the principles of mechanics and of stress-strain relationships. Schematically simplification is necessary, as when a prestressed beam is assumed to be homogeneous and elastic. A comprehensive treatment of the subject of prestressed concrete may be found in Ref. 14.1.

Consider, then, a simply supported rectangular beam subjected to a compressive prestressing force \( P \), as shown in Figure 14.1(a). The compressive stress on the beam cross-section is uniform and has an intensity

Photo 14.1  Sunshine Skyway Bridge, Tampa Bay, Florida. Designed by Figg and Muller Engineers, Inc., the bridge has a 1294-ft cable-stayed main span with a single pylon, 178-ft vertical clearance, and total length of 21,876 ft. It has twin 40-ft roadways and has 135-ft spans in precast segmental bridge with prestress, and high approaches to elevation + 100 ft. (Courtesy of Figg Engineers, Inc., Tallahassee, FL.)
Figure 14.1 Concrete fiber stress distribution in a rectangular beam with straight tendon:
(a) concentric tendon, prestress only; (b) concentric tendon, self-weight added; (c) eccentric tendon, prestress only; (d) eccentric tendon, self-weight added.

\[ f = \frac{P}{A_{c}} \]  
(14.1)

where \( A_{c} = bh \) is the cross-sectional area of a beam section of width \( b \) and total depth \( h \).

A minus sign is used for compression and a plus sign for tension throughout the text.

Also, bending moments are drawn on the tensile side of the member.

If external transverse loads are applied to the beam, causing a maximum moment \( M \) at midspan, the resulting stress becomes

\[ f' = \frac{P}{A} - \frac{Mc}{I_{y}} \]  
(14.2a)

and

\[ f_{u} = \frac{P}{A} + \frac{Mc}{I_{y}} \]  
(14.2b)
where $f_p$ = stress at the top fibers
$f_b$ = stress at the bottom fibers
$c = \frac{1}{2} h$ for the rectangular section
$I_g =$ gross moment of inertia of the section ($bh^3/12$ in this case)

Equation 14.2b indicates that the presence of prestressing-compressive stress $-P/A$ is reducing the tensile flexural stress $Mc/I$ to the extent intended in the design, either eliminating tension totally (even inducing compression) or permitting a level of tensile stress within allowable code limits. The section is then considered uncracked and behaves elastically; the concrete's inability to withstand tensile stresses is effectively compensated for by the compressive force of the prestressing tendon.

The compressive stresses in Eq. 14.2a at the top fibers of the beam due to prestressing are compounded by the application of the loading stress $-Mc/I$, as seen in Figure 14.1b. Hence the compressive stress capacity of the beam to take a substantial external load is reduced by the concentric prestressing force. In order to avoid this limitation, the prestressing tendon is placed eccentrically below the neutral axis at midspan, to induce tensile stresses at the top fibers due to prestressing (Figure 14.1c and d). If the tendon is placed at eccentricity $e$ from the center of gravity of the concrete, termed the ege line, it creates a moment $Pe$, and the ensuing stresses at midspan become

$$f_p = \frac{P}{A_e} + \frac{Pe}{I_g} \cdot \frac{Mc}{I_g}$$  \hspace{1cm} (14.3a)

$$f_b = \frac{-P}{A_e} - \frac{Pe}{I_g} \cdot \frac{Mc}{I_g}$$  \hspace{1cm} (14.3b)

Since the support section of a simply supported beam carries no moment from the external transverse load, high tensile fiber stresses at the top fibers are caused by the eccentric prestressing force. To limit such stresses, the eccentricity of the prestressing tendon profile, the ege line, is made less at the support section than at the midspan section, or eliminated altogether, or else a negative eccentricity above the ege line is used. The ege line is the profile of the center of gravity of the prestressing tendon and the ege line is the profile of the center of gravity of the concrete.

In designing prestressed concrete elements, the concrete fiber stresses are directly computed from the external forces applied to the concrete by longitudinal prestressing and the external transverse load. Equations 14.3a and 14.3b can be modified and simplified for use in calculating stresses at the initial prestressing stage and at service load levels. If $P_e$ is the initial prestressing force before stress losses and $P_e$ is the effective prestressing force after losses, then

$$\gamma = \frac{P_e}{P_e}$$

can be defined as the residual prestress factor. Substituting $\gamma$ for $P_e/A_e$ in Eq. 14.3, where $r$ is the radius of gyration of the gross section, the expressions for stress can be rewritten as follows:

1. Prestressing force only

$$f_p = -\frac{P}{A_e} \left(1 - \frac{\gamma}{r^2}\right)$$  \hspace{1cm} (14.4a)

$$f_b = \frac{P}{A_e} \left(1 + \frac{\gamma}{r^2}\right)$$  \hspace{1cm} (14.4b)
where \( c_t \) and \( c_b \) are the distances from the center of gravity of the section (the egg line) to the extreme top and bottom fibers, respectively.

2. Prestressing plus self-weight. If the beam self-weight causes a moment \( M_0 \) at the section under consideration, Eqs. 14.4a and 14.4b, respectively, become

\[
f' = \frac{P}{A} \left( 1 - \frac{c_b}{r} \right) \cdot \frac{M_0}{S}
\]

(14.5a)

and

\[
f_b = \frac{P}{A} \left( 1 + \frac{c_b}{r} \right) \cdot \frac{M_0}{S_b}
\]

(14.5b)

where \( S \) and \( S_b \) are the moduli of the sections for the top and bottom fibers, respectively.

The change in eccentricity from the midspan to the support section is obtained by raising the prestressing tendons either abruptly from the midspan to the support, a process called harping, or gradually in a parabolic form, a process called draping. Figure 14.2a shows a harped profile usually used for precasted beams and for concretes with transverse loads. Figure 14.2b shows a draped tendon usually used in posttensioning.

Subsequent to erection and installation of the floor or deck, live loads act on the structure, causing a superimposed moment \( M_f \). The full intensity of such loads normally occurs after the building is completed and some time-dependent loads in prestress have already taken place. Hence, the prestressing force used in the stress equations would have to be the effective prestressing force \( P_r \). If the total moment due to gravity loads is \( M_0 \), then

\[
M_f = M_0 + M_{10} + M_L
\]

(14.6)

where:
- \( M_0 \) = moment due to self-weight.
- \( M_{10} \) = moment due to superimposed dead load, such as flooring.
- \( M_L \) = moment due to live load, including impact and seismic loads if any.

Equations 14.5 then become

\[
f' = \frac{P}{A} \left( 1 - \frac{c_b}{r} \right) \cdot \frac{M_0}{S}
\]

(14.7a)

\[
f_b = \frac{P}{A} \left( 1 + \frac{c_b}{r} \right) \cdot \frac{M_0}{S_b}
\]

(14.7b)

Figure 14.2 Prestressing tendon profile: (a) harped tendon, (b) draped tendon.
The tensile stress in the concrete permitted at the extreme fiber of the section cannot exceed the maximum permissible in the code (e.g., $f_y = 0.8 f_k$ in the ACI Code). If it is exceeded, bonded nonprestressed reinforcement proportioned to resist the total tensile force has to be provided to control cracking at service loads so that $f_y = 0.8 f_k$ can be used.

### 14.1.1 Example 14.1: Computation of Fiber Stresses in a Prestressed Beam by Basic Principles

A prestressed simply supported T-beam has a span of 64 ft (19.51 m) and the geometry shown in Figure 14.3. It is subjected to a uniform intensity of superimposed gravity dead-load intensity $W_{ps}$, and live-load intensity $W_p$, summing to 420 psf (6.12 kN/m²). The initial prestressing stress before losses is $f_p = 0.9 f_y = 189,000$ psi (1303 MPa), and the effective prestress after losses is $f_p = 150,000$ psi (1058 MPa). Compute the extreme fiber stresses at the midspan section due to (a) the initial full prestress and the external gravity load and (b) the final service-laden conditions when prestress losses have taken place. Allowable stress data are as follows:

- $W_{ps} = 420$ psf (6.12 kN/m²)
- $W_p = 420$ psf (6.12 kN/m²)

![Diagram](image)

**Figure 14.3** Example 14.1
Chapter 14: Introduction to Prestressed Concrete

\( f'_c = 6000 \text{ psi, normal-weight (41.3 MPa)} \)
\( f_{yc} = 270,000 \text{ psi, stress relieved (1862 MPa)} = \text{specified tensile strength of the tendons} \)
\( f_p = 220,000 \text{ psi (153.7 MPa)} = \text{specified yield strength of the tendons} \)
\( f'_e = 150,000 \text{ psi (1034 MPa)} \)
\( f_t = 12 \sqrt{f'_c} = 830 \text{ psi (5.6 MPa)} = \text{maximum allowable tensile stress in concrete} \)
\( f'_{ct} = 4800 \text{ psi (33.1 MPa)} = \text{concrete compressive strength at time of initial prestress} \)
\( f_o = 0.65f'_e = 2880 \text{ psi (19.9 MPa)} = \text{maximum allowable stress in concrete at initial prestress} \)
\( f_s = 0.45f'_e = \text{maximum allowable compressive stress in concrete at service-load level} \)

Assume that ten \( rac{1}{8}-\text{in.-dia} \) seven-wire-strand (ten 12.7-mm-dia strand) tendons are used to prestress the beam and

\[
\begin{align*}
A_s &= 449 \text{ in.}^2 (2.915 \text{ cm}^2) \\
l_s &= 22.469 \text{ in.} (93.347 \text{ cm}) \\
r &= l_s/A_s = 50.04 \text{ in.}^2 \\
c_s &= 17.77 \text{ in.} (452 \text{ mm}) \\
c_r &= 6.23 \text{ in.} (158 \text{ mm}) \\
e &= 14.77 \text{ in.} (375 \text{ mm}) \\
\rho &= 7.77 \text{ in.} (197 \text{ mm}) \\
S &= 1.264 \text{ in.}^3 (20.714 \text{ cm})^3 \\
S' &= 1.607 \text{ in.}^3 (40.08 \text{ cm})^3 \\
W &= 389 \text{ psi (2.65 kN/m)}
\end{align*}
\]

Solution:

(0) Initial Conditions at Prestressing

\[
\begin{align*}
A_p &= 18 \times 0.153 = 4.33 \text{ in.}^2 \\
P &= A_p f_o = 1.53 \times 189,000 = 289,170 \text{ lb (1.287 kN)} \\
P' &= 1.53 \times 151,000 = 229,500 \text{ lb (1.020 kN)}
\end{align*}
\]

The midspan self-weight dead-load moment is

\[
M_o = \frac{wL^2}{8} = \frac{299 \text{ (64)}}{8} \times \frac{12}{2} = 2,205,696 \text{ in.-lb (249 kN-m)}
\]

From Equations 14.5 and 14.7,

\[
\begin{align*}
\frac{f'}{f_c} &= \frac{\frac{P}{P_c} \left(1 + \frac{c_s}{c_r} \right)}{\frac{M_o}{S}} \\
&= \frac{289,170}{449} \left(1 + \frac{17.77}{6.23} \right) \frac{50.04}{3.607} \\
&= +540.3 - 612.5 \text{ ft (70 ft-C)} \\
f_s &= -\frac{P}{A} \left(1 + \frac{c_s}{c_r} \right) \frac{M_o}{S} \\
&= \frac{289,170}{449} \left(1 + \frac{17.77}{6.23} \right) \frac{50.04}{1,264} \\
&= -4,022.1 + 1,745.0 = -2,277 \text{ ft-C} \\
\Rightarrow f_s &= -2,880 \text{ psi allowed, O.K.}
\end{align*}
\]
14.2 Partial Loss of Prestress

(ii) Final Condition at Service Load. Midspan moment due to superimposed dead and live load is

\[ M_{a2} = M_{l} = \frac{4238644}{5} \times 12 = 2590.483 \text{ in.-lb} \]

Total Moment, \( M_{a} = 2206.696 + 2404.060 = 4610.756 \text{ in.-lb} \) or (514 kN-m)

\[ f' = \frac{P_{c}}{A_{c}} \left( \frac{c}{r} \right) = \frac{M_{a}}{S} \]

\[ = \frac{229200}{449} \left( \frac{14.77 \times 6.33}{50.04} \right) = \frac{4786.176}{3.607} \]

\(< f_c = 0.45 \times 6.000 = 2.700 \text{ psi, O.K.} \]

\[ f_s = \frac{P_{s}}{A_{s}} \left( 1 + \frac{S}{r} \right) \]

\[ = \frac{229200}{449} \left( 1 + \frac{14.77 \times 17.77}{50.04} \right) = \frac{4786.176}{3.264} \]

\[ = -0.192 + 3.766 = 594 (7) (2.2 \text{ MPa}) \]

\[ < f_s = 12 \sqrt{f_c} = 930 \text{ psi, O.K.} \]

14.2 PARTIAL LOSS OF PRESTRESS

It is a well-established fact that the initial prestressing force applied to the concrete element undergoes a progressive process of reduction over a period of approximately 5 years. Consequently, it is important to determine the level of the prestressing force at each loading stage, from the stage of transfer of the prestressing force to the concrete, to the various stages of prestressing available at service load, up to the ultimate. Essentially, the reduction in the prestressing force can be grouped into two categories:

1. Immediate elastic loss during the fabrication or construction process, including elastic shortening of the concrete, concrete losses, and frictional losses.
2. Time-dependent losses such as creep, shrinkage, and those due to temperature effects and steel relaxation, all of which are determinable at the service-load limit state of stress in the prestressed concrete element.

An exact determination of the magnitude of these losses, particularly the time-dependent ones, is not feasible, since they depend on a multiplicity of interrelated factors. Empirical methods of estimating losses differ with the different codes of practice or recommendations, such as those of the Prestressed Concrete Institute, the AASHTO spring approach, the Comité EuroInternational du Béton (CEB), and the FIP (Fédération Internationale de la Précontrainte). The degree of rigor of these methods depends on the approach chosen and the accepted practice of record.

A very high degree of refinement of loss estimation is neither desirable nor warranted, because of the multiplicity of factors affecting the estimate. Consequently, lump-sum estimates of losses are more realistic, particularly in routine designs and under average conditions. Such lump-sum losses are summarized in Table 14.1 of AASHTO and Table 14.2 of PCI. They include elastic shortening, relaxation in the prestressing,
Table 14.1  AASHTO Lump-Sum Losses

<table>
<thead>
<tr>
<th>Type of Prestressing Steel</th>
<th>Total Loss [psi (N/mm²)]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_c = 4000$ psi (27.6 N/mm²)</td>
</tr>
<tr>
<td>Pretensioning strand</td>
<td>45,000 psi (310)</td>
</tr>
<tr>
<td>Postensioning wire or strand</td>
<td>32,000 psi (221)</td>
</tr>
<tr>
<td>Bars</td>
<td>22,000 psi (152)</td>
</tr>
</tbody>
</table>

*Losses due to Section 8.5 are excluded. Each loss should be computed according to Section 8.5 of the AASHTO specifications.

Steel, creep, and shrinkage are applicable only to routine, standard conditions of loading: normal concrete, quality control, construction procedures, and environmental conditions; and the importance and magnitude of the system. Detailed analysis has to be performed if these standard conditions are not fulfilled.

A summary of the sources of the separate prestressing losses and the stages of their occurrence is given in Table 14.1, in which the subscript j denotes "initial" and the subscript f denotes the loading stage after jacking. From this table, the total loss in prestress can be calculated for pretensioned and posttensioned members as follows.

1. Pretensioned members

$$\Delta f_{ps} = \Delta f_{res} + \Delta f_{deg} + \Delta f_{rel} + \Delta f_{con}$$  (14.8a)

where:
- $\Delta f_{res} = \Delta f_{res}(t_f, t_t)$
- $t_f = $ time at jacking
- $t_t = $ time at transfer
- $t_{con} = $ time at stabilized loss

Hence, computations for steel relaxation loss have to be performed for the time interval $t_t$ through $t_{con}$ of the respective loading stages.

As an example, the transfer stage, say, at 18 hours would result in $t_f = t_t = 18$ hours and $t_{con} = t = 0$. If the next loading stage is between transfer and 5 years (17,520 hours), which losses are considered stabilized, then $t_f = t_t = 17,520$ hours and $t_{con} = 18$ hours.

Table 14.2  Approximate Prestress Loss Values for Posttensioning

<table>
<thead>
<tr>
<th>Posttensioning Tendon Material</th>
<th>Slabs</th>
<th>Beams and Joints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stress-relieved 270K strand and stress-relieved 240K wire Bar</td>
<td>30,000 (207)</td>
<td>35,000 (241)</td>
</tr>
<tr>
<td>20,000 (138)</td>
<td>35,000 (241)</td>
<td></td>
</tr>
<tr>
<td>Low-relaxation 270K strand</td>
<td>15,000 (103)</td>
<td>20,000 (138)</td>
</tr>
</tbody>
</table>

Source: Post-Tensioning Institute.

*This table of approximate prestress losses was developed to provide a common posttensioning industry basis for determining tendon requirements on projects on which the magnitude of prestress losses is specified by the designer. These loss values are based on $f_c$ of normal-weight concrete and on average values of concrete strength, prestress level, and exposure conditions. Actual values of losses may vary significantly above or below the table values when the concrete is stressed at low strengths, when the concrete is highly prestressed, or in very dry or very wet exposure conditions. The table values do not include losses due to friction.
### Table 14.3 Types of Prestress Loss

<table>
<thead>
<tr>
<th>Type of Prestress Loss</th>
<th>Stage of Occurrence</th>
<th>Tendon Stress Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pretensioned Members</td>
<td>Post-tensioned Members</td>
</tr>
<tr>
<td>Elastic shortening of concrete (ES)</td>
<td>At transfer</td>
<td>After sequential jacking</td>
</tr>
<tr>
<td>Relaxation of tendons (R)</td>
<td>Before and after transfer</td>
<td>After transfer</td>
</tr>
<tr>
<td>Creep of concrete (CE)</td>
<td>After transfer</td>
<td>After transfer</td>
</tr>
<tr>
<td>Shrinkage of concrete (SM)</td>
<td>After transfer</td>
<td>After transfer</td>
</tr>
<tr>
<td>Friction (F)</td>
<td>—</td>
<td>After jacking</td>
</tr>
<tr>
<td>Anchor-age seating loss (A)</td>
<td>—</td>
<td>After transfer</td>
</tr>
<tr>
<td>Total</td>
<td>Life</td>
<td>Life</td>
</tr>
</tbody>
</table>

2. Post-tensioned members

$$\Delta f_{T} = \Delta f_{P} + \Delta f_{F} + \Delta f_{CE} + \Delta f_{SM} + \Delta f_{A}$$  \hspace{1cm} (14.8c)

where $\Delta f_{FCE}$ is applicable only when tendons are jacked sequentially, and not simultaneously.

In the post-tensioned case, computation of relaxation loss starts between the transfer time $t_1 = t_0$, and the end of the time interval $t_2$ under consideration. Hence

$$f_{P} = f_{T} - \Delta f_{T} - \Delta f_{F}$$  \hspace{1cm} (14.8d)

### 14.2.1 Example 14.2: Step-by-Step Computation of all Time-dependent Losses in a Post-tensioned Beam

A simply supported post-tensioned, 70-ft-span, lightweight, steam-cured, double T-beam as shown in Figure 14.4 is prestressed by twelve 2-in. (51 mm) diameter 270-k grade stressed-relieved strands. The tendons are lapped, and the eccentricity at midspan is 0.1 in. (2.5 cm) and at the end 12.98 in. (330 mm). Compute the prestress loss at the critical section in the beam at 0.40 psf (4.08 kPa) at transfer, (b) stage II after concrete topping is placed, and (c) 2 years after concrete topping is placed. Suppose the topping is 2-in. (51 mm) normal-weight concrete cast at 30 days. Suppose also that prestress transfer occurred 18 hours after casting the section and tendonsizing the strands. Assume that stress increase due to topping = 50 kPa (0.5 MPa). Given:

- $f_p = 5000$ psi (34.5 MPa), lightweight
- $f_s = 189,000$ psi (130 MPa)
- $f_c = 3500$ psi (24.1 MPa)
- $f_a = 230$ psi

and the following noncomposite section properties:
Figure 14.4  Double-T precasted beam: (a) elevation; (b) precasted section.

\[ A_c = 615 \text{ in}^2 (3968 \text{ cm}^2) \]
\[ E_c = 28 \times 10^6 \text{ psi} \]
\[ L = 59.720 \text{ in.} \cdot (1.49 \times 10^4 \text{ cm}) \]
\[ S_s = 2717 \text{ in}^2 \cdot \text{in} \]
\[ c_s = 21.96 \text{ in.} \cdot (55.8 \text{ cm}) \]
\[ c_t = 10.02 \text{ in.} \cdot (25.5 \text{ cm}) \]
\[ S = 5960 \text{ in}^2 \cdot \text{in} \]

Shrinkage loss at transfer = 6190 psi (42.7 MPa)

**Solution:**

1. **Anchorage seating loss**
   \[ \Delta_1 = \frac{1}{4} - 0.25^\circ = 0.07^\circ \]
   \[ L = 70 \text{ ft} \]

   From the ACI Code, the anchorage slip stress loss is
   \[ \Delta \sigma_s = \frac{\Delta \theta}{L} E_c = \frac{0.25}{70 \times 12} \times 28 \times 10^6 = 833 \text{ psi (5.7 MPa)} \]

2. **Elastic shortening:** Since all jacks are simultaneously tensioned, the elastic shortening will precipitate during jacking. As a result, no elastic shortening stress is taken place in the strand. Hence \( \Delta \sigma_{el} = 0 \).

3. **Frictional loss:** Assume that the parabolic tendon approximates the shape of an arc of a circle. Then, from the equation of the parabola (see Ref. 14.1),
   \[ \alpha = \frac{x}{y} = \frac{12(1.875 - 12.95)}{70 \times 12} = 0.058 \text{ rad} \]

   From the ACI Code, use \( K = 0.001 \) and \( \mu = 0.25 \). Then
   \[ f_{pl} = 199,000 \text{ psi (1360 MPa)} \]

   From the ACI Code, the stress loss in prestress due to friction is
\[
\Delta f_c = f_c (u_p - K_a)
\]
\[
= 189,000 \times (0.25 \times 0.0548 + 0.001 \times 70)
\]
\[
= 15,819 \text{ psi (104.5 MPa)}
\]

The stress remaining in the prestressing steel after all initial instantaneous losses is
\[
f_p = 189,000 - 8333 - 0 - 15,819 = 164,848 \text{ psi (1136 MPa)}
\]

Hence the net prestressing force is
\[
T = 164,848 \times 12 \times 0.153 - 296,727 lbs
\]

**Stage 1: Stress at transfer**

1. **Anchorage seating loss**
   - Anchorage loss = 8333 psi
   - Net stress = 164,838 psi

2. **Relaxation loss**
   \[
   \Delta f_{ta} = 164,848 \times \frac{\log 18}{10} \frac{164,838}{230,000} = 3450 \text{ psi (238 MPa)}
   \]

3. **Creep loss**
   \[
   \Delta f_{tc} = 0
   \]
(4) Shrinkage loss

\[ \Delta f_{sh} = 0 \]

So the tendon stress \( f_t \) at the end of stage I is

\[ 164,848 - 3,458 = 161,390 \text{ psi (11.3 MPa)} \]

Stage II: Transfer of stresses after 30 days

(1) Creep loss

\[ f_t = 161,398 \times 12 \times 0.53 = 296,327 \text{ lb} \]

\[ \tilde{f}_t = \frac{f_t}{A_0} \left( 1 + \frac{c}{pc} \right) \frac{M_{te} e}{L} \]

\[ = \frac{296,327}{615} \left( 1 + \frac{17.50^2}{97.11} \right) \frac{3,644,496 \times 17.50}{59.720} \]

\[ = -2016.2 + 1020.0 = 996.2 \text{ psi (6.94 MPa)} \]

Hence the creep loss is

\[ \Delta f_{cr} = nK_c (\tilde{f}_t - f_{te}) \]

\[ = 9.72 \times 1.6(996.2 - 519.3) = 7417 \text{ psi (51.9 MPa)} \]

(2) Shrinkage loss: Given

\[ \Delta f_{sh} = 3.590 \text{ psi (24.8 MPa) at 30 days using a shrinkage reduction coefficient} \]

\[ = 0.58 \]

(3) Steel relaxation loss at 30 days

\[ f_{te} = 161,398 \text{ psi} \]

The relaxation loss in stress becomes

\[ \Delta f_{sr} = 161,398 \text{ psi} - \frac{\log 720 - \log 18}{10} \frac{161,398}{231,000} = 5923 \text{ psi (40.0 MPa)} \]

Stage II: Total losses

\[ \Delta f_{te} = \Delta f_{cr} + \Delta f_{sh} + \Delta f_{sr} \]

\[ = 7417 + 3590 + 5923 = 16,930 \text{ (113.5 MPa) psi (120.3 MPa)} \]

The increase in stress in the strands due to the addition of topping is \( f_{ho} = 5048 \text{ psi (34.8 MPa); hence the total} \)

\[ f_{te} = f_t - \Delta f_{cr} + \Delta f_{sh} = 161,398 - 14,930 + 5048 = 151,516 \text{ psi (1045 MPa)} \]

Stage III: At end of 2 years

\[ f_{ho} = 151,516 \text{ psi} \]

\[ n = 720 \text{ hours} \]

\[ t = 17,350 \text{ hours} \]

The steel relaxation stress loss is

\[ \Delta f_{sr} = 151.516 \text{ psi} \left( \frac{\log 17,350 - \log 720}{10} \right) \frac{151,516}{231,000} = 2248 \text{ psi (15.9 MPa)} \]
14.3 Flexural Design of Prestressed Concrete Elements

Assuming that the creep shrinkage losses were maintained till stage III, the strand stress $f_a$ at the end of stage III is approximately

$$151,516 = 2248 - 149,232 \text{ psi (1029 MPa)}$$

**Summary of stresses**

<table>
<thead>
<tr>
<th>Stress Level at Various Stages</th>
<th>Steel Stress (psi)</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>After tensioning ($f_{pt}$)</td>
<td>189,000</td>
<td>100.0</td>
</tr>
<tr>
<td>Elastic shortening loss</td>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>Anchorage loss*</td>
<td>-8,319</td>
<td>-4.4</td>
</tr>
<tr>
<td>Frictional loss</td>
<td>-15,819</td>
<td>-8.4</td>
</tr>
<tr>
<td>Creep loss</td>
<td>-7,417</td>
<td>-3.9</td>
</tr>
<tr>
<td>Shrinkage loss</td>
<td>-3,590</td>
<td>-1.9</td>
</tr>
<tr>
<td>Relaxation loss (3528 + 400e + 2056)</td>
<td>-9,657</td>
<td>-5.1</td>
</tr>
<tr>
<td>Increase due to topping</td>
<td>-5,048</td>
<td>-2.7</td>
</tr>
<tr>
<td>Final net stress $f_a$</td>
<td>149,232</td>
<td>79.0</td>
</tr>
</tbody>
</table>

Percentage of total losses = 100% - 77.9% = 22.1%, say, 22% for this posttensioned beam

*Frictional and anchorage safety losses are included in this table since the total jacking stress is given as 189,000 psi otherwise, the tendons would have to be jacked an additional stress of such magnitude as to neutralize the frictional and anchorage safety losses.

14.2.2 Example 14.3: Lump-sum Computation of Time-Dependent Losses in Prestressing

Solve Example 14.2 by the approximate lump-sum method and compare the results.

**Solution:** From Table 14.2, the total loss $\Delta_P = 35,000$ psi (241.3 MPa). So the net final strand stress by the lump-sum method is

$$f_a = 189,000 - 45,000 = 144,000 \text{ psi (993 MPa)}$$

**step-by-step** $f_a$ value = 149,232 psi

percent difference = \[
\frac{149,232 - 144,000}{140,000} \times 100 = 3.72\%
\]

The difference between the step-by-step "exact" method and the approximate lump-sum method is quite small, indicating that in normal, standard cases both methods are equally reliable.

14.3 FLEXURAL DESIGN OF PRESTRESSED CONCRETE ELEMENTS

Flexural stresses are the result of external, or imposed, bending moments. In most cases, they control the selection of the geometrical dimensions of the prestressed concrete section regardless of whether it is pretensioned or posttensioned. The design process starts with the choice of a preliminary geometry, and by trial and adjustment it converges to a final section with geometrical details of the concrete cross-section and the sizes and alignments of the prestressing strands. The section satisfies the flexural (bending) requirements of concrete stress and steel stress limitations. Thereafter, other factors, such as shear and torsion capacity, deflection, and cracking, are analyzed and satisfied.
While the input data for the analysis of sections differ from the data needed for design, every design is essentially an analysis. We assume the geometrical properties of the section to be prestressed and then proceed to determine whether the section can safely carry the prestressing forces and the required external loads. Hence a good understanding of the fundamental principles of analysis and the alternatives presented thereby significantly simplifies the task of designing the section. The basic mechanics of materials, principles of equilibrium of internal forces, and strain principles of superposition have to be adhered to in all stages of loading.

It suffices in the flexural design of reinforced concrete members to apply only the limit states of stress at failure for the choice of the section, provided that other requirements such as serviceability, shear capacity, and bond are met. In the design of prestressed members, however, additional checks are needed at the load-transfer and limit state at service load, as well as at the limit state at failure, with the failure load indicating the reserve strength for overload conditions. All these checks are necessary to ensure that at service load cracking is negligible and the long-term effects on deflection or camber are well controlled.

In view of the preceding, this section covers the major aspects of both the service-loading flexural design and the ultimate-load flexural design check. Note that a logical sequence in the design process entails first the service-load design of the section required in flexure and then the analysis of the available moment strength $M_u$ to compensate for the limit state at failure. A positive sign (+) is used to denote compressive stress, and a negative sign (-) is used to denote tensile stress in the concrete section. A convex or hogging shape indicates positive bending moment and a concave or sagging shape, negative bending moment, as shown in Figure 14.5.

Unlike the case of reinforced concrete members, where dead load and partial live load are applied to the prestressed concrete member at varying concrete strengths at various loading stages, these loading stages can be summarized as follows:

1. Initial prestress force $P$ is applied; then, at transfer, the force is transmitted from the prestressing strands to the concrete.
2. The self-weight $W_o$ acts on the member together with the initial prestressing force, provided that the member is simply supported, that is, there is no intermediate support.
3. The full superimposed dead load $W_{ud}$, including topping for composite action, is applied to the member.
4. Most short-term losses in the prestressing force occur initially, leading to a reduced prestressing force $P_o$.

Figure 14.5 Sign convention for flexure stress and bending moment: (a) negative bending moment; (b) positive bending moment.
5. The member is subjected to the full service load, with long-term losses due to creep, shrinkage, and steel strand relaxation taking place and leading to a net prestressing force $P_p$.

6. Overloading of the member occurs under certain conditions up to the limit state at failure.

A typical loading history and corresponding stress distribution across the depth of the critical section are shown in Figure 14.6, while a schematic plot of load versus deformation (number of deflection) is shown in Figure 14.7 for the various loading stages from the self-weight effect up to rupture.

14.3.3 Selection of Geometrical Properties of Section Components

14.3.3.1 General Guidelines. Under service-load conditions, the beam is assumed to be homogeneous and elastic. Since it is also assumed (because expected) that the prestress compressive force transmitted to the concrete closes the crack that might develop at the tensile fibers of the beam, beam sections are considered uncracked. Stress analysis of prestressed beams under these conditions is no different from stress analysis of a steel beam or, more accurately, a beam-column. The axial force due to prestressing is always present regardless of whether bending moments do or do not exist due to other external or self-loads.

It is advantageous to have the alignment of the prestressing tendons eccentric at the critical sections, such as the midspan section in a simple beam and the support section in a continuous beam. As compared to a rectangular solid section, a non-symmetrically flanged section has the advantage of efficiently using the concrete material and of concentrating the concrete in the compressive zone of the section where it is most needed.

Equations 14.9 to 14.11, to be subsequently presented, are stress equations that are convenient in the analysis of stresses in the section once the section is chosen. For design, it is necessary to transpose the three equations into geometrical equations so that the student and the designer can readily choose the concrete section. A logical transpo-
14.3.1.2. Minimum Section Modulus. To design or choose the section, a determination of the required minimum section modulus, \( S_{min} \), has to be made first. If

\[ f_c = \text{maximum allowable compressive stress in concrete immediately after transfer and prior to losses} \]
\[ = 0.60f'_c \]

\[ f_t = \text{maximum allowable tensile stress in concrete immediately after transfer and prior to losses} \]
\[ = 3 \sqrt{f'_c} \] (the value can be increased to \( 6 \sqrt{f'_c} \) at the supports for simply supported members)

\[ f_c = \text{maximum allowable compressive stress in concrete after losses at service-load level} \]
\[ = 0.45f'_c \text{ or } 0.60f'_c \text{ as stipulated in ACI 318 Code} \]

\[ f_t = \text{maximum allowable tensile stress in concrete after losses at service-load level} \]
\[ = 6 \sqrt{f'_c} \] (the value can be increased in one-way systems to \( 12 \sqrt{f'_c} \) if long-term deflection requirements are met)

Then, for any load, the fiber stresses in the concrete must not exceed the values listed.
14.3 Flexural Design of Prestressed Concrete Elements

Using the uncracked unsymmetrical section, a summary of the equations of stress for the various loading stages is as follows:

1. Stress at transfer

\[ f' = \frac{P_i}{A_t} \left( 1 - \frac{e_t}{r} \right) - \frac{M_{0b}}{S} \leq f_t \]  
\[ f_h = \frac{P_i}{A_t} \left( 1 + \frac{e_b}{r} \right) + \frac{M_{0b}}{S} \leq f_t \]  

(14.9a)

(14.9b)

where \( P_i \) is the initial prestressing force, \( f_t \) would be the horizontal component of \( P_i \) it is reasonable for all practical purposes to disregard such refinement.

2. Effective stresses after losses

\[ f'' = \frac{P_i}{A_t} \left( 1 - \frac{e_t}{r} \right) - \frac{M_{fy}}{S} \leq f_t \]  
\[ f_h = \frac{P_i}{A_t} \left( 1 + \frac{e_b}{r} \right) + \frac{M_{fy}}{S} \leq f_t \]  

(14.10a)

(14.10b)

3. Service load final stresses

\[ f' = \frac{P_i}{A_t} \left( 1 - \frac{e_t}{r} \right) - \frac{M_{fy}}{S} \leq f_t \]  
\[ f_h = \frac{P_i}{A_t} \left( 1 + \frac{e_h}{h} \right) + \frac{M_{fy}}{S} \leq f_t \]  

(14.11a)

(14.11b)

where \( M_{fy} = M_{f0} + M_{f0} + M_f \)

\( P_i \) = initial prestress

\( P_e \) = effective prestress after losses

\( e \) = eccentricity of tendons from the concrete section center of gravity, cgc

\( r^2 \) = square of radius of gyration

\( E_N \) = top/bottom section modulus value of concrete section

The decomposition stage denotes the increase in steel strain due to the increase in load from the stage when the effective prestress \( P_e \) acts alone to the stage when the additional load causes the compressive stress in the concrete at the cgc level to reduce to zero (see Figure 14.7). At this stage, the change in concrete stress due to decomposition is

\[ f_{comp} = \frac{P_i}{A_t} \left( 1 + \frac{e^2}{r^2} \right) \]  

(14.11c)

This relationship is based on the assumption that the strain between the concrete and the prestressing steel bonded to the surrounding concrete is such that the gain in the steel stress is the same as the decrease in the concrete stress.

14.3.1.3. Beams with Variable Tendon Eccentricity. Beams are prestressed with either draped or harped tendons. The maximum eccentricity is usually at the midspan controlling section for the simply supported case. Assuming that the effective prestressing force is

\[ P_e = \gamma P_i \]  

\( \gamma \) = factor to account for the variable tendon eccentricity.
where \( \gamma \) is the residual prestress ratio, the loss of prestress is
\[
P_1' = P_1 - (1 - \gamma)P_1. \tag{a}
\]

If the actual concrete extreme fiber stress is equivalent to the maximum allowable stress, the change in stress after losses, from Eqs. 14.5a and 14.5b, is given by
\[
\Delta f^p = (1 - \gamma) \left( f_0 + \frac{M_0}{S} \right) \tag{b}
\]
\[
\Delta f = (1 - \gamma) \left( -f_1 - \frac{M_0}{S} \right). \tag{c}
\]

From Figure 14.8, as the superimposed dead-load moment \( M_{pl} \) and live-load moment \( M_0 \), act on the beam, the net stress at the top fibers is
\[
f_1 = f_0 - \Delta f^p - f_1. \tag{d}
\]
or
\[
f_2 = \eta f_0 - (1 - \gamma) \frac{M_{pl}}{S} - f_2. \tag{e}
\]

Figure 14.8 Maximum fiber stresses in beams with dropped or tilted tendons; (a) critical section such as midspan; (b) support section of singly supported beam (\( f' = 0 \) at tendon moves to c.g.)
The net stress at the bottom fibers is

$$f_n = f_i - f_i' - \Delta \gamma$$

or

$$f_n = f_i - \gamma f_i - (1 - \gamma) \frac{M_y}{S_y}$$  \(\text{(c)}\)

From Eqs. (5) and (c), the chosen section should have section moduli values

$$S_i = \frac{(1 - \gamma) M_D + M_{oi} + M_t}{\gamma f_i - f_i}$$  \(\text{(14.12a)}\)

and

$$S_{ei} = \frac{(1 - \gamma) M_D + M_{oi} - M_t}{\gamma f_i - f_i}$$  \(\text{(14.12b)}\)

The required eccentricity of the prestressing tendon at the critical section, such as the midspan section, is

$$e_i = (\gamma f_i - \gamma) \frac{S}{P_i} + \frac{M_D}{P_i}$$  \(\text{(14.12c)}\)

where $\gamma$ is the concrete stress at transfer at the level of the centroid ctc of the concrete section and

$$P_i = \gamma A_i$$

Thus

$$\Delta f_i = \frac{\gamma}{\gamma} (\gamma f_i - f_i')$$  \(\text{(14.12d)}\)

### 14.3.1.4. Beams with Constant Tendon Eccentricity

Beams with constant tendon eccentricity are beams with straight tendons, as is normally the case in precast, moderate-span, simply supported beams. Because the tendon has a large eccentricity at the support, creating large tensile stresses at the top fibers without any reduction due to superimposed $M_D$, $M_{oi}$, in such beams smaller eccentricity of the tendon at midspan has to be used compared to a similar beam with a draped tendon. In other words, the controlling section is the support section, for which the stress distribution at the support is shown in Figure 14.9. Hence:

$$\Delta f_0 = (1 - \gamma) f_n$$  \(\text{(a')}\)

and

$$\Delta f_0 = (1 - \gamma) (-f_i')$$  \(\text{(b')}\)

The net stress at the service-load condition after losses at the top fibers is

$$f_i = f_i - \Delta f_0 - f_i'$$

or

$$f_i = \gamma f_i - f_i'$$  \(\text{(c')}\)

where $f_i$ is the actual service-load stress in concrete. The net stress at service load after losses at the bottom fibers is

$$f_n = f_i - f_i' - \Delta \gamma$$

or

$$f_n = f_i - \gamma f_i$$  \(\text{(d')}\)

From Eqs. (c) and (d), the chosen section should have section moduli values
Figure 14.9 Maximum fiber stresses in strain section of beams with straight tendons (stress distribution in midspan section similar to that of Figure 14.8).

\[
S = \frac{M_o + M_{0y} + M_i}{f_{ct}} \quad (14.13a)
\]

and

\[
S_p = \frac{M_o + M_{0y} + M_i}{f_{ct}} \quad (14.13b)
\]

The required eccentricity value at the critical section, such as the support for an ideal beam section having properties close to those required by Eqs. 14.13a and 14.13b is

\[
e = \left( \bar{d} - \bar{d}_e \right) \frac{S}{S_p} \quad (14.13c)
\]

Table 14.4 gives the section moduli of standard PCI rectangular sections. Tables 14.5 and 14.6 give the geometrical outer dimensions of standard PCI T sections and AASHTO I sections respectively, as well as the top-section moduli of those sections needed in the preliminary choice of the section in the service-load analysis. Table 14.7 gives dimensional details of the actual as-built geometry of the standard PCI and AASHTO sections.

<table>
<thead>
<tr>
<th>Designation</th>
<th>12RB16</th>
<th>12RD20</th>
<th>12RB24</th>
<th>12RB28</th>
<th>12RB32</th>
<th>12RD36</th>
<th>12RB40</th>
<th>16RA40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Section modulus, ( S ) (in.(^3))</td>
<td>1532</td>
<td>800</td>
<td>1153</td>
<td>1508</td>
<td>2046</td>
<td>2502</td>
<td>2771</td>
<td>3456</td>
</tr>
<tr>
<td>Width, ( b ) (in.)</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>16</td>
</tr>
<tr>
<td>Depth, ( d ) (in.)</td>
<td>18</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>
### 14.3 Geometrical Outer Dimensions and Section Moduli of Standard PCI 1 Sections

<table>
<thead>
<tr>
<th>Designation</th>
<th>Top-Bottom Section Module (in.²)</th>
<th>Flange Width, t₁ (in.)</th>
<th>Flange Width, t₂ (in.)</th>
<th>Flange Depth, h (in.)</th>
<th>Web Width, b₂ (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8D12</td>
<td>1001/116</td>
<td>98</td>
<td>1</td>
<td>12</td>
<td>9.5</td>
</tr>
<tr>
<td>8D14</td>
<td>1307/429</td>
<td>96</td>
<td>1</td>
<td>14</td>
<td>9.5</td>
</tr>
<tr>
<td>8D16</td>
<td>1630/556</td>
<td>94</td>
<td>2</td>
<td>16</td>
<td>9.5</td>
</tr>
<tr>
<td>8D18</td>
<td>2330/580</td>
<td>96</td>
<td>2</td>
<td>20</td>
<td>9.5</td>
</tr>
<tr>
<td>8D20</td>
<td>3303/1224</td>
<td>96</td>
<td>2</td>
<td>24</td>
<td>9.5</td>
</tr>
<tr>
<td>8D24</td>
<td>5140/2613</td>
<td>96</td>
<td>2</td>
<td>32</td>
<td>9.5</td>
</tr>
<tr>
<td>8D28</td>
<td>5960/2717</td>
<td>120</td>
<td>2</td>
<td>32</td>
<td>12.5</td>
</tr>
<tr>
<td>*8D36</td>
<td>10456/3340</td>
<td>144</td>
<td>4</td>
<td>34</td>
<td>12.5</td>
</tr>
<tr>
<td>*8D40</td>
<td>15138/4274</td>
<td>180</td>
<td>4</td>
<td>34</td>
<td>12.5</td>
</tr>
</tbody>
</table>

*Excepted

#### 14.3.2 Example 14.4: Service-load Design Example

Design a simply supported prestressed beam with a span of 65 ft (19.8 m) using the AASHTO Building Code. Statically stabilize the beam with a superimposed dead load of 1.100 kip (6.4 kN/m) and a superimposed live load of 100 kip (441 kN/m) and have no concrete reinforcement. Assume that the beam is made of normal-weight concrete with f_c = 2,000 psi (13.8 MPa) and the concrete strength of 71% of the cylinder strength. Assume also that the time-dependent losses of the initial prestress are 18% of the initial prestress and f_y = 270,000 psi (1,862 MPa) for stress-relieved tendons.

### 14.5 Geometric Outer Dimensions and Section Moduli of Standard AASHTO Ridge Sections

<table>
<thead>
<tr>
<th>Designation</th>
<th>Type 1</th>
<th>Type 2</th>
<th>Type 3</th>
<th>Type 4</th>
<th>Type 5</th>
<th>Type 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>x, in.</td>
<td>276</td>
<td>369</td>
<td>560</td>
<td>779</td>
<td>3013</td>
<td>3085</td>
</tr>
<tr>
<td>Moment of inertia, Z, in.⁴</td>
<td>23.790</td>
<td>36.979</td>
<td>123.500</td>
<td>260.731</td>
<td>521.130</td>
<td>735.520</td>
</tr>
<tr>
<td>Top-bottom section modulus, M</td>
<td>1.476</td>
<td>2.527</td>
<td>5.070</td>
<td>8.008</td>
<td>16.790</td>
<td>20.307</td>
</tr>
<tr>
<td>Top flange width, b₁ (in.)</td>
<td>12</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>42</td>
<td>42</td>
</tr>
<tr>
<td>Top flange average thickness, t₁ (in.)</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>Bottom flange width, b₂ (in.)</td>
<td>10</td>
<td>18</td>
<td>22</td>
<td>26</td>
<td>28</td>
<td>28</td>
</tr>
<tr>
<td>Bottom flange average thickness, t₂ (in.)</td>
<td>7</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>Total depth, h (in.)</td>
<td>28</td>
<td>36</td>
<td>65</td>
<td>54</td>
<td>63</td>
<td>72</td>
</tr>
<tr>
<td>Web width, b₂ (in.)</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>c₁ (in.)</td>
<td>15.41</td>
<td>20.17</td>
<td>24.7</td>
<td>29.27</td>
<td>31.04</td>
<td>35.62</td>
</tr>
<tr>
<td>c₂ (in.)</td>
<td>12.90</td>
<td>15.83</td>
<td>20.37</td>
<td>24.75</td>
<td>31.96</td>
<td>36.38</td>
</tr>
<tr>
<td>p, in.²</td>
<td>82</td>
<td>132</td>
<td>224</td>
<td>330</td>
<td>314</td>
<td>676</td>
</tr>
</tbody>
</table>

### Notes
- The AASHTO specifications are used for the design example.
- The beam is subjected to both dead and live loads, with appropriate considerations for time-dependent losses and concrete strength.
- The material properties are specified for normal-weight concrete and prestressing steel, with appropriate strength and modulus values.
- The section properties are calculated to ensure structural integrity and compliance with design guidelines.
### Table 14.7 Geometrical Details of As-built PCI and AASHTO Sections

<table>
<thead>
<tr>
<th>Designation</th>
<th>( b_0 ) (in.)</th>
<th>( b_1 ) (in.)</th>
<th>( h_1 ) (in.)</th>
<th>( h_0 ) (in.)</th>
<th>( h ) (in.)</th>
<th>( b ) (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>EDT12</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>12.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT14</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>14.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT16</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>16.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT18</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>18.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT20</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>20.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT24</td>
<td>96.2</td>
<td>5.75</td>
<td>3.75</td>
<td>24.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT32</td>
<td>96.2</td>
<td>7.75</td>
<td>4.75</td>
<td>32.0</td>
<td>48.0</td>
<td></td>
</tr>
<tr>
<td>EDT34</td>
<td>120.4</td>
<td>7.75</td>
<td>4.75</td>
<td>32.0</td>
<td>60.0</td>
<td></td>
</tr>
<tr>
<td>EDT34</td>
<td>144.4</td>
<td>7.75</td>
<td>4.75</td>
<td>34.0</td>
<td>60.0</td>
<td></td>
</tr>
<tr>
<td>I2EDT34</td>
<td>144.4</td>
<td>7.75</td>
<td>4.75</td>
<td>34.0</td>
<td>60.0</td>
<td></td>
</tr>
</tbody>
</table>

### Solution

\[
\gamma = 100 - 18 = 82 \%
\]

\[
f_0 = 0.75 \times 3000 = -3750 \text{ psi}(25.9 \text{ MPa})
\]

\[
f_1 = 0.50 \times 3750 = -1875 \text{ psi}(13.1 \text{ MPa})
\]

\[
f_2 = (3750 - 350) = 184 \text{ psi (midspan)}
\]

\[
m = 0.75 \times 3750 = 2813 \text{ psi (support)}
\]

\[
f = 0.43 \times 3000 = -1290 \text{ psi}(8.9 \text{ MPa})
\]

Use \( f = (3750 - 350) = 184 \text{ psi} \) as the maximum stress in tension, and assume a self-weight of approximately 85 psi (0.6 kN/m). Then the self-weight moment is given by

\[
M_1 = \frac{wL^2}{8} = \frac{85(85)^2}{8} \times 12 = 5.388 \times 10^4 \text{ in.-lb} (687.7 \text{ kN-m})
and the superimposed load moment is

\[ M_{\text{super}} + M_f = \frac{(1100 + 100)(65)^2}{8} \times 12 = 7,655,000 \text{ in.-lb} (859.4 \text{ kN-m}) \]

Since the tendon is harped, the critical section is close to the midspan, where dead-load and superimposed dead-load moments reach their maximum. The critical section is in many cases taken at 0.42L from the support, where L is the beam span. From Eqs. 14.12a and 14.12b,

\[ S' = \frac{(1 - \gamma)M_{ps} + M_{ps} + M_f}{\gamma f_c - f_t} \]

\[ S' = \frac{(1 - 0.82)5,386,875 + 7,603,000}{0.82 \times 184 + 2250} = 3572 \text{ in.}^2 (58,535 \text{ cm}^2) \]

\[ S_0 = \frac{(1 - 0.82)5,386,875 + 7,603,000}{425 + (6.82 \times 2250)} = 3777 \text{ in.}^2 (51,892 \text{ cm}^2) \]

required \( S' = 3572 \text{ in.}^2 (58,535 \text{ cm}^2) \)
required \( S_0 = 3777 \text{ in.}^2 (51,892 \text{ cm}^2) \)

Since the section moduli at the top and bottom fibers are almost equal, a symmetrical section is adequate. Next, analyze the section in Figure 14.10 chosen by trial and adjustments.

**Analysis of stresses at transfer**

From Eq. 14.12d.

\[ \bar{f}_t = f_t - \frac{S_0}{A_0} \left( f_t - f_t^* \right) \]

\[ = 184 - \frac{3777}{40} \left( -1104 + 2250 \right) = -1104 \text{ psi (7.6 MPa)} \]

\[ P_t = A_t \bar{f}_t = 377 \times 1104 = 416,218 \text{ lb (1831 kN)} \]

\[ M_{nt} = \frac{393(65)^2}{8} \times 12 = 2,490,638 \text{ in.-lb (281 kN-m)} \]

![Figure 14.10 I-beam section in Ex. 14.4.](image-url)
From Eq. 14.12a, the eccentricity required at the section of maximum moment at midspan is
\[ e = (s - t_0) \frac{S'}{P} = \frac{M_0}{P} \]
\[ = (184 + 1104) \frac{3572}{416.208} = \frac{2490.03}{416.208} \]
\[ = 11.05 + 5.98 = 17.04 \text{ in. (433 mm)} \]
Since \( t_0 = 18.84 \text{ in.} \) and assuming a cover of 3.75 in., try \( e = 18.84 + 3.75 = 15.0 \text{ in. (381 mm)} \).

required area of tendons \( A_p = \frac{P}{f_p} = \frac{416.208}{185,000} = 2.2 \text{ in.}^2 (14.2 \text{ cm}^2) \)

number of strands = \( \frac{2.2}{0.152} = 14.38 \)

Try thirteen 1-in. strands, \( A_p = 1.99 \text{ in.}^2 (12.8 \text{ cm}^2) \), and an actual \( P = 185,000 = 1.99 = 376,110 \text{ ft} (1675 \text{ kN}) \), and check the concrete extreme fiber stress. From Eq. 14.2b,
\[ f_c = \frac{P}{A \left( e - \frac{a}{E}\frac{M_0}{S} \right)} \]
\[ = \frac{376,110}{377} \left( \frac{1}{15.0 \times 31.16} \right) \frac{2490.03}{3340} \]
\[ = 691.2 - 725.7 = -54.5 \text{ psi (C), no tension at transfer} \]
\[ \text{O.K.} \]

From Eq. 14.9b,
\[ f_s = \frac{P}{\chi \left( 1 + \frac{a}{E}\frac{M_0}{S} \right)} \frac{M_n}{S_0} \]
\[ = \frac{376,110}{377} \left( \frac{1 + 15 \times 19.84}{167.5} \right) \frac{2490.03}{3750} \]
\[ = -2501.3 + 664.5 = -1837.1 \text{ psi (C) < } f_s = 2250 \text{ psi} \]
\[ \text{O.K.} \]
14.3 Flexural Design of Prestressed Concrete Elements

Analysis of stresses at service load

From Eq. 14.11a,

\[ f' = \frac{P_i}{A_i} \left( 1 + \frac{e_i}{r_i} \right) - \frac{M_i}{S_i} \]

where

- \( P_i = 13 \times 0.153 \times 154,980 = 308,235 \text{ lb (1372 kN)} \)
- Total moment \( M_i = M_{t0} + M_{l0} = 2,905,638 + 7,605,000 = 10,505,638 \text{ in-lb (1141 kNm)} \)
- \( f' = \frac{308,235}{377} \left( 1 - \frac{150 \times 21.16}{187.5} \right) = \frac{10,505,638}{3750} \)
- \( = 566.6 < 3022.6 = -258.4 \text{ psi (C) } \Rightarrow f_p = -2250 \text{ psi} \)

Hence either enlarge the depth of the section or use higher-strength concrete. Using \( f' = 6000 \text{ psi}, \)

\[ f_p = 0.45 \times 6000 = 2700 \text{ psi } \Rightarrow O.K. \]

\[ f_s = \frac{P_i}{A_i} \left( 1 + \frac{e_i}{r_i} \right) - \frac{M_i}{S_i} = \frac{308,235}{377} \left( 1 - \frac{150 \times 18.84}{187.5} \right) = \frac{10,505,638}{3750} \]

\[ = 2050 + 2092.2 = 642 \text{ psi (T) } \Rightarrow O.K. \]

Check support section stresses

- \( f_u = 0.75 \times 6000 = 4500 \text{ psi} \)
- \( f_t = 0.60 \times 4500 = 2700 \text{ psi} \)
- \( f_s = 3 \sqrt{f_u} = 201 \text{ psi for span} \)
- \( f_s = 6 \sqrt{f_u} = 402 \text{ psi for support} \)
- \( f_s = 0.45f_u = 2250 \text{ psi} \)
- \( f_s = 2 \sqrt{f_u} = 920 \text{ psi} \)

(a) At transfer: Support section compressive fiber stress,

\[ f_o = \frac{P_i}{A_i} \left( 1 + \frac{e_i}{r_i} \right) = 0 \]

\[ P_i = 376,110 \text{ lb} \]

or

\[ -2700 = \frac{376,110}{377} \left( 1 + \frac{e \times 18.84}{187.5} \right) \]

so that

\[ e = 16.98 \text{ in}. \]

Accordingly, try \( e = 12.49 \text{ in.} \):

\[ f' = \frac{376,110}{377} \left( 1 - \frac{12.49 \times 21.16}{187.5} \right) = 0 \]

\[ = 469 \text{ psi (T) } \Rightarrow f_p = 469 \text{ psi} \]

Thus, use mild steel at the top fibers at the support section to take all tensile stresses in the concrete, or use a higher-strength concrete for the section, or reduce the eccentricity.
Chapter 14  Introduction to Prestressed Concrete

(b) At service load

\[
f' = \frac{508.225}{377} \left( 1 - \frac{12.49 \times 21.16}{187.5} \right) = 0 = 334.9 \text{ psi} \quad (T) < 465 \text{ psi} \quad \text{O.K.}
\]

\[
f_b = \frac{508.225}{377} \left( 1 + \frac{12.49 \times 18.84}{187.5} \right) = 0 = -1843 \text{ psi} \quad (C) < -2700 \text{ psi} \quad \text{O.K.}
\]

Hence adopt the 40-in. (102-cm)-deep I-section prestressed beam of \( f' \), equal to 6000 psi (41.4 MPa) normal-weight concrete with thirteen 1-in. strands having midspan eccentricity \( c_x = 15.0 \text{ in.} \) (381 mm) and end section eccentricity \( c_y = 12.5 \text{ in.} \) (318 mm). An alternative to this solution is to continue using \( f' = 5000 \text{ psi} \), but change the number of strands and eccentricities.

14.3.3 Flowchart for Service-load Flexural Design of Prestressed Beams

Figure 14.11 shows a flowchart for the service-load flexural design of prestressed beams.
14.4 SERVICEABILITY REQUIREMENTS IN PRESTRESSED CONCRETE MEMBERS

Prestressed concrete flexural members are classified into 3 classes in the new ACI 318 Code.

(a) Class U:

\[ f_s \leq 7.5 \sqrt{f_c} \]  

(14.14a)

In this class, the gross section is used for section properties when both stress computations at service loads and deflection computations are made. No skin reinforcement needs to be used.

(b) Class T:

\[ 7.5 \sqrt{f_c} < f_s \leq 12 \sqrt{f_c} \]  

(14.14b)

This class is a transition between uncracked and cracked sections. For stress computations at service loads, the gross section is used. The cracked bilinear section is used in the deflection computations. No skin reinforcement needs to be used.

(c) Class C:

\[ f_s > 12 \sqrt{f_c} \]  

(14.14c)

This class denotes cracked sections. Hence, a cracked section analysis has to be made for evaluation of the stress level at service, and for deflection. Computation of \( \Delta \varepsilon \) or \( f_s \), for crack initiation and growth, where \( f_s \) is stress at first peak of the stress-strain curve.
and $f_e =$ stress in the mild reinforcement when mild steel reinforcement is also used.

Crank control provisions for distribution of mild steel reinforcement is as follows:

$$s = \frac{540}{f_e} - 2.5c_e$$  \hspace{1cm} (14.15)

where $s =$ mild reinforcement spacing, in.
$f_e =$ service load stress level in the reinforcement of one-way members = 0.60 $f_e$, ksi
$c_e =$ clear cover from the nearest surface in tension to the surface of the tension reinforcement, in.

If longitudinal skin reinforcement has to be used at the vertical side faces in the case of very deep beams, the size and spacing of bars or wires are to be chosen as given in Section 8.12 stipulating the size and spacing of skin reinforcement.

The ACI Code does not give guidance for evaluating the developed crack width, and whether it is within tolerable limits, for overstress $\Delta f_e$ beyond the decompression state. The author's extensive work (Ref. 14.7) and the provisions of the ACI 224 Report on cracking (Ref. 14.8) recommend the following expressions for crack width evaluation in prestressed concrete members:

(a) Pretensioned beams:

$$w_{cr} = 5.85 \times 10^{-6} \frac{A_e}{\Sigma_0} (\Delta f_e)$$  \hspace{1cm} (14.16a)

(b) Post-tensioned bonded beams:

$$w_{cr} = 6.51 \times 10^{-6} \frac{A_e}{\Sigma_0} (\Delta f_e)$$  \hspace{1cm} (14.16b)

For non-bonded beams, the factor 6.51 becomes 6.83.

$A_e =$ area of concrete in tension, in.$^2$
$\Sigma_0 =$ sum of perimeters of all reinforcing elements, both mild and prestressing reinforcement, in.
$\Delta f_e =$ increase in stress in the prestressing reinforcement above the decompression state level

For high-strength prestressed concrete beams where $6.000 < f_e' \approx 12.000$, a factor of 2.75 is to be used instead. For more refined values, a modifying factor for particular $f_e'$ values can be obtained from the following expressions:

For pretensioned beams, the reduction multiplier $\lambda_p$ is

$$\lambda_p = \frac{2}{(0.75 + 0.06 \sqrt{f_e}) \sqrt{f_e'}}$$  \hspace{1cm} (14.17a)

For post-tensioned beams, the reduction multiplier $\lambda_p$ is

$$\lambda_p = \frac{1}{0.75 + 0.06 \sqrt{f_e'}}$$  \hspace{1cm} (14.17b)

where $f_e'$ and the reinforcement stress are in ksi.

### 14.5 ULTIMATE-STRENGTH FLEXURAL DESIGN OF PRESTRESSED BEAMS

#### 14.5.1 Rectangular Sections

The actual distribution of the compressive stress in a section at failure has the form of a rising parabola, as shown in Figure 14.12c. It is time-consuming to evaluate the volume of the compressive stress block if it has a parabolic shape. An equivalent rectangular stress
block due to Whitney can be used with ease and without loss of accuracy to calculate the compressive force and hence the flexural moment strength of the section. This equivalent stress block has a depth \(a\) and an average compressive strength \(0.85f'_c\). As seen from Figure 14.12d, the value of \(a\) is determined by using a coefficient \(B\), such that the area of the equivalent rectangular block is approximately the same as that of the parabolic compressive block, resulting in a compressive force \(C\) of essentially the same value in both cases.

The value \(0.85f'_c\) for the average stress of the equivalent compressive block is based on the core test results of concrete in the structure at a minimum age of 28 days. Based on exhaustive experimental tests, a maximum allowable strain of 0.003 in./in. was adopted by the ACI as a safe limiting value. Even though several forms of stress blocks, including the trapezoidal, have been proposed to date, the simplified equivalent rectangular block is accepted as the standard in the analysis and design of reinforced concrete. The behavior of the steel is assumed to be elastoplastic.

Using all the preceding assumptions, the stress distribution diagram shown in Figure 14.12c can be redrawn as shown in Figure 14.12d. We can easily deduce that the compression force \(C\) can be written \(0.85f'_c/ba\), that is, the volume of the compressive block at or near the ultimate when the tension steel has yielded \(s > s_c\). The resultant force \(T\) can be written as \(A_{o_f}f'_o\); thus the equilibrium equation can be written as

\[
A_{o_f}f'_o = 0.85f'_c/ba
\]

Hence

\[
a = \beta_o s = \frac{A_{o_f}f'_o}{0.85f'_c}
\]

The nominal moment strength is obtained by multiplying \(C\) or \(T\) by the moment arm \(r_i = a/2\), yielding
14.6 Ultimate-Strength Flexural Design of Prestressed Beams

\[ M_u = A_p f_y \left( d_e - \frac{d}{2} \right) \]  \hspace{1cm} (14.19a)

where \( d_e \) is the distance from the compression fibres to the center of the prestressed reinforcement. The steel percentage \( p_s = A_s / b h' \) gives nominal strength of the prestressing steel only as follows

\[ M_s = p_s f_y b h' \left( 1 - 0.59 p_s \frac{f_y}{f_c} \right) \]  \hspace{1cm} (14.19b)

If \( p_s \) is the reinforcement index = \( g, f_y, f_c \). Eq. 14.19b becomes

\[ M_s = p_s f_y b h' \left( 1 - 0.59 p_s \right) \]  \hspace{1cm} (14.19c)

The contribution of the mild steel tension reinforcement should be similarly treated, so that the depth \( d \) of the compressive block is

\[ d = \frac{A_{m, f_m} + A_{f_c}}{0.85 f_c b} \]  \hspace{1cm} (14.20a)

If \( c = a \beta_{11} \), the strain at the level of the mild steel is (Fig. 14.12)

\[ \epsilon_c = \epsilon \frac{d}{c} \]  \hspace{1cm} (14.20b)

Equation 14.19a, for rectangular sections but with mild tension steel and no compression steel accounted for, becomes

\[ M_s = w_c f_y b h' \left( 1 - 0.59 w_c \frac{f_y}{f_c} \right) + w_c b h' \left( 1 - 0.59 w_c \frac{f_y}{f_c} \right) \]  \hspace{1cm} (14.21a)

or can be rewritten as either

\[ M_s = A_{p, f} \left( 1 - 0.59 \left( w_c + \frac{d}{d_e} \right) \right) + A_{f_c} \left( 1 - 0.59 \left( d_e \frac{d}{d_e} w_c + \epsilon \right) \right) \]  \hspace{1cm} (14.21b)

---

**Photo 14.5** Flexural cracks at failure of prestressed T-beams. (Tests by Navy and Peppard.)
where \( w = \pi f_y f_c' \) or
\[
M_c = A_{p_r} f_y \left( \frac{d_r}{2} \right) + A_{f_r} \left( d - \frac{d_r}{2} \right) \quad (14.21c)
\]
The contribution from compression reinforcement can be taken into account provided it has been assumed to have yielded
\[
a = A_{p_r} + A_{f_r} \frac{f_y}{0.85 f_p} \quad (14.22)
\]
where \( b \) is the section width at the compression face of the beam.

Taking moments about the center of gravity of the compressive block in Figure 14.13, the nominal moment strength in Eq. 14.21 becomes
\[
M_c = A_{p_r} f_y \left( \frac{d_r}{2} \right) + A_{f_r} \left( d - \frac{d_r}{2} \right) + A_{f_c} \left( \frac{d}{2} - a \right) \quad (14.23)
\]

14.5.2 Nominal Moment Strength of Flanged Sections

When the compression flange thickness \( h_f \) is less than the neutral axis depth \( c \) and equivalent rectangular block depth \( d \), the section can be treated as a flanged section as in Figure 14.14. From the figure,
14.5 Ultimate-Strength Flexural Design of Prestressed Beams

\[ T_p + T_e = T_{pm} + T_{ef} \]  

(14.24)

where

\( T_p \) = total prestressing force = \( A_p f_p \)

\( T_e \) = ultimate force in the nonprestressed steel = \( A_f f_e \)

\( T_{pm} \) = part of the total force in the tension reinforcement required to develop the web = \( A_{pm} f_{pm} \)

\( A_{pm} \) = total reinforcement area corresponding to the force \( T_{pm} \)

\( T_{ef} \) = part of the total force in the tension reinforcement required to develop the flange = \( C_e = 0.85f_e (b - b_o) h_y \)

\( C_e = 0.85f_e h_y \)

Substituting in Eq. 14.20, we obtain

\[ T_{re} = A_{pm} f_{pm} + A_{ef} f_e - 0.85f_e (b - b_o) h_y \]  

(14.25)

Summing up all forces in Fig. 14.14c and d, we have

\[ T_{pm} + T_{ef} = C_e + C_f \]

giving

\[ \sigma = \frac{A_{pm} f_{pm}}{0.85f_e h_y} \]  

(14.26a)

or

\[ \sigma = \frac{A_{pm} f_{pm} + A_{ef} f_e - 0.85f_e (b - b_o) h_y}{0.85f_e h_y} \]  

(14.26b)

Equation 14.23 for a beam with compression reinforcement can be rewritten to give the nominal moment strength for a flanged section where the neutral axis falls outside the flange and \( \sigma > h_y \) as follows, taking moments about the center of the prestressing steel:

\[ M_e = A_{pm} f_{pm} \left( d_o - \frac{d}{2} \right) + A_{ef} f_e (d - d_o) + 0.85f_e (b - b_o) h_y \left( d_o - \frac{b}{2} \right) \]  

(14.27)

The design moment in all cases would be

\[ M_e = 0.9 M_e \]  

(14.28)

where \( \phi = 0.90 \) for flexure.

In order to determine whether the neutral axis falls outside the flange, requiring a flanged section analysis, we must determine, as discussed in Chapter 5, whether the total compressive force \( C_e \) is larger or smaller than the total tensile force \( T_e \). If \( T_p + T_e \) in Fig. 14.14 is larger than \( C_e \), the neutral axis falls outside the flange and the section has to be treated as a flanged section. Otherwise, it should be treated as a rectangular section of the width \( b \) of the compression flange.

Another method of determining whether the section can be considered flanged is to calculate the value of the equivalent rectangular block depth \( d \) from Eq. 14.26(b), thereby determining the neutral-axis depth \( c = d h_y \).

14.5.3 Determination of Prestressing Steel Nominal Failure Stress \( f_p \)

The value of the stress \( f_p \) of the prestressing steel at failure is not readily available. However, it can be determined by strain compatibility through the various loading stages up to the limit state at failure, as defined in Eqs. 14.15. Such a procedure is required if

\[ f_p = P_{ps} < 0.50 f_{pm} \]  

(14.29a)

Approximate determination is allowed by the ACI 318 building code provided that
\[ f_{re} = \frac{f_u}{A_{re}} = 0.50f_u \]

with separate equations for \( f_{re} \) given for bonded and nonbonded members.

**Bonded Tendons.** The empirical expression for bonded members is

\[ f_{re} = f_{reD} \left( 1 - \frac{\gamma_r}{\gamma_n} \left[ \frac{1}{\phi_1} \left( \frac{f_{rc}}{f_{ci}} + \frac{d_{p}}{d_{p}} (\omega - \omega') \right) \right] \right) \]

where the reinforcement index for the compression nonprestressed reinforcement is \( \omega' = \frac{\omega}{b} \frac{f_{ci}}{f_{ci}} \). If the compressive reinforcement is taken into account when calculating \( f_{re} \) by Eq. 14.26, the term \( \left( \frac{f_{rc}}{f_{ci}} + \frac{d_{p}}{d_{p}} (\omega - \omega') \right) \) should not be less than 0.17 and \( d_{p} \) should not be greater than 0.15d. Also, \[ \gamma_r = 0.55 \text{ for } f_{reD}/f_u \text{ not less than 0.80} \]
\[ = 0.40 \text{ for } f_{reD}/f_u \text{ not less than 0.65} \]
\[ = 0.28 \text{ for } f_{reD}/f_u \text{ not less than 0.90} \]

The value of the factor \( \gamma_r \) is based on the criterion that \( f_{re} = 0.80f_u \) for high-strength prestressing bars, 0.85 for stress-relieved strands, and 0.90 for low-relaxation strands.

**Unbonded Tendons.** For a span-to-depth ratio of 35 or less,

\[ f_{re} = f_{reD} + 10,000 + 0.001 \frac{f_{ci}}{f_u} \]

where \( f_{reD} \) shall not be greater than \( f_{reD} \) or \( f_{reD} \) + 60,000.

For a span-to-depth ratio greater than 35,

\[ f_{re} = f_{reD} + 10,000 + \frac{f_{ci}}{500f_u} \]

where \( f_{reD} \) shall not be greater than \( f_{reD} \) or \( f_{reD} \) + 30,000. Code requirements for maximum and minimum reinforcement index \( \omega \) have to be observed.

AASHTO expressions for the ultimate design strength \( f_u \) of the prestressing reinforcement are given in Chapter 15.

The ultimate analysis and design of prestressed concrete members follow the strain limits approach detailed in Chapter 5. The net tensile strain limits for compression, tension-controlled sections shown in Fig. 5.5 are equally applicable to prestressed concrete sections. They replace the maximum-reinforcement limits used in code provisions prior to the 2005 ACI 318 Code. The net tensile strain for tension-controlled sections, may still be stated in terms of the reinforcement index, \( \omega_{re} \) embodied in Eq. 14.30, where the maximum index is

\[ [\omega_{re} + \frac{d_{p}}{d_{p}'} (\omega - \omega')] \]

The net tensile strain limit of 0.005 corresponds to \( \omega_{re} = 0.22b \) for prestressed rectangular sections.

It should be noted that the total amount of prestressed and nonprestressed reinforcement should be adequate to develop a factored load of at least 1.2 times the cracking load computed on the basis of the modulus of rupture \( f_c \). This provision in ACI 318 Code is permitted to be waived for (a) two-way, unbonded post-tensioned slabs and (b) flexural members with shear and flexural strength at least twice the load level causing the first cracking moment \( M_{cr} \).

A flowchart for strength flexural analysis is shown in Figure 14.15.
Figure 14.15 Flowchart for ultimate load flexural analysis of rectangular and flanged prestressed sections based on cgs profile depth.
14.6 EXAMPLE 14.5: ULTIMATE-STRENGTH DESIGN OF PRESTRESSED SIMPLY SUPPORTED BEAM BY STRAIN COMPATIBILITY

Design the tendon beam in Ex. 14.4 by the ultimate-load theory using nonprestressed reinforcement to partially carry part of the factored loads. Use strain compatibility to evaluate $f_{pc}$ given:

- $f_p = 270,000$ psi (1862 MPa)
- $f_c = 0.85 f_p$ for stress relieved strands
- $f'_c = 60,000$ psi (414 MPa)
- $f'_c = 5,000$ psi (34.2 MPa), normal-weight concrete

Use seven-wire 0.89-in.-diameter strands. The nonprestressed partial mild steel is to be placed with $1$-in. clear cover and so compression steel is to be accounted for. No wind or earthquake is taken into consideration. The stress-strain diagram of the prestressing tendons is given in Figure 14.16.

**Solution:** From Ex. 14.4,

- service $W_s = 1100$ plf (16.1 kN/m)
- service $W_{sl} = 500$ plf (1.46 kN/m)
- assumed $W_p = 393$ plf (5.74 kN/m)
- beam span $= 65$ ft (19.8 m)

**Factored moment**

$$W_c = 1.2 (W_s + W_{sl}) + 1.6 W_p$$
$$= 1.2 (1100 + 500) + 1.6 (393) = 2352$$ plf (35.2 kN/m)

The factored moment is given by

$$M_c = \frac{W_c f^2}{8} = \frac{2352 (65)^2}{8} = 14,905,800$$ in.-lb (1164 kN-m)

![Stress-strain diagram of seven-wire 270-K prestressing strand](Figure 14.16)

**Figure 14.16** Typical stress-strain relationship of seven-wire 270-K prestressing strand.
and the required nominal moment strength is

\[ M_n = \frac{M_{u}}{\phi} = \frac{14,005,800}{0.9} = 15,622,000 \text{ in.-lb (1872 kN-m)} \]

**Choice of preliminary section**

Assuming a depth of 0.6 in. per foot of span, we can have a flange section depth \( h = 0.6 \times 60 = 36 \) in. (102 cm). Then assume a mild high-strength No. 6 @ 4\times 0.44 = 1.76 in. \( ^2 \) (11.4 cm\(^2\)). Empirical expressions \( A_t \) and \( A_s \) can be used as follows (see Ref. 14.1):

\[ A_t = \frac{M_n}{0.65 \sqrt{f_y \cdot b \cdot d}} = \frac{15,622,000}{0.65 \times 270,000 \times 40} = 222 \text{ in.}^2 \]

Assume a flange width of 18 in. Then the average flange thickness \( t = \frac{18.5}{18} = 1.02 \) in. (26 mm). So suppose the web \( b_w = 6 \) in. (125 mm), subsequently to be verified for shear requirements.

\[ A_n = \frac{M_n}{0.70 \sqrt{f_y \cdot b \cdot d}} = \frac{15,622,000}{0.70 \times 270,000 \times 40} = 252 \text{ in.}^2 \]

and the number of 1-in. stress-relieved wire strands is \( \frac{222}{252} = 0.88 \). So try thirteen 1-in. strands.

\[ A_n = 13 \times 0.88 = 11.99 \text{ in.}^2 \]

Calculate the stress \( f_n \), in the prestressing tendon at nominal strength using the strain compatibility approach.

The geometrical properties of the flange section are very close to the assumed dimensions for the depth \( h \) and the top flange width \( b \). Hence use the following data for the purpose of the example:

\[ A_t = 377 \text{ in.}^2 \]
\[ c_t = 21.16 \text{ in.} \]
\[ d_t = 15 + c_t = 15 + 21.16 = 36.16 \text{ in.} \]
\[ r_t = 187.5 \text{ in.} \]
\[ r = 15 \text{ in. at midspan} \]
\[ c = 225 \text{ in.} \]
\[ \frac{c}{r} = \frac{225}{187.5} = 1.20 \]
\[ E_s = 57,000 \sqrt{5000} = 4.93 \times 10^6 \text{ psi (32.8 \times 10^6 \text{ MPA)}} \]
\[ E_n = 28 \times 10^6 \text{ psi (159 \times 10^6 \text{ MPA)}} \]

The maximum allowable compressive strain \( \epsilon_{\text{concrete}} \) at failure is \( 0.002 \text{ in./in.} \). Assume \( f_{pm} \) that the effective prestress at service load is \( f_{pm} = 150,000 \text{ psi (1069 MPa)} \).

\[ \epsilon_{\text{concrete}} = \epsilon_{\text{strain}} - \frac{f_{pm}}{E_n} = \frac{28 \times 10^6}{28 \times 10^6} = 0.0005 \text{ in./in.} \]

\[ P_n = 13 \times 0.153 \times 155,000 = 308,295 \text{ lb} \]

The increase in prestressing steel strain as the concrete is deformed by the increased external load (see Figure 14.7 and Eq. 14.11a) is given as

\[ \epsilon_s = \frac{P_n}{A_s \cdot f_y} \left( \frac{1 + \epsilon_{\text{concrete}}}{1 + \epsilon_s} \right) = \frac{308,295}{377 \times 4.03 \times 10^6 \times (1 + 1.20)} = 0.0004 \text{ in./in.} \]
Assume that the stress \( f_p = 205,000 \text{ psi} \) as a first trial. Suppose the neutral axis inside the flange is verified. Then, from Eq. 14.20a,

\[
\sigma = \frac{A_p f_p + A_f f_f}{0.85 \cdot h} = \frac{1.99 \times 205,000 + 1.76 \times 60,000}{0.85 \times 5000 \times 18} = 6.71 \text{ in.} \quad (17 \text{ cm}) < h_i = 7.5 \text{ in.}
\]

Hence the equivalent compressive block is inside the flange and the section is to be treated as rectangular.

Accordingly, for 5000 psi concrete:

\[
p_l = 0.85 \cdot 0.05 = 0.8
\]

\[
c = \frac{d}{b_i} = \frac{6.71}{0.80} = 8.39 \text{ in.} \quad (22.7 \text{ cm})
\]

\[
d = 40 \left( 1.5 + \frac{1}{2} \text{ in.} \right) \text{ for stress} \quad \frac{5}{16} \text{ in.} \text{ (or hr.)} = 37.6 \text{ in.}
\]

The increment of strain due to overloading, from Eq. 14.20b, is

\[
\varepsilon_s = \frac{d - c}{c} = \frac{0.05}{0.05} = \frac{37.6 - 8.39}{8.39} = 0.0104 \text{ in.} / \text{in.} > 0.005 \text{, hence tension-controlled ductile behavior. O.K.}
\]

The total strain is

\[
\varepsilon_{total} = \varepsilon_1 + \varepsilon_2 + \varepsilon_s = 0.0055 + 0.0004 + 0.0104 = 0.0163 \text{ in./in.}
\]

From the stress-strain diagram in Figure 14.16, the \( f_p \) corresponding to \( \varepsilon_s = 0.0163 \) is 230,000 psi.

**Second trial for \( f_p \) value:** Assume that

\[
f_p = 229,000 \text{ psi}
\]

\[
a = \frac{1.99 \times 229,000 + 1.76 \times 60,000}{0.85 \times 5000 \times 18} = 7.34 \text{ in.}
\]

\[
c = \frac{7.34}{0.80} = 9.17 \text{ in.}
\]

\[
\varepsilon_s = \frac{37.6 - 9.17}{9.17} = 0.0993 > 0.005 \text{, tension-controlled. O.K.}
\]

Then the total strain is \( \varepsilon_{total} = 0.0055 + 0.0004 + 0.0093 = 0.0152 \text{ in./in.} \) From Fig. 14.16, \( f_p = 229,000 \text{ psi (1.576 MN)} \); see

\[
A_s = 4 \text{ No. 1} \text{ in.} = 1.76 \text{ in.}^2
\]

**Available moment strength:**

From Eq. 14.17b, for the neutral axis falling within the flange,

\[
M_s = 1.99 \times 229,000 \left( 36.16 - \frac{7.34}{2} \right) + 1.76 \times 5000 \times 18 \left( 37.6 - \frac{7.34}{2} \right)
\]

\[
= 14,906.87 + 3,653,088 = 18,399,025 \text{ in.-lb (2072 kN-m)}
\]

\[
> \text{required} \quad M_s = 16,507,180 \text{ in.-lb (O.K.}
\]

The percentage of moment resisted by the nonprestressed steel is

\[
\frac{3,853,088}{16,507,180} = 22\%
\]
14.6 Example 14.5: Ultimate-strength Design of Prestressed Simply Supported Beam by Strain Compatibility

Check for minimum and maximum reinforcement

Minimum \( A_t = 0.0044 \), where \( A_t \) is the area of the part of the section between the tension face and the cpc. From the cross section of Figure 14.10,

\[
A = 377 - 18 \left( 4.12 + \frac{1.375}{2} \right) = 6(21.16 - 5.5) = 201 \text{ in.}^2
\]

minimum \( A_t = 0.0044 \times 201 = 0.89 \text{ in.}^2 \text{ < 1.76 used} \) O.K.

The maximum steel index, from Ref. 14.1, is

\[
\lambda = \sqrt{\frac{f_y}{f_c}} (v - \omega) < 0.36\beta \leq 0.29 \text{ for } \beta = 0.80
\]

and the actual total reinforcement index is

\[
\lambda_t = \frac{1.99 \times 229,000}{18 \times 35.16 \times 5000} = \frac{37.6}{36.16} \times 37.5 \times 5000 = 0.14 + 0.03 = 0.17 < 0.29 \text{ O.K.}
\]

An alternative check in accordance with ACI 318-02, where \( \chi_t > 0.015 \), satisfies the minimum limit on the reinforcement allowed for ductile behavior.

Choice of section for ultimate load

The section in Ex. 14.4 with the modifications shown in Figure 14.17 has the normal moment strength \( M_n \) that can carry the factored load, provided that four No. 6 nonprestressed bars are used at the tension side as a partially prestressed section. So we can adopt the section for fixture, as it also satisfies the service-load flexural stress requirements both at midspan and at the support. Note that the section could only develop the required nominal strength \( M_n = 16,502,000 \text{ in.-lb} \) by the addition of the nonprestressed bars at the tension face to resist 22% of the total required moment strength. Note also that this section is adequate with a concrete \( f_c = 5000 \text{ psi} \), while the section in Ex. 14.4 has to have \( f_c = 6000 \text{ psi} \) strength in

![Figure 14.17 Midspan section of the beam in Ex. 14.5.](image-url)
14.7 WEB REINFORCEMENT DESIGN PROCEDURE FOR SHEAR

The following is a summary of a recommended sequence of design steps:

1. Determine the required nominal shear strength value \( V_s = V_{s0} \) at a distance \( d_s/2 \) from the face of the support, where \( d = 0.75 \).

2. Calculate the nominal shear strength \( V_s \) that the web has by one of the following two methods.
   (a) ACI short method if \( f_{yd} > 0.40f_{cs} \)
   \[ V_s = 0.65a \sqrt{f_t} \frac{b_s d_s}{M_o} \]
   where \( 2a \sqrt{f_t} b_s d_s \leq V_s \leq 5a \sqrt{f_t} b_s d_s \) and where \( M_o = 0.75 \) and \( V_s \) is calculated at the same section for which \( M_o \) is calculated.
   If the average tensile splitting strength \( f_{yd} \) is specified for lightweight concrete, then \( \lambda = f_{yd} / V_s \) with \( V_s \) to not exceed a value of 100.
   (b) Detailed analysis where \( V_s \) is the lesser of \( V_{s0} \) and \( V_s \)
   \[ V_0 = 0.65a \sqrt{f_t} b_s d_s + V_s = \frac{V_s}{M_{max}} \geq 1.7 \sqrt{f_t} b_s d_s \]
   \[ V_s = 3.5a \sqrt{f_t} + 0.3f_{yd} b_s d_s + V_s \]
   using \( d_s \) or \( 0.5b \), whichever is larger, and where \( M_{max} = (f_{yd}) \cdot (5a \sqrt{f_t} + f_{yd} - f_t) \) or
   \[ M_{max'} = 5a \sqrt{f_t} + f_{yd} \] where \( M_{max} \) is due to externally applied load
   \( V_s \) = factored shear force at section due to externally applied loads
   occurring simultaneously with \( M_{max} \)

3. If \( V_s/d_s \leq 1 \), no web steel is needed. If \( V_s/d_s > 1 \), provide minimum reinforcement. If \( V_s/d_s > V_s \) and \( V_s = V_s/d_s - V_s \leq 8 \sqrt{f_t} b_s d_s \), design the web steel. If \( V_s = V_s/d_s - V_s > 8 \sqrt{f_t} b_s d_s \) or if \( V_s > 0(V_s = 8 \sqrt{f_t} b_s d_s) \), enlarge the section.

4. Calculate the required minimum web reinforcement. The spacing is \( s \leq 0.15 \) or \( 24 \) in., whichever is smaller:
   \[ A_{min} = \frac{50f_{yd}}{f_t} \] (conservative)
   or \[ A_{min} = 0.75 \sqrt{f_t} \frac{d_s}{f_t} \] whichever is larger
If \( f_{sh} \geq 0.40 f_{y} \), \( A_s \) is the smaller of

\[
\frac{A_{sh} f_{sh}}{0.85 f_{y}} \frac{d_{s}}{h_{w}} \sqrt{V_{c} / V_{u}}
\]

where \( d_{s} \leq 0.85 h_{w} \)

5. Calculate the required web reinforcement size and spacing. If \( V_{c} = (V_{c}/\phi - V_{c}) \leq 4 h_{w} / f_{y} \), then the stirrup spacing \( s \) is as required by the design expressions in step 6. If \( V_{c} = (V_{c}/\phi - V_{c}) > 4 h_{w} / f_{y} \), the stirrup spacing \( s \) is half the spacing required by the design expressions in step 6.

6. \( s = \frac{A_{s} f_{s} / d_{s}}{V_{c} / V_{u} - V_{c} / 4 h_{w}} \leq 0.75 h_{w} = 24 \text{ in.} \approx \text{maximum } s \) from step 4

7. Draw the shear envelope over the beam span, and mark the bond requiring web steel.

8. Sketch the size and distribution of web stirrups along the span using No. 3 or No. 4 size stirrups as preferable, but no larger size than No. 6 stirrups.

9. Design the vertical dowel reinforcement in cases of composite sections.
   (a) \( V_{d} \leq 85 d_{s} f_{d} \) for both roughened contact and no vertical ties or dowels, and nonroughened but with minimum vertical ties, use
   \[
   A_{d} = \frac{50 d_{s} f_{d}}{f_{y}}
   \]
   (b) \( V_{d} \leq 500 d_{s} f_{d} \) for a roughened contact surface with full amplitude 1 in.
   (c) For cases where \( V_{d} > 500 d_{s} f_{d} \), design vertical ties for \( V_{d} = A_{d} f_{d} \),

   where \( A_{d} = \text{area of frictional steel dowels} \)
   \[\mu = \text{coefficient of friction} = 1.0 \text{ for intentionally roughened surface, where } \lambda = 1.0 \text{ for normal-weight concrete. In all cases,} \]
   \[V_{c} = V_{d} = 0.5 f_{p} A_{p} \leq 800 A_{p}, \text{where} A_{p} = b_{p} d_{s}. \]

An alternative method of determining the dowel reinforcement area \( A_{d} \) is by computing the horizontal force \( F_{h} \) at the concrete contact surface such that

\[
F_{h} = \mu_{c} A_{d} f_{d} = V_{d}
\]

where

\[
\mu_{c} = \frac{1000 f_{p} f_{d}}{f_{y}} = 2.0
\]

14.7.1 Example 14.6: Shear Strength and Web-Dowel Steel Design in a Prestressed Beam

Design the bonded beam of Ex. 14.4 to be safe against shear failure, and proportion the required web reinforcement by the ACI short method.

Solution: Data and nominal shear strength determination

\[
\begin{align*}
&T_{w} = 270,000 \text{ psi (1862 MPa)} \quad d_{s} = 26.16 \text{ in. (66.4 cm)} \\
&f_{c} = 5000 \text{ psi (34.5 MPa)} \quad d = 37.6 \text{ in. (95.5 cm)} \\
&f_{y} = 60,000 \text{ psi (414 MPa)} \quad b_{x} = 6 \text{ in. (15 cm)} \\
&f_{c} = 155,000 \text{ psi (1069 MPa)} \quad t_{c} = 15 \text{ in. (38 cm)} \\
&F_{c} = 5000 \text{ psi normal-weight concrete} \quad e_{c} = 15 \text{ in. (38 cm)}
\end{align*}
\]
\[ A_s = \frac{13 \text{ seven-wire } \frac{1}{8} \text{ in. strands}}{1.99 \text{ in.}^2} = 12.5 \text{ in.} (32 \text{ cm}) \]
\[ l_s = \frac{70,700 \text{ in.}^3 (2.94 \times 10^6 \text{ cm}^3)}{12.8 \text{ cm}^2} \]
\[ A_t = 4 \text{ Na. 6 bars } = 1.76 \text{ in.}^2 (11.4 \text{ cm}^2) \]
\[ C_t = 287.5 \text{ in.}^2 (12.1 \text{ cm}^2) \]
\[ \sigma_s = 18.84 \text{ in.} (48 \text{ cm}) \]
\[ \sigma_t = 21.16 \text{ in.} (54 \text{ cm}) \]
\[ P_t = 368,255 \text{ lb} (1,670 \text{ kN}) \]
\[ h = 40 \text{ in.} (101.6 \text{ cm}) \]

Factored load \( W_{O} = 1.2O + 1.6L = 1.2(100 + 393) + 1.6 \times 1100 = 2552 \text{ lbf} \)

Factored shear force at face of support \( V_o = \frac{W_{O} \times h}{2} = \frac{2552 \times 40}{2} = 50,440 \text{ lb} \)

Required \( V_c = \frac{50,440}{0.75} = 101,213 \text{ lb at support} \)

\text{Plane at } k, \text{ from face of support}

\text{Nominal shear strength } V_c \text{ of web (steps 2, 3)}

\[ \frac{1}{2} d_e = \frac{36.16}{2 \times 12} = 1.5 \text{ ft} \]

Required \( V_c = 101,213 \times \frac{(65/2) - 1.5}{65/2} = 97,216 \text{ lb} \)

\[ f_s = 355,000 \text{ psi} \]

\[ 0.45f_s = 0.45 \times 355,000 = 159,750 \text{ psi (1096 MPa)} \]

\text{Use ACI short method. Since } d_e > 0.89, \text{ use } d_e = 38.16 \text{ in. assuming that } 1/8 \text{ in. prestressing}

\text{strands continue straight to the support.}

\[ V_c = \left( 0.66b_s \sqrt{f} + 300 \frac{b_s d_p}{M_{p}} \right) b_s d_p + 21 \sqrt{f} b_s d_p = 5a \sqrt{f} b_s d_p \]

\[ \lambda = 1.0 \text{ for normal-weight concrete} \]

\[ M_{p} \text{ at } \frac{d_e}{2} \text{ from face } = \text{reaction } \times \frac{1.5}{2} = \frac{W_{O} (1.5)^2}{2} = 70,440 \times 1.5 - 2552(1.5)^2 \]

\[ = 112,014 \text{ ft lb} = 1,344,168 \text{ in. lb} \]

\[ V_{c,d} = \frac{22,912 \times 36.16}{1,344,168} = 0.96 > 1.0 \]

So use \( V_c / M_p = 1.0 \). Then

\[ \text{Minimum } V_c = 21 \sqrt{f} b_s d_p = 2 \times 1.0 \sqrt{3600} \times 6 \times 36.16 = 30,883 \text{ lb} \]
14.7 Web Reinforcement Design Procedure for Shear

Maximum \( V_i = 8 \times \sqrt{f'} b_d a_d = 76.707 \) lb

\[ V_i = (0.60 \times 10.0 \times \sqrt{5000}) = 970 \times 1.06 \times 36.16 = 161.077 \text{ lb} \]

\[ \text{max} \ V_i = 76.707 \text{ lb} \]

Then \( V_i = 76.707 \) lb controls (541 kN). Also, \( V_{jk} > V_i \); hence web steel is needed. Accordingly,

\[ V_{jk} = \frac{V_i}{d} - V_i = 97.216 - 76.707 = 23.509 \text{ lb} \]

\[ 8 \times \sqrt{f'} b_d a_d = 8 \times 1.0 \times \sqrt{5000} \times 6 \times 36.16 = 122.713 \text{ lb (544 kN)} > V_{jk} = 23.509 \text{ lb} \]

So the section depth is adequate.

Minimum web steel (step 4): From Equation 5.22b.

\[ \frac{A_w f_{uk}}{f_{yd} d_{yd}} \frac{d_d}{d} = \frac{1.99 \times 270.000}{90 \times 60.000 \times 36.15} \sqrt{\frac{56.16}{6}} = 0.0097 \text{ in}^2/\text{in} \]

Required web steel (step 5.6):

\[ s = \frac{A_w f_{yd} d_{yd}}{(V_{jk}/8)} = 0.75s = 34 \text{ in.} \]

or

\[ A_w = \frac{V_{jk}}{f_{yd} d_{yd}} = \frac{23.509}{90 \times 60.000 \times 36.15} = 0.0095 \text{ in}^2/\text{in} \]

Use a minimum required web shear size \( A_w = 0.0095 \text{ in}^2/\text{in} \), although 0.0067 in. \( \text{in}^2/\text{in} \) could be used since \( f_{yd} > 40 \% f_{uk} \). No, trying No. 3 U stirrups, \( A_w = 0.11 \times 0.22 \text{ in.}^2 \), and we get 0.0067 = 0.22, so that the maximum spacing is

\[ s = \frac{0.22}{0.0067} = 33 \text{ in. (89 cm)} \]

and

\[ 4 \times \sqrt{f'} b_d a_d = 4 \times 1.0 \times \sqrt{5000} \times 6 \times 34.16 = 61.366 \text{ lb} > V_i \]

Hence we do not need to use it. Now

\[ 0.75s = 0.75 \times 40.3 = 30.2 \text{ in.} \]

Thus use No. 3 U web-shear reinforcement at 20 in. center to center (0.5-mm diameter at 50 cm center to center).

Plane at which no web steel is needed.

Assume that such a plane is at distance \( s \) from support. By similar triangles,

\[ V_i = 76.707 = 119.920 \times \frac{65}{2} - \frac{x}{65/2} \]

or

\[ \frac{65}{2} - \frac{x}{65/2} = 76.707 \times \frac{65}{119.920} \times \frac{65}{2} \]
Chapter 14  Introduction to Prestressed Concrete

giving

\[ x = 8.04 \text{ ft} (2.44 \text{ m}) = 96 \text{ in.} \]

Therefore, adopt the design in question, using No. 3 U at 20 in. center to center over a street width of approximately 96 in., with the first stirrup to start at 18 in. from the face of support. Extend the stirrups to the midspan if composite action doweling is needed.

SELECTED REFERENCES

14.8. ACI Committee 224, Control of Cracking in Concrete Structures, Committee Report, American Concrete Institute, Farmington Hills, MI, 2002, 75 pp.

PROBLEMS FOR SOLUTION

14.1. An AASHTO prestressed simply supported beam has a span of 34 ft. (10.4 m) and is 36 in. (91.4 cm) deep. Its cross section is shown in Figure 14.18. It is subjected to a live load intensity \( W_L = 3000 \text{ psf} \) (52.6 kN/m). Determine the required 5-in.-diameter, stress-relieved, seven-wire strands to resist the applied gravity load and the self-weight of the beam, assuming that the tendon eccentricity at midspan is \( e_c = 13.12 \text{ in.} \) (333 mm). Maximum permissible stresses are as follows:

![Figure 14.18](image-url)


\[ f_e' = 6000 \text{ psi} \quad (41.4 \text{ MPa}) \]
\[ f_e = 0.45 f_e' \]
\[ f_e = 2700 \text{ psi} \quad (18.6 \text{ MPa}) \]
\[ f_{c'} = 2700 \text{ psi} \quad (18.6 \text{ MPa}) \]
\[ f_c = 10900 \text{ psi} \quad (75.7 \text{ MPa}) \]
\[ f_{c'} = 10900 \text{ psi} \quad (75.7 \text{ MPa}) \]

The section properties, given these stresses, are
\[ A_t = 349 \text{ in}^2 \]
\[ I_y = 50.979 \text{ in}^4 \]
\[ r^2 = \frac{I_y}{A_t} = 138 \text{ in}^2 \]
\[ c_e = 13.83 \text{ in.} \]
\[ S_y = 3220 \text{ in}^3 \]
\[ F = 3237 \text{ in} \]
\[ W_{n0} = 384 \text{ plf} \]
\[ W_t = 3600 \text{ plf} \]

14.2. A simply supported pretensioned beam has a span of 75 ft (22.9 m) and the cross-section shown in Figure 14.19. It is subjected to a uniform gravitationlive-load intensity \( W_t = 1200 \text{ plf} \quad (75.7 \text{ kN/m}) \) in addition to its self-weight and is pretensioned with 20 stress-relieved 9-in. (12.7 mm) diameter 7-wire strands. Compute the total prestress losses by the step-by-step method, and compare them with the values obtained by the lump-sum method. Take the following values as given
\[ f_e' = 6000 \text{ psi} \quad (41.4 \text{ MPa}), \text{ normal-weight concrete} \]
\[ f_e = 0.45 f_e' \]
\[ f_c = 2700 \text{ psi} \quad (18.6 \text{ MPa}) \]
\[ f_{c'} = 2700 \text{ psi} \quad (18.6 \text{ MPa}) \]
\[ f_o = 0.70 f_{c'} \]

Relaxation time \( t = 15 \text{ years} \)
\[ e = 0.19 \text{ in.} \quad (483 \text{ mm}) \]
relative humidity, RH = 75 \%
\[ \frac{V}{V_t} = 3.0 \text{ in.} \quad (76.2 \text{ cm}) \]

Figure 14.19 (a) elevation; (b) section.
143. For service-load and ultimate-load conditions, design a pretensioned symmetrical I-section beam to carry a superimposed dead load of 75 plf (1.00 kN/m) and a service live load of 1500 plf (21.00 kN/m) on a span of 15.2 m (50 ft) simply supported at the ends. Assume that the sectional properties are:

\[ \begin{align*}
  f_e &= 270,000 \text{ psi (1862 MPa)} \\
  E_e &= 28.5 \times 10^6 \text{ psi (196 \times 10^6 MPa)} \\
  f_c &= 5000 \text{ psi (34.5 MPa)}; \text{ normal-weight concrete} \\
  f_p &= 3500 \text{ psi (24.1 MPa)} \\
  f_s &= 12,000 \text{ psi (82.7 MPa)} \\
  \theta &= 0.04 \text{ ft} \\
  \phi &= 0.36 \\
  \lambda &= 0.50, \text{ and } \beta = 0.40, \text{ and suppose the following data are given} \\
  &\text{Sketch the design details, including the anchorage zone reinforcement, and arrangement of strands for (a) the straight tendon case and (b) a harelquin at the third span points with an eccentricity zero. Assume total prestress losses of 22%.} \\
  &\text{144. A post-tensioned bonded prestressed beam has the cross section shown in Figure 14.20. It has a span of 75 ft (22.9 m) and is subjected to a service superimposed dead load } W_{0d} = 450 \text{ plf (6.5 kN/m)} \text{ and a superimposed service live load } W_{0l} = 300 \text{ plf (4.3 kN/m)}. \text{ Design the web reinforcement necessary to prevent shear cracking (a) by the detailed design method and (b) by the alternative method at a section 15 ft (4.6 m) from the face of the support. The profile of the prestressing tendon is parabolic. Use No. 3 stirrups in your design, and detail the section. The following data are given:} \\
  &\text{\( A_e = 876 \text{ in.}^2 (5653 \text{ cm}^2) \)} \\
  &\text{\( I_e = 433,555 \text{ in.}^4 (7.09 \times 10^9 \text{ cm}^4) \)} \\
  &\text{\( I_t = 495 \text{ in.}^2 (8.19 \text{ cm}^4) \)} \\
  &\text{\( c_t = 25 \text{ in. (63.5 cm)} \)} \\
  &\text{\( S_t = 7300 \text{ in.}^2 (2.83 \times 10^8 \text{ cm}^4) \)} \\
  &\text{\( \gamma_t = 38 \text{ in. (96.5 cm)} \)} \\
  &\text{\( S_L = 11,400 \text{ in.}^2 (1.86 \times 10^8 \text{ cm}^4) \)} \\
  &\text{\( W_{0d} = 910 \text{ plf (13.3 kN/m)} \)} \\
  &\text{\( e_t = 32 \text{ in. (81.3 cm)} \)} \\
\end{align*} \]

![Figure 14.20](image-url)
Given:

- \( e = 2 \text{ in. (5 cm)} \)
- \( f_s = 5000 \text{ psi (44.5 MPa)}, \text{ normal-weight concrete} \)
- \( f_p = 3500 \text{ psi (24.1 MPa)} \)
- \( f_{	ext{stirrups}} = 60,000 \text{ psi (414 MPa)} \)
- \( f_{	ext{w}} = 270,000 \text{ psi (1862 MPa)}, \text{ low-relaxation strands} \)
- \( f_r = 157,500 \text{ psi (1086 MPa)} \)
- \( A_p = \text{twenty-four 1-in. (12.7-mm) diameter seven-wire strands} \)
15.1 LRFD TRUCK LOAD SPECIFICATIONS

The design of prestressed concrete elements of a bridge is governed by requirements of the American Association of Highway and Transportation Officials (AASHTO). The traffic lanes and the loads they contain for the design of the bridge superstructure have to be chosen and placed in such numbers and positions on the roadway that they produce the maximum stress in the constituent members.

The bridge live loadings should consist of standard truck or lane loads that are equivalent to truck trains. For railway bridges, the requirements are set by the American Railway Engineering Association (ARES). Requirements for structural proportioning of supporting members are essentially based on the ACI and PCI standards. Except for truck wheel loads and geometry, LRFD requirements differ in some areas from the standard AASHTO requirements. For comparison, it is useful to present also the major AASHTO expressions, particularly since they have been used so long by the design profession.

Photo 15.1 West Kowloon Expressway Viaduct, Hong Kong, 1997. A 4.2 km dual three-lane causeway connecting Western Harbor Crossing to new airport (Courtesy Institution of Civil engineers, London)
15.1.1 Loads

There are four standard classes of highway loading: H 20, H 15, HS 20, and HS 15. Loading HS 15 is 75 percent of HS 20. If loadings other than these are to be considered, they should be obtained by proportionally adjusting the weights for the standard trucks and the corresponding lane loads. Bridges supporting interstate highways should be redesigned for HS 20-44 loading or an alternate military loading of two axles 4 ft apart, with each axle weighing 24,000 lb, whichever loading produces the larger stress value.

Figure 15.1 shows the standard H truck loading, while Figure 15.2 shows the standard HS truck loading giving wheel spacing and load distribution. Figure 15.3 gives the equivalent lane loading for both the H and HS 20-44 and the H and HS 15-44 categories (Ref. 15.1). Figure 15.4 gives an overview of the different bridge deck systems in common use.

(i) Impact. Movable loads require impact allowance as a fraction of the live load stress. It can be expressed by AASHTO as:

$$ I = \frac{50}{L + 125} \leq 30\% $$

(15.1)

where $I = $ Impact fraction

$L = $ Length in feet of the portion of the span that is loaded resulting in maximum stress in that member.

The loaded length $L$ for transverse members, such as floor beams, is the span length of the member center to center of the supports.

Figure 15.1 Wheel loads and geometry for H trucks.
(ii) Longitudinal Forces. Provision should be made for the effect of a longitudinal force of 5% of the live load in all lanes carrying traffic headed in the same direction. All lanes should be loaded in the case of bridges which could likely become one-directional in the life of the structure. The load area, without impact, should be as follows:

Lane load + concentrated load so placed on the span as to produce maximum stress. The concentrated load and uniform load should be considered as uniformly distributed over a 10 foot width on a line normal to the centerline of the lane. The center of gravity of the longitudinal force is to be assumed located 6 feet above the floor slab.

Figure 15.3 Equivalent lane loading for H and HS trucks.
<table>
<thead>
<tr>
<th>SUPPORTING COMPONENTS</th>
<th>TYPE OF DECK</th>
<th>TYPICAL CROSS-SECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cast-in-place Concrete Slab</td>
<td>Cast-in-place concrete slab</td>
<td></td>
</tr>
<tr>
<td>Open Steel or Precast Concrete Boxes</td>
<td>Cast-in-place concrete slab, precast concrete deck slab</td>
<td></td>
</tr>
<tr>
<td>Precast Solid, Welded or Carved Concrete Beams with Shear Keys</td>
<td>Cast-in-place concrete overlay</td>
<td></td>
</tr>
<tr>
<td>Precast Solid, Tapped or Cellular Concrete Box with or without Transverse Prestressing</td>
<td>Integral concrete, prestressed tension</td>
<td></td>
</tr>
<tr>
<td>prestressed Concrete Channeled Sections with Shear Keys</td>
<td>Cast-in-place concrete overlay</td>
<td></td>
</tr>
<tr>
<td>Precast Concrete Double Taper Section with Shear Keys and Transverse Prestressing</td>
<td>Integral concrete, prestressed tension</td>
<td></td>
</tr>
<tr>
<td>Precast Concrete Tee Section with Shear Keys and Transverse Prestressing</td>
<td>Integral concrete, prestressed tension</td>
<td></td>
</tr>
<tr>
<td>Precast Concrete Taper Bump Tee Section</td>
<td>Cast-in-place concrete, prestressed tension</td>
<td></td>
</tr>
</tbody>
</table>

Figure 16.4 Cross sections of typical bridge deck structures (Ref. 12.11).
A reduction factor should be applied when a number of traffic lanes are simultaneously loaded, as in Section (iv) to follow.

(iii) Centrifugal Horizontal Force. This force is produced by vehicle motion on curves. It is a percentage of the live load, without impact, as follows:

\[ C = 0.001175 \frac{D}{R} - 0.695 \]  \hspace{1cm} (15.2)

where

- \( C \) = centrifugal force in percent of the live load without impact
- \( S \) = design speed in miles per hour
- \( D \) = degree of curve
- \( R \) = radius of curve in feet.

(iv) Reduction in Load Intensity. When maximum stresses are produced in any member by loading a number of traffic lanes simultaneously, a reduction in the live load intensity can be made as follows:

<table>
<thead>
<tr>
<th>Percent</th>
<th>One or two lanes</th>
<th>Three lanes</th>
<th>Four lanes or more</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>90</td>
<td>75</td>
</tr>
</tbody>
</table>

15.1.2 LFD Standard AASHTO Wheel Load Distribution on Bridge Decks

(i) Shear. No longitudinal distribution of wheel loads can be made for wheel or axle load adjacent to the end when computing end shears and reactions in transverse or longitudinal beams.

(ii) Bending Moments: Longitudinal Beams. In computing bending moments in longitudinal beams or stringers, no longitudinal distribution of the wheel loads is permitted. In the case of interior stringers, the live load bending moment for each stringer should be determined by applying to the stringer a fraction of the wheel load as follows for prestressed concrete elements.

<table>
<thead>
<tr>
<th>Bridge designed for one traffic lane</th>
<th>Bridge designed for two or more traffic lanes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prestressed concrete girders</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>if ( S \leq 6 \text{ ft.} )</td>
<td>if ( S &gt; 12 \text{ ft.} )</td>
</tr>
<tr>
<td>Non-attached Concrete Box girders</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>5%</td>
</tr>
<tr>
<td>if ( S &gt; 6 \text{ ft.} )</td>
<td>if ( S &gt; 12 \text{ ft.} )</td>
</tr>
</tbody>
</table>

*If \( S \) exceeds denominator, the load on the beam should be the reaction of the wheel loads assuming the spacing between beams to act as a simple beam.

\( S \) = spacing of floor beams in feet.

(iii) Side by Side Precast Beams in Multi-Beam Decks. A multi-beam bridge is constructed with prestressed or posttressed concrete beams that are placed side by side on the supports. The interaction between the beams is developed by longitudinal shear keys used in combination with transverse tie assemblies which may, or may not, be prestressed, such as bolts, rods, or posttressing strands, or other mechanical means. Full-depth rigid end diaphragms are needed to ensure proper load distribution for channel, single- and multi-stemmed tee beams.
In computing bending moments in multi-beam precast concrete bridges, conventional or prestressed, no longitudinal distribution of wheel load shall be assumed. The live load bending moment for each section is determined by applying to the beam the fraction of a wheel load (both front and rear) determined by the following equation:

\[
\text{Load Fraction} = \frac{S}{D}
\]

where,

\[
S = \text{width of precast member;}
\]

\[
D = \begin{cases} 
(5.75 - 0.2N_c) + 0.7N_c(1 - 0.2C)^2 & \text{when } C \leq 5 \\
(5.75 - 0.2N_c) & \text{when } C > 5 
\end{cases}
\]

\[
C = \frac{K}{W/L}
\]

where,

\[
W = \text{overall width of bridge measured perpendicular to the longitudinal girders in feet;}
\]

\[
L = \text{span length measured parallel to longitudinal girders in feet; for girders with cast-in-place end diaphragms, use the length between end diaphragms;}
\]

\[
K = \left(1 + \frac{\mu}{2}\right)^{1.5}
\]

If the value of \(\sqrt{\frac{I}{J}}\) exceeds 5.0, the live load distribution should be determined using a more precise method, such as the Articulated Plate Theory or Grillage Analysis.

where,

\[
f = \text{moment of inertia;}
\]

\[
J = \text{Saint-Venant torsion constant;}
\]

\[
\mu = \text{Poisson's ratio for girders.}
\]

In lieu of more exact methods, "1" may be estimated using the following equations:

For Non-voided Rectangular Beams, Channels, Tee Beams:

\[
J = \sum [b(1/3)br^2(1 - 0.65b/r)]
\]

where,

\[
h = \text{the length of each rectangular component within the section,}
\]

\[
r = \text{the thickness of each rectangular component within the section.}
\]

The flanges and stems of stemmed or channel sections are considered as separate rectangular components whose values are summed together to compute "J". Note that for "Rectangular Beams with Circular Voids" the value of "J" can usually be approximated by using the equation above for rectangular sections and neglecting the voids.

For Box-Section Beams:

\[
J = \frac{2m(d - f)(d - t_f)}{br + d_l - r = r_f}
\]

where,

\[
h = \text{the overall width of the box,}
\]

\[
d = \text{the overall depth of the box,}
\]

\[
r = \text{the thickness of either web,}
\]

\[
t_f = \text{the thickness of either flange.}
\]
The formula assumes that both flanges are the same thickness and uses the thickness of only one flange. The same is true for the webs.

For preliminary design, the following values of \( K \) may be used:

<table>
<thead>
<tr>
<th>Bridge type</th>
<th>Beam Type</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multi beam</td>
<td>Non-voided rectangular beams</td>
<td>0.7</td>
</tr>
<tr>
<td></td>
<td>Rectangular beams with circular voids</td>
<td>0.8</td>
</tr>
<tr>
<td></td>
<td>H-beam section beams</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>Channel, single- and multi-stemmed tee beams</td>
<td>2.2</td>
</tr>
</tbody>
</table>

(iv) Stresses in Concrete

Case I: All Loads including Prestress \((D + L + P/S)\)

\[ f_c = 0.60 f'_c \]

\[ f_c = 0.6 \sqrt{f'_c} \]

Case II: Prestress + All Dead Loads \((D + P/S)\)

\[ f_c = 0.40 f'_c \]

\[ f_c = 0.6 \sqrt{f'_c} \]

Case III: [Prestress + Dead] + Live Load \(0.5 \cdot (D + P/S) + L\)

\[ f_c = 0.30 f'_c \]

\[ f_c = 0.6 \sqrt{f'_c} \]

15.1.3 LFD Standard AASHTO Bending Moments in Bridge Deck Slabs

There are two categories for bending moment calculations: category A and category B for reinforcement perpendicular and parallel respectively to the traffic.

\( S = \) effective span length in feet

\( E = \) width of slab in feet over which a wheel load is distributed

\( P = \) load on one rear wheel of truck \((P_{15} \text{ or } P_{20})\)

\( P_{15} = 12,000 \text{ lbs. for H 15 loading} \)

\( P_{20} = 16,000 \text{ lbs. for H 20 loading} \)

(a) Case A—Main Reinforcement Perpendicular to Traffic (spans 2 to 24 feet)

The live load moments for simple spans are to be determined in accordance with the following expressions:

H 20 Loading,

\[ M_l = \left( \frac{S + 2}{32} \right) P_{20} \quad (15.3a) \]

H 15 Loading,

\[ M_l = \left( \frac{S + 2}{32} \right) P_{15} \quad (15.3b) \]

where \( M_l \) is in ft-lbf or slab width.
15.1 LRFD Truck Load Specifications

In slabs continuous over three or more supports, a continuity factor of 0.8 should be applied to Equations 15.3(a) and 15.3(b).

(b) Case B—Main Reinforcement Parallel to Traffic

For wheel loads, the distribution width, $L$, should be $L = 4 + 0.06S$ ft. Lane loads are distributed over a width $2L$ as follows:

- **H 20 Loading**
  - $S = 50$ ft: $M_x = 900$kip
  - $S = 50 - 100$ ft: $M_x = 1000$kip

where $M_x$ is in ft-kip

For H 15 loading, reduce the values in Equations 15.3(c) and 15.3(d) by 25 percent.

15.1.4 Wind Loads

In accounting for the wind loads, the exposed area is equal to the sum of the areas of all members including floor system and railings as seen in an elevation 90 degrees to the longitudinal axis of the structure. Design should be based on a wind velocity $V = 100$ miles per hour. The area may be reduced as stipulated in Ref. 15.2.

15.1.5 Seismic Forces

Both the equivalent static force method and the response spectrum method can be used for the design of structures with supporting members of approximately equal stiffness. Details are given in Ref. 15.2. Additional basic discussion of earthquake response, the fundamental period of vibration and the International Building Code (IBC 2000) are given in Ref. 15.5.

15.1.6 LRFD Load Combinations

The load combinations using the LRFD specifications differ from the standard specifications. The following tables: 15.1 to 15.2, give the required load combinations, and Tables 15.4 to 15.6 the shear and moment expressions to be used in design. Section 15.2.2 gives the LRFD resistance factors, $\phi$, which differ from the standard reduction factor $\delta$. It should be noted that in the standard specifications, either the lane load or the truck load is used in the live-load calculations. The LRFD specifications require that the combined lane and truck load be used in the live-load computations.

- Strength I: Basic load combination, no wind
- Strength II: Load on bridge with owner-specified design, no wind
- Strength III: Load includes wind
- Strength IV: Very high ratio of dead to live load
- Service I: Normal operational use load combinations with deflection and crack control
- Service II: Load combinations with control of yielding of steel structures
- Service III: Load combinations relating only to tension in prestressed concrete

The LRFD Resistance Factor $\phi$ values are given in Table 15.8 to follow. The following expressions in Tables 15.4 and 15.5 (Ref. 15.2) may be used to compute the maximum bending moments and the maximum shear force per lane of any point in a span for H20 truck, with the limitations indicated in the table. The computed values have to be multiplied by a factor of $\delta$ in order to obtain the shear force and moment per lane of wheels.

The expressions in the tables are limited to simply supported spans and do not include the impact factors.
## Table 15.1 LRFD Load Combinations and Load Factors

| Limit State | DC | DD | DW | EH | EV | ES | WA | WS | WL | FR | TU | CR | SH | SE | ED | IC | CT | CV |
|-------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| STRENGTH-I* | γ_c | 1.75 | 1.00 | - | - | 1.00 | 1.00 | 0.50 | 1.20 | γ_Ek | γ_Ek | - | - | - | γ_Ek | - | - | - | - |
| STRENGTH-II | γ_c | 1.25 | 1.00 | - | - | 1.00 | 1.00 | 0.50 | 1.20 | γ_Ek | γ_Ek | - | - | - | γ_Ek | - | - | - | - |
| STRENGTH-III | γ_c | - | 1.00 | 1.00 | 1.00 | 0.50 | 1.20 | γ_Ek | γ_Ek | - | - | - | γ_Ek | - | - | - | - |
| STRENGTH-IV | γ_c | 1.50 | 1.00 | - | - | 1.00 | 1.00 | 0.50 | 1.20 | - | - | - | - | - | γ_Ek | γ_Ek | - | - | - |
| STRENGTH-V | γ_c | 1.35 | 1.00 | 0.40 | 1.00 | 0.50 | 1.20 | γ_Ek | γ_Ek | - | - | - | γ_Ek | γ_Ek | - | - | - | - |
| EXTREME EVENT-I | γ_p | γ_Ek | 1.00 | - | - | 1.00 | - | - | - | 1.00 | - | - | - | γ_Ek | γ_Ek | - | - | - | - |
| EXTREME EVENT-II | γ_p | 0.50 | 1.00 | - | - | 1.00 | - | - | - | - | - | - | γ_Ek | γ_Ek | - | - | - | - |
| SERVICE-I | 1.00 | 1.00 | 1.00 | 0.30 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | γ_Ek | γ_Ek | - | - | - | γ_Ek | γ_Ek | - | - | - |
| SERVICE-II | 1.00 | 1.30 | 1.00 | - | - | 1.00 | 1.00 | 1.00 | 1.00 | - | - | - | - | - | - | - | - | - |
| SERVICE-III | 1.00 | 0.80 | 1.00 | - | - | 1.00 | 1.00 | 1.00 | 1.00 | γ_Ek | γ_Ek | - | - | - | γ_Ek | γ_Ek | - | - | - |
| FATIGUE-LL, IM & CE ONLY | - | 0.75 | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - | - |

where: γ_c = Load factor for permitted loads
*For load combinations,
Maximum P = 1.25 DC + 1.50 DW + 1.75 (LL + EM)
Minimum Q = 0.90 DC + 0.65 DW + 1.75 (LL + EM)

### Permanent Loads
- DD = dead load of structural components and associated attachments
- DW = dead load of wearing surfaces and utilities
- EH = horizontal earth pressure load
- ES = earth subsurface load
- EV = vertical pressure from dead load of earth fill
- EL = load in eccentricity at one

### Transient Loads
- P_E = vehicular braking force
- C_E = vehicular centrifugal force
- C_R = creep
- C_T = vehicular radiation force
- C_V = vehicular collision force
- E_E = earthquake
- F_F = fraction
- I_C = ice load
- D_C = vehicular dynamic load allowance
- L_L = vehicular live load
- L_S = live load
- P_L = pedestrian live load
- E_E = eccentric
- S_H = shrinkage
- T_T = temperature
- U_T = uniform air temperature
- W_A = water load
- W_L = wind load

### Strengths
- Strength I: Basic load combination, no wind
- Strength II: Load on bridge with owner-specified design, no wind
- Strength III: I load includes wind
- Strength IV: Very high ratio of dead to live load
- Service I: Normal operational use load combinations with deflection and crack control
- Service II: Load combinations with control of yielding of steel structures
- Service III: Load combinations relating only to tension in prestressed concrete
The maximum bending moments and maximum shear forces per lane at any point on a span for a lane load of 0.64 kip/ft may be computed from the following simplified expressions:

\[
\text{Maximum } V_{e} = \frac{0.54(L - x)^{3}}{2L} \text{ kip} \tag{15.4a}
\]

\[
\text{Maximum } M_{e} = \frac{0.64(L - x)^{3}}{2} \text{ ft-kip} \tag{15.4b}
\]

where:
- \( x \) = distance from left support, ft
- \( L \) = beam span, ft
- \( L = \text{lane load} \)

The LRFD specifications require a higher impact factor than the standard specifications. They also require consideration of the fatigue design limits. For fatigue, a special truck load is considered. It consists of a single design truck which has the same axle weight.
Table 15.2  LRFD Permanent Loads

<table>
<thead>
<tr>
<th>Type of Load</th>
<th>Maximum</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC: Compressed and Attachments</td>
<td>1.25</td>
<td>0.90</td>
</tr>
<tr>
<td>D2: Downstream</td>
<td>1.80</td>
<td>0.45</td>
</tr>
<tr>
<td>DW: Wearing surface and utilities</td>
<td>1.80</td>
<td>0.65</td>
</tr>
<tr>
<td>EN: Horizontal Earth Pressure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Active</td>
<td>1.50</td>
<td>0.90</td>
</tr>
<tr>
<td>At-Rest</td>
<td>1.35</td>
<td>0.90</td>
</tr>
<tr>
<td>EL: Locked-in Earth Pressure</td>
<td>1.90</td>
<td>1.00</td>
</tr>
<tr>
<td>EV: Vertical Earth Pressure</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Overall Stability</td>
<td>1.50</td>
<td>0.90</td>
</tr>
<tr>
<td>Retaining Structure</td>
<td>1.35</td>
<td>1.00</td>
</tr>
<tr>
<td>Rigid Buried Structure</td>
<td>1.30</td>
<td>0.90</td>
</tr>
<tr>
<td>Rigid Frame</td>
<td>1.35</td>
<td>0.90</td>
</tr>
<tr>
<td>Flexible Buried Structure other than Metal Box</td>
<td>1.95</td>
<td>0.90</td>
</tr>
<tr>
<td>ES: Earth Surcharge</td>
<td>1.50</td>
<td>0.75</td>
</tr>
</tbody>
</table>

used at all other limit states, but with a constant spacing of 30 ft between the 32-kip axles. Table 15.6a gives the impact factor $I_M$ for the various types of limit states. Table 15.7 (Ref 15.3) gives expressions for computing the maximum bending moments per lane due to HL-93 fatigue truck loading. The values obtained from the table have to be multiplied by a factor of $\delta$ in order to obtain the values per line of wheels.

The LRFD design live load is an HL-92 truck configuration which consists of a combination of:

(a) design truck or design tandem with dynamic allowance. The design truck is the same as the HS20 design truck specified in the Standard AASHTO specifications. The design tandem consists of a pair of 25-kip axles spaced at 4-ft apart.

(b) design lane load of 0.64 kip/ft without dynamic allowance.

15.2 FLEXURAL DESIGN CONSIDERATIONS

15.2.1 Strain $\epsilon$ and Factor $\phi$ Variations: The Strain Limits Approach

For ductile behavior of sections, the reinforcement percentage has to be considerably smaller than the balanced strain limit in flexure as detailed in Sec. 5.3.1. No upper limits on the amount of reinforcement needed to be used in a beam provided that the strain limit

Table 15.3a  Distribution of Live Load Per Lane for Shear in Interior Beams

<table>
<thead>
<tr>
<th>Section</th>
<th>One Design Lane</th>
<th>Two or More Design Lanes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete Box Beams in Multidrum Decks</td>
<td>$\frac{h}{10L}$</td>
<td>$\frac{h}{10L}$</td>
</tr>
<tr>
<td>Concrete Deck, I-, T- and Double-T Sections</td>
<td>$0.30 \gamma + \frac{x}{12.5}$</td>
<td>$0.20 \gamma + \frac{x}{22.5}$</td>
</tr>
</tbody>
</table>

$1$ Range for $h, d, L < S$, $\delta$ are given in Ref. 15.3.
$2$ From $h, \gamma$, Ref. 15.3, Section 15.3.
### Table 15.3b Distribution of Live Load Per Lane For Moment In Invert Beams

<table>
<thead>
<tr>
<th>Section</th>
<th>One Design Lane</th>
<th>Two or More Design Lanes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete Box Beams in</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multi-beam Decks</td>
<td>$k = \frac{b}{35.74} \left( \frac{h}{L} \right)^{0.25}$</td>
<td>$k = \frac{b}{305} \left( \frac{h}{L} \right)^{0.25}$</td>
</tr>
<tr>
<td>Concrete Deck: L, T,</td>
<td></td>
<td></td>
</tr>
<tr>
<td>and Double-T Sections</td>
<td>$0.06 + \left( \frac{b}{2} \right)^{0.15} \left( \frac{K_n^2}{L^2} \right)^{0.6}$</td>
<td>$0.075 + \left( \frac{b}{2} \right)^{0.15} \left( \frac{K_n^2}{L^2} \right)^{0.6}$</td>
</tr>
</tbody>
</table>

1. Ranges for $b, d, L, E, K_n$ are given in Ref. 15.3.
2. For exterior beams, see Ref. 15.3, Section 4.6.1.
3. Notations:
   - $b$ = Beam width, in.
   - $J = \frac{St. Vicente's terminal constant, in.t}{A}$
   - $K_n = \text{Longitudinal stiffness parameter distribution factor for multi-beam bridges, where}$
   - $K_n = d + A_n^2$
   - $x_n = \text{distance between center of gravity of members}$
   - $h = 2.0(N_n)^{0.65}$ where $N_n = \text{number of beams}$
   - $A_n = \text{cross-sectional area}$
   - $L = \text{span, ft}$
   - $A_e = \text{area enclosed by extremities of the beam elements}$
   - $x = \text{length of an element of box beam}$

### Table 15.4 Maximum Shear Force per Lane for HS20 Truck Load ($V_{st}$)

<table>
<thead>
<tr>
<th>Load Type</th>
<th>Formula for maximum shear, kips</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS20 Truck</td>
<td>$\frac{22}{L}[(L - x) - 4.87]$</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>$\frac{22}{L}[(L - x) - 4.33]$</td>
<td>0</td>
<td>42</td>
</tr>
</tbody>
</table>

*x* is the distance from left support to the section being considered. It $L = $ truck load

### Table 15.5 Maximum Bending Moment per Lane for HS20 Truck Load ($M_{st}$)

<table>
<thead>
<tr>
<th>Load Type</th>
<th>Formula for maximum bending moment, ft-kips</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS20 Truck</td>
<td>$\frac{720}{L}[(L - x) - 9.33]$</td>
<td>14</td>
</tr>
<tr>
<td>Truck</td>
<td>$\frac{720}{L}[(L - x) - 4.67]$</td>
<td>0</td>
</tr>
</tbody>
</table>

*x*, *y* = the distance from left support to the section being considered. $L = $ truck load
Table 15.6 Impact Factors

<table>
<thead>
<tr>
<th>Component</th>
<th>IM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deck Joints—All Limit States</td>
<td>15%</td>
</tr>
<tr>
<td>All other Components</td>
<td></td>
</tr>
<tr>
<td>Fatigue and Fracture Limit States</td>
<td>15%</td>
</tr>
<tr>
<td>All Other Limit States</td>
<td>33%</td>
</tr>
</tbody>
</table>

Table 15.7 Fatigue Bending Moment per Lane

<table>
<thead>
<tr>
<th>Load Type</th>
<th>χx</th>
<th>Formula for maximum bending moment, ft-kips</th>
<th>Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fatigue</td>
<td>0.241</td>
<td>(72\chi x \left( L - x \right) \div L )</td>
<td>(a) 44</td>
</tr>
<tr>
<td>Truck Loading (LRFD)</td>
<td>0.241-0.500</td>
<td>(72\chi x \left( L - x \right) \div L )</td>
<td>14 28</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*\(\chi\) is the distance from left support to the section being considered. \(L\): LT = truck load

is not exceeded and the appropriate factor is used. A tensile strain \(\epsilon_t = 0.005\) in./in., as the limiting strain is comparable to a 75% of the balanced reinforcement percentage and is the basis of this approach (Figure 15.5). This minimum limiting strain is considered at the extreme tensile steel reinforcement level, namely, at the centroid of the layer closest to the tensile face of the section. More precisely, \(\epsilon_t = 0.0045\) corresponds to \(f_s = 320,000\) psi in the prestressing steel.

In the LRFD procedure, a limiting value of the ratio of the neutral axis depth, \(c\), to the effective beam depth, \(d_e\), to the centroid of the reinforcement is taken as 0.42 in this strain limits approach, invariably called as a unified approach (Paar. 15.14-15.16).

\[\epsilon_c = 0.003\%\]
\[c = 0.375d\]
\[\epsilon_t = 0.005\%\]
\[\epsilon_c = 0.002\%\]
\[c = 0.8d\]

Figure 15.6 Strain limits (a) tension controlled; (b) compression controlled.
A strain value \( \varepsilon_c \) considerably higher than 0.005 in./in. has to be used, such as 0.007 to 0.009 in./in. The lower compression-controlled limit for beam-column sections is \( \varepsilon_c = 0.002 \) in./in. The \( \varepsilon_c = 0.002 \) is used as a basis for first yield strain. \( \varepsilon_y = f_y/E_y = 0.002 \), although this value can vary depending on the type of reinforcement used. Fig. 15.6 gives on this basis the limits of strain for tension-controlled and compression-controlled concrete sections for all cases, reinforced and prestressed, as in Figure 15.6.

When the net tensile strain in the extreme tension reinforcement is sufficiently large (equal to or greater than 0.005), the section is defined as tension-controlled where ample warning of failure with extensive deflection and cracking can occur. When the net tensile strain in the extreme tension reinforcement is small (less than or equal to the compression-controlled strain limit), a brittle failure condition is expected to develop, with little warning of impending failure.

A balanced strain condition develops at a section when the maximum strain at the extreme compression fiber just reaches 0.003 in./in. simultaneously with the first yield strain \( \varepsilon_y = f_y/E_y \) in the tension reinforcement corresponding to a net tensile strain in the tension reinforcement set in this method at a value \( \varepsilon_c = 0.002 \) in./in.

This condition cannot be used in the flexural design of beams not subjected to compression. In such members, a strain \( \varepsilon_c \) in the extreme tensile reinforcement should be about 0.0075 in./in. for practical purposes.

### 15.2.2 Factored Flexural Resistance

The factored flexural resisting moment,

\[
M_t = \phi M_r
\]

(15.5)

where the resistance factor \( \phi = 1.0 \).

It is recommended in strain compatibility analysis that \( \phi \) be reduced from a value \( \phi = 1.0 \) for net tensile strain of 0.005 in./in. to \( \phi = 0.7 \) for net tensile strain of 0.002 in./in., in the extreme tension steel, namely.
Table 15.8 LRFD Resistance Factors $\phi$

<table>
<thead>
<tr>
<th></th>
<th>$\phi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexure and tension of reinforced concrete</td>
<td>0.90</td>
</tr>
<tr>
<td>Flexure and tension in prestressed concrete</td>
<td>1.00</td>
</tr>
<tr>
<td>Shear and torsion:</td>
<td></td>
</tr>
<tr>
<td>normal density concrete</td>
<td>0.90</td>
</tr>
<tr>
<td>low-density concrete</td>
<td>0.70</td>
</tr>
<tr>
<td>Axial compression with spirals of tie</td>
<td>0.75</td>
</tr>
<tr>
<td>Bearing on concrete</td>
<td>0.70</td>
</tr>
<tr>
<td>Compression in strain and tie models</td>
<td>0.70</td>
</tr>
<tr>
<td>Compression in anchorage zone: normal density concrete</td>
<td>0.80</td>
</tr>
<tr>
<td>low-density concrete</td>
<td>0.65</td>
</tr>
<tr>
<td>Tension in steel in anchorage zones</td>
<td>1.00</td>
</tr>
<tr>
<td>For prewly prestressed components in flexure with or without tension, where PFR = $A_{pfr}/(A_{pfr} + A_{fr})$</td>
<td>$0.90 = 0.6(0.8)$</td>
</tr>
</tbody>
</table>

$$0.7 \leq \phi = 0.50 + 0.30 \left( \frac{d_{wor}}{e} - 1 \right) \leq 1.0$$ (15.6)

where $d_{wor}$ is $d_e$ of the extreme layer of reinforcement, namely the one closest to the extreme tension fibers of the prewly prestressed concrete section.

The limits of the LRFD resistance factors for the various types of stress conditions are given in Table 15.8.

15.3.3 Flexural Design Parameters

The expression for computing the nominal moment strength of the prestressed sections by the LRFD method are similar to the standard AASHTO and ACI 318 strength design procedures given in Section 4.11 of Chapter 4. The ultimate design strength $f_{pu}$ of the reinforcement can be computed either by strain compatibility procedures such as in Example 4.19 or by an approximate method using the following expression:

$$f_{pu} = f_{pu} \left(1 - k \frac{d_1}{d_e}\right)$$ (15.7a)

where,

$$k = 2 \left(1.04 - \frac{f_{pu}}{f_{pu}} \right)$$ (15.7b)

= 0.28 for low relaxation steel

In the Standard AASHTO specifications:

$$f_{pu} = f_{pu} \left(1 - \frac{y}{y_{pu}} \right)$$ (15.7c)

A first estimate of the average stress in the prestressing steel may be made from the following:

$$f_c = f_{pu} - 15 \text{ ksi}$$ (15.7d)
The depth, \( c \), of the neutral axis is obtained from the following expressions:

(a) **Doubly reinforced sections:**

\[
\begin{align*}
\frac{c}{d_e} &= \frac{A_{pm}f_{pm} + A_{pm}f_{p} - A_{pm}f_{p}}{0.85\beta_1 + kA_p\frac{f_{pm}}{\beta_1}} \\
&= \frac{A_{pm}f_{pm} + A_{pm}f_{p} - 0.85\beta_1(h - b_c)S_f}{0.85\beta_1 + kA_p\frac{f_{pm}}{\beta_1}}
\end{align*}
\]

(b) **Flanged Sections:**

\[
\begin{align*}
\frac{c}{d_e} &= \frac{A_{pm}f_{pm} + A_{pm}f_{p} - 0.85\beta_1(h - b_c)S_f}{0.85\beta_1 + kA_p\frac{f_{pm}}{\beta_1}}
\end{align*}
\]

where \( f_p \) = yield strength of the compression reinforcement

\( d_e \) = distance from the extreme compression fiber to the centroid of the prestressing tendons.

15.2.4 Reinforcement Limits

(a) **Maximum reinforcement limit**

The maximum amount of prestressed and non-prestressed reinforcement should be such that,

\[
\frac{d_e}{d_c} = 0.42
\]

where \( d_c = \frac{A_{pm}f_{pm} + A_{pm}f_{p}}{A_{pm} + A_{pm}} \)

(b) **Minimum reinforcement**

At any section, the amount of prestressed and non-prestressed reinforcement should be adequate to develop a factored flexural resistance, \( M_{pc} \), at least equal to the lesser of 1.2 \( M_e \), determined on the basis of elastic analysis or 1.33 times the factored moment required by the applicable strength load combinations.

\[
M_{pc} = (f_p + f_c)S_0 - M_{oc} \left[ \frac{S_{oc}}{S_0} - 1 \right]
\]

where,

\( M_{pc} \) = moment due to non-composite dead loads

\( S_0 \) = non-composite section modulus

\( S_{oc} \) = composite section modulus

\( f_p \) = modulus of rupture = 7.5 Vf_p psi = 0.24 Vf_p ksi

\( f_c \) = compressive stress in the concrete due to effective prestress only, after losses, at the extreme tensile fibers of the section where tensile stresses are caused by external loads.

15.3 SHEAR DESIGN CONSIDERATIONS

15.3.1 The Modified Compression Field Theory

When torsion exists, it is assumed that concrete carries no tension after cracking and the field of diagonal compressive struts carries the torsional shear. The inclination angle \( \theta \) of
Chapter 15  LRFD AASHTO Design of Concrete Bridge Deck Structures

These struts vary depending on the longitudinal, transverse, and principal strains (Ref. 12.6) in the web such that:

\[ \tan \theta = \frac{\varepsilon_t - \varepsilon_v}{\varepsilon_v - \varepsilon_t} \]  

(15.13)

where:
- \( \varepsilon_t \) = longitudinal strain of web, tension positive
- \( \varepsilon_v \) = transverse strain, tension positive
- \( \varepsilon_p \) = principal compression strain, negative

Figure 15.7 shows the stress field in the web of a non-prestressed beam before and after cracking. Before the beam cracks, the stress is equally carried by the diagonal tensile and diagonal compressive stresses acting at a 45° angle (Figure 15.7(a)). After cracking, the diagonal cracks from the tensile stresses in the concrete are considerably reduced (Ref. 15.6, 15.7(b); also the shear and tension equilibrium theory by Hsu in Ref. 12.8, 12.9).

In the compression field theory, the assumption is made that the principal tensile stress, \( f_t \), equals zero as in Figure 12-8(d) after the concrete has cracked. The modified compression field theory takes into account the contribution of the tensile stresses in the concrete between the cracks as in Figure 15.7(c). From Mohr’s stress circle in Chapter 5.2(b), in Chapter 6, in conjunction with Figure 15.7(c), the following expression can be obtained:

\[ f_t = \tan \theta + \cot \theta \alpha - f_v \]  

(15.14a)

where the applied shear stress is:

\[ \alpha = \frac{V}{b_0 d} = \frac{(V_m - 8V_v)}{\phi b_0 d} \]  

(15.14b)

\( d = (d - w) \) and \( b_0 \) = effective web width. The tension web reinforcement, \( A_v \), required to balance the compressive stresses would have to be expressed as:

\[ A_v f_v = \left( f_v \sin \theta \right) \theta - f_t \cos \theta \]  

(15.15)

where \( A_v f_v \) is the vertical component of the balancing tensile force to close the diagonal crack inclined at angle \( \theta \) and \( f_v \) is the average stress in the vertical stirrups. Substituting for \( f_t \) in equation 15.14(a) into equation 15.15 gives:

\[ V = f_v b_0 d \cos \theta \frac{A_v f_v}{\theta} d \cot \theta \]  

(15.16)

where \( V \) represents \( V_v \) and is equal to \( (V_v + V_t) \), \( V_v \) being the shear force taken by the vertical stirrups.

![Figure 15.7 Stress fields in web of reinforced concrete beam (Ref. 12.6) (a) before cracking \( f_t = f_v = 45° \), (b) compression field theory, \( f_v = 0 \), (c) modified compression field theory, \( f_v = 0 \).](image-url)
15.3.2 Design Expressions

As discussed in detail in Ref. 15.6, by making simplifying assumptions, the basic
equations of the modified compression field theory can be rearranged so that the
nominal shear resistance, \( V_{c} \), in a prestressed beam can be evaluated, where:

\[
V_{c} = V_{l} + V_{t} + V_{p}
\]

where:
- \( V_{l} \) = nominal shear strength provided by the tensile stresses in the concrete
- \( V_{t} \) = nominal shear strength provided by the tensile stresses in the web reinforce-
- \( V_{p} \) = nominal shear strength provided by the vertical component of the harped or
deeped longitudinal tendons.

15.3.3 LRFD Specifications. The LRFD provisions recognize two methods:
(a) Strut-and-tie model applicable to any section geometry with regular or discontinu-
    ity features
(b) Modified compression field model (Ref. 15.3, 15.6). This model is based on variable
    angle truss model in which the inclination of the diagonal compression field is al-
    lowed to vary.

The nominal resistance is taken as the lesser of:

\[
V_{c} = V_{l} + V_{t} + V_{p}
\]

or:

\[
V_{c} = 0.25f_{b}b_{w}d_{w}
\]

where:
- \( b_{w} \) = effective web width
- \( d_{w} \) = effective shear depth = \( d_{w} - aw \)
- \( a \) = depth of the compressive block

This critical section for shear is located at distance \( d_{s} \) or (0.5d, 0.6f) whichever is larger.
The value of \( d_{s} \) is taken from midspan flexural capacity computations.

The nominal shear resistance of the plain concrete, \( V_{c} \), in psi is:

\[
V_{c} = 0.85f_{c}b_{w}d_{w}
\]

and in ksi:

\[
V_{c} = 0.0316b_{w}
\]

The factor 0.0316 is \( 1/\sqrt{1000} \), which converts the expression from psi to ksi.

The contribution of the vertical web reinforcement is taken as:

\[
V_{w} = A_{w}f_{y,con}
\]

Transverse shear reinforcement should always be provided when the factored shear, \( V_{r} \),
exceeds the plain concrete shear capacity, namely when:

\[
V_{r} > 0.5f_{c}b_{w}(V_{l} + V_{t})
\]

Additionally, when the beam reaction induces compression into the ends of the members
as occurs in the majority of cases, the critical section for shear is taken as the larger of:
0.5d, 0.6f or \( d_{w} \), measured from the face of the support.

In order to determine the nominal shear resistance of the prestressed member, the
design engineer has to determine the values of \( \beta \) and \( b_{w} \) needed for computing \( V_{l} \) and \( V_{t} \) in
equations 15.19 and 15.20. For non-prestressed concrete sections use $\beta = 2.0$ and $\theta = 45^\circ$.
For prestressed concrete sections, lower variable $\beta$ values are to be used by trial and adjustment. AASHTO Table 15.9 gives the values of $\beta$ and $\theta$ for various values of strain $\varepsilon$. The strain, $\varepsilon$, in the tensile reinforcement is obtained from the following expression:

$$
\varepsilon = \frac{M_y}{d_y} + 0.5N_y + 0.5V_y\cos\theta - A_{ps}f_{ps} \left[ \frac{E_y A_y}{E_{ps} A_{ps}} \right]
$$

The strain $\varepsilon_y$ is shown in Figure 15.8.

$$f_{ps} = f_y \left( \frac{E_{ps}}{E_y} \right)$$

The stress $f_{ps}$ represents the stress in the prestressing strands at jacking for pretensioned members, and, conservatively, the average stress in post-tensioned tendons. For usual levels of prestressing, use $f_{ps} = 0.70 f_{pe}$ for both pretensioned and post-tensioned members.

$f_{ps}$ = concrete compressive stress at the centroid of the composite section resisting live load or at the junction of the web and the flange if it lies within the flange due to both prestress and the bending moments resisted by precast section acting alone, namely, prior to composite action.

$f_{pe}$ = effective stress in the posttensing steel after losses. $f_{ps}$ can conservatively be taken as the effective prestress $f_{ps'}$.

### Table 15.9 Values of $\beta$ and $\theta$ for Sections with Traverse Reinforcement

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\beta \times 1,000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq 0.20$</td>
<td>$\leq 0.10$</td>
</tr>
<tr>
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</tr>
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<td>24.7</td>
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<tr>
<td>26.1</td>
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<td>27.5</td>
<td>28.6</td>
</tr>
<tr>
<td>2.39</td>
<td>2.39</td>
</tr>
</tbody>
</table>
If the strain in the tensile reinforcement is negative, $\varepsilon_t$ should be multiplied by the factor $F_t$ in the following expression:

$$F_t = \frac{E_A + E_{tr}A_{tr}}{E_A + E_{tr}A_{tr} + \frac{1}{2}E_{tr}A_{tr}}$$

where $A_c$ = area of the concrete of the flexural tension side of the member as shown in the shaded portion of Figure 15.8.

The longitudinal reinforcement should be so proportioned that each beam section has to satisfy the following expression:

$$A_{rf} + A_{fr} = \frac{M}{d \phi} + 0.5 \frac{I}{\phi} \left( \frac{V_t}{\phi} + 0.5V + V_r \right) \cos \theta$$

From the foregoing AASHTO expression, the variable $\beta$ is an essential determinant for evaluating the nominal shear resistance $V_s$, as in Equation 15.18. A plot of the values in Table 15.8, which is based on the modified compression field theory, seems to indicate that the tabulated values are insensitive for ratios ($v_{cf}$) in excess of 0.125 when the strain is less than 0.005 in./in. Hu's discussion in Ref. 15.21 points to this difficulty, partly arising from assigning a numerical value to the crack shear stress, $v_{cr}$, namely, the ability of the crack interface to transmit a shear stress value dependent on the crack width, $w$, in the following expression:

$$v_{cr} = \frac{2.16 \sqrt{f_t}}{24w} \text{ psi, in.}$$

$$v_{cr} = \frac{0.18 \sqrt{f_t}}{24w} \text{ Mpa, mm}$$

Hu's work (Ref. 15.21, 15.22) proposes using $v_{cr} = 0$ in order to maintain equilibrium and compatibility. Also, the crack angle $\theta$ in the $V_s$ term of Equation 15.22 is the angle between the longitudinal steel and the principal compression stress (strain) of concrete. As such, the shear stress along the principal axis is zero. This discussion also applies to the LRFD provision for the case of combined shear and torsion. Future AASHTO modifications might become necessary in order to rectify the discrepancy.

15.3.4 Maximum Spacing of Web Reinforcement. The maximum allowable spacing, $s$, of the web reinforcement is the smaller of

$$s \leq 0.75 \frac{h}{b} \text{ or } \frac{24}{h}$$

If $V_s > 4 \sqrt{f_t} b_d$, the maximum allowable spacing is reduced by 50 percent.
15.4 HORIZONTAL INTERFACE SHEAR

15.4.1 General Principles

The horizontal interface shear at service and ultimate load levels is defined in the basic mechanics expression

\[ V_h = \frac{VQ}{I_b} \quad (5.29) \]

where
- \( V_h \) = horizontal shear stress at the required section
- \( V \) = design vertical shear acting on the composite section
- \( Q \) = moment of area about the c.g.c. of the segment of the beam cross-section above or below the c.g.c.
- \( I_b \) = moment of inertia of the entire composite section
- \( b \) = contact width of the precast section web, or width of section at which horizontal shear is being computed.

This basic expression for the horizontal shear stress determines the magnitude of the dowel reinforcement needed to tie the precast section to the situ-slab slab over it forming the full composite beam cross-section. The following section gives the AASHTO expressions for proportioning the dowel reinforcement.

15.4.2 LRFD Requirements for Dowel Reinforcement

The LRFD specifications do not give guidance for computing the horizontal shear \( V_{hd} \). The following expression can be used:

\[ V_{hd} = \frac{V_e}{d_e} \quad (15.30) \]

where
- \( V_{hd} \) = horizontal factored shear stress
- \( V_e \) = factored vertical shear
- \( d_e \) = distance between resultants of tensile and compressive forces
- \( d_e = (\delta - w/2) \)
- \( d_e \) = interface width

LRFD specifies that the nominal shear resistance of the interface surface, \( V_i \), be computed using the following expression:

\[ V_i = \xi A_i + \mu (A_r f_y + P_i) \quad (15.31) \]

and that

\[ V_{hd} A_i \leq d V_i \quad (15.32) \]

where
- \( c \) = cohesion factor
- \( \mu \) = friction factor
- \( A_i \) = interface area of concrete engaged in shear transfer
- \( A_r \) = interface area of shear reinforcement crossing the shear plane within area \( A_i \)
- \( P_i \) = permanent net compressive force normal to the shear plane (may be conservatively neglected)
- \( d \) = total dowel reinforcement.
Typically, the top surface of the precast element is intentionally roughened to an amplitude of 1 in. Hence, for normal weight concrete, LRFD recommends simplifying equations 15.31 and 15.32 as follows, with units in ksi:

$$v_n = \phi \left( 0.1 + \frac{A_{v_n}}{A_{v'}} \right)$$  \hspace{1cm} (15.33)

where the minimum

$$A_{v'} = \frac{(0.056 \cdot s)}{f_s}$$  \hspace{1cm} (15.34)

and the nominal shear resistance is to be taken as the lesser of

$$V_{n} = 0.2f'c/A_{v}$$  \hspace{1cm} (15.35)

or

$$V_{n} = 0.8f_{a}A_{v}$$  \hspace{1cm} (15.36)

The cohesion factor $c$ and the friction factor $\mu$ in equation 15.31 have the following values for the particular conditions of the interacting surfaces:

(a) Monolithically placed concrete:

$$c = 145 \text{ psi} \quad \mu = 1.4 \lambda$$

(b) Concrete placed against clean, hardened concrete with surface intentionally roughened

$$c = 100 \text{ psi} \quad \mu = 1.0 \lambda$$

(c) Concrete placed against hardened concrete clean and free of laitance but not intentionally roughened

$$c = 75 \text{ psi} \quad \mu = 0.6 \lambda$$

(d) Concrete anchored to as-rolled structural steel by headed studs or by reinforcing bars, where all steel in contact with the concrete is clean and free of paint

$$c = 25 \text{ psi} \quad \mu = 0.7 \lambda$$

where $\lambda = 1.0$ for normal-density concrete

- 0.85 for sand-low-density concrete
- 0.75 all other low-density concrete.

While the LRFD AASHTO specifications require the minimum reinforcement to be provided regardless of the stress level at the interface, designers may choose to limit this reinforcement to cases in which $V_{n}/\phi$ is greater than 100 psi (0.7 MPa). Doing so would be consistent with the ACI 318 code and the standard AASHTO specifications.

15.4.3 Maximum Spacing of Dowel Reinforcement

The maximum allowable spacing of the dowels is:

(i) If $V_{n} < 0.1 f'c h$, $d_{c}$, maximum $s \leq 0.8 d_{c} \leq 24$ in.

(ii) If $V_{n} > 0.1 f'c h$, $d_{c}$, maximum $s \leq 0.4 d_{c} \leq 12$ in.
15.5 COMBINED SHEAR AND TORSION

The shear stresses due to torsion and shear are assumed in this hypothesis to add on one side of the section and counteract on the opposite side. The transverse closed tie reinforcement is designed for the side in which the combined shear and torsional effects are additive.

The external loading which causes the highest torsional moment is not the same as the loading that causes the highest shear at the critical section. The tendency by the designers is to combine the highest value of torsion and the highest value of shear in the design of the web reinforcement. This is, naturally, conservative. It is possible to utilize the fact that the two loads are different and thus design the transverse reinforcement for the highest torsion and its concurrent shear or the highest shear and its concurrent torsion, whichever leads to a higher resistance capacity. The LRFD uses the same nominal torsional resisting moment as the ACI:

\[ T_s = \frac{2A_d f_{ctd} \phi}{s} \]  \hspace{1cm} (15.37)

where

- \( A_d \) = cross-section area enclosed by the shear flow path, including area of holes
- \( A_i \) = area of one leg of the enclosed transverse torsion reinforcement
- \( \phi \) = variable angle of transverse reinforcement

In order to determine the value of \( \phi \), the strain, \( \epsilon_s \), in the tensile reinforcement is obtained from equation 15.25, except that \( V_s \) should be replaced by

\[ V_s = \frac{V_v^2 + \left( \frac{0.9 P_d T_s}{2A_d} \right)^2}{V_r} \]  \hspace{1cm} (15.38)

The required amount of transverse reinforcement for shear is obtained from equation 15.21(a) in conjunction with equation 15.23, namely,

\[ V_r = \beta \sqrt{f_{cd}} b_d l + \frac{A_d f_{cd} \phi}{s} + V_s \] \hspace{1cm} (15.39)

so that for shear in lb units and stress in psi,

\[ \frac{A_d}{s} = \frac{V - \left( \beta \sqrt{f_{cd}} b_d l + V_s \right)}{f_{cd} \phi} \] \hspace{1cm} (15.40a)

If ksi units are used, multiply \( \beta \) by 0.0316 and for torsion, from equation 15.37,

\[ \frac{A_t}{s} = \frac{T_s}{2A_d f_{cd} \phi} \] \hspace{1cm} (15.40b)

the total area of web reinforcement would be:

\[ \frac{A_{tw}}{s} = \frac{A_d}{s} + 2 \frac{A_t}{s} \] \hspace{1cm} (15.40c)

The angle \( \phi \) is obtained from Table 15.9 using shear, \( V \), as follows:

(a) Box sections:

\[ V = \frac{V_v - \Phi V_s}{A_d f_{cd} \phi} + \frac{T_s P_d}{s A_d f_{cd} \phi} \] \hspace{1cm} (15.41)
(b) Other Sections:

\[ V = \sqrt{\left( \frac{V_o}{\phi b_d t} \right)^2 + \left( \frac{T_x P_x}{\phi A_{tu}} \right)^2} \]  

(15.42)

where

- \( P_o \) = perimeter of the center line of the enclosed transverse torsion reinforcement
- \( A_{tu} \) = area enclosed by the center line of the outermost closed torsional reinforcement
- \( A_{he} \) = gross area enclosed by the shear flow path (see Figure 5.45 for graphical representation of \( A_{he} \) and \( A_{tu} \) where \( A_{he} = 0.85 A_{tu} \))
- \( T_x \) = factored torsional moment
- \( \phi \) = resistance factor

The value of \( \beta \) in equation 15.39 for determining the shear capacity, \( V_o \), of the plain concrete in the web is obtained from Table 15.8. In order to avoid yielding of the longitudinal reinforcement, a check has to be made that the flexural reinforcement on the tension face is so proportioned as to satisfy the following condition:

\[ \phi (A_{he} + A_{tu}) \leq \frac{M_o}{d_o} + 0.5N_o + \cot \theta \left( V_o - 0.5V_s - V_s \right) + \left( 0.45T_x P_x \right) \]  

(15.43)

where

- \( \theta \) = Perimeter of the shear flow path
- \( N_o \) = Applied axial force. Taken as positive if compressive
15.6 STEP-BY-STEP LRFD DESIGN PROCEDURES

The following is a summary of a recommended sequence of design steps:

1. Determine whether or not partial prestressing is to be chosen.
2. Select the shear forces and bending moments using Tables 15.4 and 15.5.
3. Follow the step sequence for flexural design of the member described in Section 15.2. Generally, \( d = (d_s + d_f) \).
4. Determine the factored shear force \( V_s \) due to all applied loads at the critical section located at a distance \( d_s \) or \( 0.5 d_f \) from the face of the support, whichever is larger, where
   \[
   d = \text{effective depth} = d_f \text{ if no mild steel is used.}
   \]
5. Compute the tendon shear component \( V_t \). The factored shear stress is:
   \[
   V = \frac{V_s - \Delta V_t}{2b_d d_f}
   \]
   The nominal available shear stress \( V_t = V_t^0 \).
6. Compute the quantity \( V/V_t \) and assume a value of \( \theta \). A good initial assumption for prestressed beams is \( \theta = 25^\circ \sim 30^\circ \).
7. Compute the strain in the tensile reinforcement in order to enter Table 15.9 to obtain a trial value of \( \theta \) and \( \beta \):
   \[
   \varepsilon_s = \frac{M_s}{d_s^2} + 0.5 \varepsilon_c + 0.5 V_t \cot \theta - \frac{A_t f_{st}}{2 (E_i A_i + E_p A_p)}
   \]
   where
   \[
   f_{st} = 0.70 f_{ct},
   \]
   \( f_{ct} \) = compressive stress in the concrete at the centroid of the tension reinforcement considering both prestressing load after losses and all permanent loads.

If the strain in the tensile reinforcement is negative, \( \varepsilon_s \) should be multiplied by the factor \( F_p \):
   \[
   F_p = \frac{\varepsilon_c A_c + \varepsilon_p A_p}{\varepsilon_c A_c + \varepsilon_p A_p}
   \]
   \( A_c \) = area of the concrete in the flexural tension side of the member.
8. Enter LRFD Table 15.9 again with the value of \( V/V_t \) and strain \( \varepsilon_s \) if the strut angle \( \theta \) is not close to the one assumed in the first trial. In order to obtain an adjusted...
value of $\beta$. Otherwise, compute $V_c$ from Equation 15.20, namely $V_c = \beta \sqrt{f_t} b_i d_i$
(b) or $V_c = 0.0316 b_i \sqrt{f_t} b_i d_i$ (kip) using the $\beta$ value obtained from Table 15.9.

9. Compute $V_c$ for the web reinforcement after the value of $V_c$ has been determined. Find the corresponding shear reinforcement spacing from:

$$A_s = 0.036 \sqrt{f_t} b_i x$$

10. In regions of high shear stresses, ensure that the amount and development of the longitudinal reinforcement $A_s$ and $A_{ps}$ should satisfy the following expression:

$$A_s f_s + A_{ps} f_{ps} \geq \left[ \frac{M_s}{d_0} + 0.5 \frac{N_0}{d_0} + \left( \frac{V_s}{d_0} - 0.5 V_c - V_c \right) \cot \theta \right]$$

It is recommended that this check be made at the face of the bearing which lies within the transfer length of the strands where the effective prestressing force is not fully developed.

11. When torsion exists combined with shear and flexure, the following steps need to be followed:

nominal torsion $T_e = \frac{2A_t A_s f_s \cot \theta}{s}$

Strain in tensile reinforcement:

$$\varepsilon_t = \left[ \frac{M_i}{d_i^2} + 0.5 \frac{N_i}{d_i} + 0.5 \cot \theta \frac{V_s + (T_e f_s)}{2A_s} \right]$$

where $f_{ps} = 0.70 f_{ps}$

Nominal shear resistance:

$$V_* = V_c + V_s$$

where $d_i = (d_i - a_2)$

Shear reinforcement:

$$A_s = \frac{V_s}{f_s d_i, \cot \theta}$$

Forces are in kips and the stresses in ksi. For using lb and psi units, remove factor 0.0316.

Torsion reinforcement:

$$A_t = \frac{T_e}{f_t A_s f_s, \cot \theta}$$
Total web closed ties reinforcement:

\[ \frac{A_{cw}}{s} = \frac{A_p}{s} + \frac{3}{2} \frac{A_t}{s} \]

Shear stress \( \nu \) for obtaining angle \( \theta \):

(a) Box sections:

\[ \nu = \frac{V_Y - \phi V_p}{\phi b_d c} + \frac{R}{\phi A_{ch} h^2} \]

(b) Other sections:

\[ \nu = \sqrt{\left( \frac{V_Y - \phi V_p}{\phi b_d c} \right)^2 + \left( \frac{R}{\phi A_{ch} h^2} \right)^2} \]

For avoiding yield of the longitudinal tensile reinforcement:

\[ \phi (A_{lf} + A_{mf} f_m) \geq \frac{M_s}{d} + 0.5 N_v + \cos \theta \left( V_v - 0.5 V_p - V_P \right)^2 + \left( \frac{0.45 T_{fs} A_{fs}^2}{2 A_{fs}} \right) \]

12. Check the horizontal interface shear:

\[ \nu_w A_{sw} \leq \phi V_s \]

where

\[ V_s = c A_{sw} + \mu (A_{sw} f_m) \]

\[ \nu_w = \phi \left( 0.1 + \frac{A_{sc}}{A_{sw}} \right) \]

where

\[ A_{sc} = 0.05 b_d t \]

(f_m is in ksf)

Take the nominal shear resistance as the lesser of:

\[ V_s = 0.20 f_s A_{sw} \]

or

\[ V_s = 0.80 A_{sw} \]

c = cohesion factor
\( \mu \) = friction factor
\( A_{sw} \) = concrete interface area = \( b_d t \)
\( A_{sc} \) = area of shear reinforcement crossing the shear plane within area \( A_{sw} \)
\( \phi \) = strength reduction factor = 0.90.

Limit \( A_{sw} \) to cases in which \( V_s/\phi \) is greater than 100 psi.

13. Maximum allowable spacing of web shear reinforcement:

\[ s = 0.75 h = 24 \text{ in.} \]

If \( V_s > 4 \sqrt{f_s} b_d t \), reduce spacing by 50%

For Dowel reinforcement spacing:
15.7 LRFD Design of Bulb-Tee Bridge Deck

If $V < 0.1 f_d b_d, s \leq 0.3 f_d \leq 24$ in.
If $V > 0.1 f_d b_d, s \leq 0.4 f_d \leq 12$ in.
where $b_d$ = width of contact for horizontal shear

15.7 LRFD DESIGN OF BULB-TEE BRIDGE DECK: EXAMPLE 15.1

Design for a rectangular beam of a 120 ft. (36.6 m) simply supported AASHTO-PCI influence composite bridge deck with no shear (adapted from Ref. 15.11). The superstructure is composed of six post-tensioned beams at 9'-0" (2.74 m) on centers as shown in Figure 15.9. The bridge has an 8-in. (203-mm) cast-in-place concrete deck with the top 4-in. to be considered as wearing surfac. The design live load is the HL-93 AASHTO-LRFD fatigue loading.

Assume the bridge is to be located in a low seismic zone.

Given:

Maximum allowable stresses:
Deck $f_d = 4000$ psi, normal weight
f = 0.60 $f_d = 2400$ psi
Bulb-tee $f_t = 6500$ psi
f = 0.60 $f_t = 3900$ psi

Allowable Stresses

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<th>$f_t$</th>
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Section Properties

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<th>$c_i$</th>
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<td>56.90 in.</td>
<td>55.40 in.</td>
<td>14.915 in.</td>
<td>798 plf.</td>
</tr>
</tbody>
</table>

Solution:

1. Transformed Deck slab controlling width

Compute the transformed flange width:

$$E_s = 33 \times (5)^{0.5} \times 4000 = 3300 \text{ ksf}$$

$$E, \text{ at transfer} = 5(5)^{0.5} \times 5500 = 4500 \text{ ksf}$$

![Diagram of Bulb-Tee Bridge Deck Cross Section](Figure 15.9)

**Figure 15.9** Bulb-tee Bridge Deck Cross Section in Example 12.1 (Ref. 15.11)
Chapter 15 LRFD AASHTO Design of Concrete Bridge Deck Structures

\[ E_u \text{ at service} = 33(1.5)^{1.5} \sqrt{5500} = 4990 \text{ psi} \]

Effective flange width is the lesser of:

1. (i) span \( = 12 \times 12 = 360 \text{ in.} \)

2. (ii) \( 12 \times b \) greater of web thickness or \( t \)-beam top flange width, \( b = 12 \times 7.5 = 0.5 \times 42 = 11 \text{ in.} \)

3. (iii) average spacing between beams \( = 9 \times 12 = 108 \text{ in.} \)

hence, controlling flange width = 108 in.

Modular ratio \( n = \frac{E_u}{E_c} = \frac{3830}{4690} = 0.78 \)

Transformed width \( b_n = n \cdot b = 0.78 \times 108 = 84 \text{ in.} \)

2. Properties of composite section

Disregard as insignificant the contribution of the deck concrete to \( f_r \), which is needed because of the percent element camber.

\[ A' = 1.397 \text{ in.}^2 \]

\[ h = 80 \text{ in.} \]

\[ I_{og} = 1,095,296 \text{ in.}^4 \]

\[ c_w = 54.6 \text{ in.} \text{ in the bottom fibers} \]

\[ c_n = 72 - 54.6 = 17.4 \text{ in.} \text{ in the bottom fibers} \]

\[ c_n = 60 - 54.6 = 5.4 \text{ in.} \text{ in the deck top} \]

\[ S_{nc} = \frac{1,095,296}{34.6} = 31,000 \text{ in.}^3 \]

\[ S'_{nc} = \frac{1,095,296}{17.4} = 62,900 \text{ in.}^3 \]

\[ S_{n} = 1,095,296 \times 25.4 \times 0.78 = 55,284 \text{ in.}^3 \]

3. Bending moments and shear forces

Static: \( W_{nc} = \frac{8}{12} \times 5 \times 150 = 900 \text{ lb/ft} \)

Barrier weight: \( W_{bc} = \frac{2 \text{ barriers (300 lb/ft)}}{6 \text{ beams}} = 100 \text{ lb/ft} \)

2 in. future wearing surface: \( W_{pc} = \frac{2}{12} \times 48 \text{ ft} \times 150 = 200 \text{ lb/ft} \)

Live load (truck load) in LRFD would be based on HL-93 truck fatigue loading.

Clear width from Figure 15.9 = 48 ft (14.6 cm)

Number of lanes \( = \frac{48}{12} = 4 \text{ lanes} \)

(a) Distribution factor for moment

For two or more lanes loaded (Ref. 15.3), the distribution factor for bending moment (Table 15.3b)

\[ DFM = 0.075 + \left( \frac{S}{S_{nc}} \right)^{1/4} \left( \frac{S}{S_{nc}} \right)^{1/4} \left( \frac{K}{12C/L} \right)^{1/2} \]

provided that
beam spacing: 3.5 ≤ S ≤ 16
slab: 4.5 ≤ T ≤ 12
span: 20 ≤ l ≤ 124
no of beams: N_o ≥ 4

\( r_s = \text{distance between the center of gravity of the beam and the slab} \)
\( \frac{r_s}{L} = 0.5 + 33.4 = 33.95 \text{ in} \)

\( n = \frac{E}{K_e} = \frac{4890}{300} = 1.63 \)

\( K_e = nL = nL \)
\( = 1.28 \times \left[ \frac{345,804 + 767 (39.65^2)}{2,242,191} \right] \)

therefore,

\[ DFM = 0.073 + \left[ \frac{9(24)}{9(24)} \right] \left( \frac{2,242,191}{12(7.5)} \right)^{0.1} \]

\[ = 0.732 \text{ lanes/beam} \]

For one design lane loaded from Table 15.3b,

\[ DFM = 0.06 + \left( \frac{S}{L} \right)^{0.1} \left( \frac{K_e}{12} \right)^{0.1} \]

\[ = 0.06 + \left( \frac{S}{L} \right)^{0.1} \left( \frac{2,242,191}{12(7.5)} \right)^{0.1} \]

therefore, the case of two or more lanes loaded controls so that the DFM = 0.732 lanes/beam.

Fatigue Moments:

The moment is taken for a single design truck having the same axle weight as in all other limit states, but with a constant spacing of 30 ft between the 32-kip axles. A multiple lane factor of 1.2 for fatigue is used to reduce the controlling DFM factor. From Table 15.1, the load factor is 0.75 and the impact factor (IM) = 15%.

Hence, the fatigue truckload bending moment becomes:

\( M_f = (\text{bending moment per lane}) \times (\text{DFM} / 1.2)(1 + IM) \)

or

\( M_f = (\text{bending moment per lane}) \times \left( \frac{0.439}{1.2} \right) (1 + 0.15) \)

\( = \text{bending moment per lane} \times \left( \frac{0.415(1 + 0.15)}{1.2} \right) \)

\( = (0.475) \text{ (bending moment per lane)} \)

(b) Distribution factor for shear

From Table 15.3(a).

For two or more lanes loaded

\[ DFV = 0.2 + \left( \frac{S}{12} \right) - \left( \frac{S}{20} \right) \]

provided that:

beam spacing: 3.5 ≤ S ≤ 16
slab: 4.5 ≤ T ≤ 12
span: 20 ≤ l ≤ 124
10,000 ≤ A_c ≥ 7,400

\[ \text{Actual } S = 9 \text{ ft}. \]
\[ \text{Actual } T = 7.5 \text{ in}. \]
\[ \text{Actual } L = 120 \text{ ft}. \]

\[ \text{Actual } M_o = 6 \text{ O.K.} \]

\[ \text{Actual } K_e = 2,242,191 \text{ in.} \]
hence, \[ DFV = 0.2 + \left( \frac{5}{12} \right) \left( \frac{5}{30} \right) = 0.887 \text{ lanes/beam}. \]

For one design lane loaded (Table 15.3a)

\[ DFV = 0.36 \left( \frac{S}{25.0} \right) = 0.36 \left( \frac{9.0}{25.0} \right) = 0.172 \text{ lanes/beam}; \]

consequently, the case of two or more lanes loaded controls and DFV = 0.887 lanes per beam.

4. Load Combinations
Total factored load, \( Q = \gamma \cdot \eta \cdot q \),
where \( \gamma \) = factor relating to ductility, redundancy, and operational importance,
\( \eta \) = load factors
\( q \) = special loads.

use \( \eta = 1.0 \) for all practical purposes in this example.

investigate all the load combinations from Tables 15.1 and 15.2. The cases that control are as follows:
(a) Service I for compressive stresses in the prestressed concrete components:
\( Q = 1.0 \cdot (DC + DW + 0.8 \cdot (LL + IM)) \)
(b) Service III for tensile stresses in the prestressed concrete components:
\( Q = 1.0 \cdot (DC + DW + 0.8 \cdot (LL + IM)) \)
(c) Strength I for ultimate strength:
Maximum: \( Q = 1.25 \cdot DC + 1.50 \cdot DW + 1.75 \cdot (LL + IM) \)
Minimum: \( Q = 0.90 \cdot DC + 1.05 \cdot DW + 1.75 \cdot (LL + IM) \)
(d) Fatigue for checking stress range in the straddles:
\( Q = 0.75 \cdot (LL + IM) \)
(The fatigue \( Q \) is a special load combination for checking the tensile stress range in the straddles due to live load and dynamic allowance.)

5. Unfactored shear forces and bending moments
(a) Truck Loads
Truck load shear force:
\[ V_{1x} = (\text{shear force per lane}) \cdot DFV \cdot (1 + IM) \]
\[ = (\text{shear force per lane}) \cdot 0.887 \cdot 1.91 \]
\[ = 1.807 \text{ (shear force per lane) kips.} \]

Truck load bending moments:
\[ M_{1x} = (\text{moment per lane}) \cdot DFV \cdot (1 + IM) \]
\[ = (\text{moment per lane}) \cdot 0.752 \cdot 1.94 \]
\[ = 0.974 \text{ (moment per lane) ft-kips.} \]

LT = Truck live load

(b) Lane Loads
For lane loads, no dynamic allowance is applied, hence,
\[ V_{1x} = (\text{shear force per lane}) \cdot DFV \]
\[ = (\text{shear force per lane}) \cdot 0.887 \text{ kips.} \]
\[ M_{1x} = (\text{moment per lane}) \cdot DFV \]
\[ = (\text{moment per lane}) \cdot 0.752 \text{ ft-kips.} \]

The lane loads from Figure 15.3, the load on this bridge is as follows in Figure 15.10.
6. Computation of moments and shears

(a) Lane live load (DFV = 0.884, DFM = 0.732)

(i) Support section:
Shear at the left support (x = 0) from equation 15.4(a) and Figure 15.9:

\[ V_{1a} = \frac{0.64}{L} (L - x)^2 \text{ (DFV)} \]
\[ = \frac{0.64}{2} (120)^2 (0.887) = 34.1 \text{ kips} \]

From equation 15.4(b) and DFM = 0.732

\[ M_{1a} = \frac{0.64x(L - x)}{2} \text{ (DFM)} = 0 \text{ ft-kip} \]

(ii) Section at 24 ft from support:
As an example, find \( V_{1a} \) and \( M_{1a} \) at \( x = 24 \) ft from the left support.

\[ V_{1a} = \frac{0.64}{2} (120 - 24)^2 (0.887) = 21.8 \text{ kips} \]
\[ M_{1a} = \frac{0.64 \times 24 \times 120}{2} (0.732) = 530.7 \text{ ft-kip} \]

(b) Truck live load (DFV = 1.180, DFM = 0.974)
Here, the impact factor IM = 33% has to be included, hence larger DFV and DFM values.

(i) Support section:
From Tables 15.4 and 15.5.

\[ V_{1a} = \frac{72[(L - x) - 9.33]}{L} \text{ (DFV)} \]
\[ = \frac{72[120 - 0 - 9.33]}{120} (1.180) = 78.1 \text{ kips} \]

From Table 15.5,

\[ M_{1a} = \frac{72x[(L - x) - 9.33]}{L} \text{ (DFM)} \]
\[ = 0 \text{ ft-kip for the support moment.} \]

(ii) Section at 24 ft from support:

\[ V_{1a} = \frac{72[(120 - 24) - 9.33]}{120} (1.180) = 61.4 \text{ kips} \]
\[ M_{y} = \frac{72(24)\{120 - 24\} - 9.33}{120} (0.974) = 1215.9 \text{ ft-kip} \]

6. Fatigue moment at 24 ft (DFF = 0.478)
From Table 15.7,
\[ M_{y} = \frac{72(24)\{120 - 6\} - 18.22}{120} \quad \text{(DFF)} \]
From before, DFF = 0.478
hence,
\[ M_{y} = \frac{72(24)\{120 - 24\} - 18.22}{120} (0.478) = 535.8 \text{ ft-kips} \]

7. Shears and moments due to dead loads:
The loads to be considered are beam weight (W_{p},), plus deck slab and haunches (W_{p,d},), and future wearing surface (W_{f,s}).
The beam is simply supported, hence, the shear and moment at any cross section along the span are:
\[ V_{r} = W_{p}(0.5L - x) \]
\[ M_{r} = 0.5W_{p}(L - x) \]
As an example, consider a section at 24 ft from the left support and compute the shear and moment due to self-weight W_{p} = 0.799 kips:
\[ V_{r} = 0.799(0.5 \times 120 - 24) = 26.8 \text{ kips} \]
\[ M_{r} = 0.5 \times 0.799 \times 24(120 - 24) = 920.4 \text{ kips} \]
Tables 15.9, 15.10, and 15.11 (Ref. 15.11) list the forces and moments required for the design of the interior beam elements. It should be noted that long-hand computations to develop such a table are time-consuming. Computer programs developed by several state DOTs are available, now on the internet, such as the Washington State DOT Program.

7. Design of the Ball-tee prestressed interior beam

1. Selection of Prestressing Strands
For Service-III load combination, bottom fiber stress f_{b} in:
\[ f_{b} = \frac{M_{u} + M_{f} + M_{t} + 0.8(M_{f} + M_{t})}{S_{u}} \]
where
\[ M_{u} = \text{unfactored self-weight moment, ft-kip} \]
\[ M_{f} = \text{unfactored moment due to slab and haunch weight, ft-kip} \]
\[ M_{t} = \text{unfactored barrier moment, ft-kip} \]
\[ M_{f,t} = \text{unfactored future wearing surface moment, ft-kip} \]
\[ M_{t,l} = \text{unfactored truck load moment, ft-kip} \]
From before, \( S_{u} = 14.915 \text{ in.} \)
\( S_{u} = 20.090 \text{ in.} \)
From Tables 15.10 and 15.11, Midspan stresses at bottom fibers at service:
\[ f_{b} = \frac{1438.2 + 1659.6}{1438.2 + 1659.6 + 180 + 360 + 0.8(1830.3 + 843.2)} \]
\[ = 2.59 + 1.60 = 4.19 \text{ kips/ft (T)} \]
15.7 LRFD Design of Bulb-Tee Bridge Deck

### Table 15.10 LRFD Service Shear and Moment Due to Dead Load

<table>
<thead>
<tr>
<th>Distance X</th>
<th>Section X/2</th>
<th>Beam Weight W_b</th>
<th>(Slab + Haunch) Weight W_s</th>
<th>Barrier Weight W_bar</th>
<th>Wearing Surface W_w</th>
<th>Shear Moment M_s</th>
<th>Shear Moment M_b</th>
<th>Shear Moment M_w</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>47.9 kips</td>
<td>0.0 kips</td>
<td>0.0 kips</td>
<td>6.0 kips</td>
<td>12.0 ft-kips</td>
<td>12.0 ft-kips</td>
<td>12.0 ft-kips</td>
</tr>
<tr>
<td>6.00&quot;</td>
<td>0.05</td>
<td>43.1 kips</td>
<td>274.3 kips</td>
<td>59.5 kips</td>
<td>135.3 kips</td>
<td>34.2 ft-kips</td>
<td>34.2 ft-kips</td>
<td>34.2 ft-kips</td>
</tr>
<tr>
<td>12</td>
<td>0.1</td>
<td>33.4 kips</td>
<td>574.3 kips</td>
<td>49.5 kips</td>
<td>106.1 kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
</tr>
<tr>
<td>24</td>
<td>0.2</td>
<td>28.8 kips</td>
<td>928.4 kips</td>
<td>132.1 kips</td>
<td>115.2 kips</td>
<td>7.2 ft-kips</td>
<td>7.2 ft-kips</td>
<td>7.2 ft-kips</td>
</tr>
<tr>
<td>36</td>
<td>0.3</td>
<td>20.2 kips</td>
<td>1,204.1 kips</td>
<td>221.1 kips</td>
<td>172.8 kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
</tr>
<tr>
<td>48+</td>
<td>0.4</td>
<td>12.6 kips</td>
<td>1,307.7 kips</td>
<td>1,111 kips</td>
<td>345.6 kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
<td>4.8 ft-kips</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>0.0 kips</td>
<td>1,038.2 kips</td>
<td>0.0 kips</td>
<td>180.0 kips</td>
<td>0.0 ft-kips</td>
<td>0.0 ft-kips</td>
<td>0.0 ft-kips</td>
</tr>
</tbody>
</table>

*Critical section for shear + Harp post*

The 4.10 ksi (7) will be neutralized by the torsion of the beam. Maximum allowable tensile stress:

\[ f_t = 6,000 \sqrt{f_y} = 6,000 \sqrt{60} = 484 \text{ psi} = 0.484 \text{ ksi} \]

Required prestress compressive stress at the bottom fibers:

\[ f_p = (4.1 - 9.48) = 3.62 \text{ ksi} \]

Assume the distance from the centroid of the compressing reinforcement and the section bottom fiber is 0.05 ft.

\[ = 0.05(72) = 3.6 \text{ in.} \text{ use 4.0 in., hence } e_s = 36.6 - 4.0 = 32.6 \text{ in.} \]

### Table 15.11 LRFD Service Shear and Moment Due to Truck and Lane Loads

<table>
<thead>
<tr>
<th>Distance X</th>
<th>Section X/2</th>
<th>Truck Load with Impact W_tr</th>
<th>Lane Load W_l</th>
<th>Fatigue Truck with Impact W_f</th>
<th>Shear V_tr</th>
<th>Moment M_tr</th>
<th>Shear V_l</th>
<th>Moment M_l</th>
<th>Moment M_f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
<td>78.1 kips</td>
<td>33.9 kips</td>
<td>0.0 kips</td>
<td>78.1 kips</td>
<td>33.9 kips</td>
<td>0.0 kips</td>
<td>0.0 kips</td>
<td>0.0 kips</td>
</tr>
<tr>
<td>6.00&quot;</td>
<td>0.05</td>
<td>73.8 kips</td>
<td>267.8 kips</td>
<td>30.6 kips</td>
<td>160.2 kips</td>
<td>160.2 kips</td>
<td>30.6 kips</td>
<td>30.6 kips</td>
<td>30.6 kips</td>
</tr>
<tr>
<td>12</td>
<td>0.1</td>
<td>69.6 kips</td>
<td>691.6 kips</td>
<td>27.5 kips</td>
<td>303.6 kips</td>
<td>303.6 kips</td>
<td>27.5 kips</td>
<td>27.5 kips</td>
<td>27.5 kips</td>
</tr>
<tr>
<td>24</td>
<td>0.2</td>
<td>61.4 kips</td>
<td>1,235.0 kips</td>
<td>21.8 kips</td>
<td>599.7 kips</td>
<td>599.7 kips</td>
<td>21.8 kips</td>
<td>21.8 kips</td>
<td>21.8 kips</td>
</tr>
<tr>
<td>36</td>
<td>0.3</td>
<td>52.7 kips</td>
<td>1,579.2 kips</td>
<td>16.5 kips</td>
<td>701.3 kips</td>
<td>701.3 kips</td>
<td>16.5 kips</td>
<td>16.5 kips</td>
<td>16.5 kips</td>
</tr>
<tr>
<td>48+</td>
<td>0.4</td>
<td>44.2 kips</td>
<td>1,778.6 kips</td>
<td>12.2 kips</td>
<td>809.5 kips</td>
<td>809.5 kips</td>
<td>12.2 kips</td>
<td>12.2 kips</td>
<td>12.2 kips</td>
</tr>
<tr>
<td>60</td>
<td>0.3</td>
<td>35.7 kips</td>
<td>1,830.2 kips</td>
<td>8.5 kips</td>
<td>843.3 kips</td>
<td>843.3 kips</td>
<td>8.5 kips</td>
<td>8.5 kips</td>
<td>8.5 kips</td>
</tr>
</tbody>
</table>

*Critical section for shear + Harp post*
to give prestressing force \( P_e = 1037 \text{ kips} \)
Assume total prestress loss = 22%:
\[
P_e = \frac{1037}{1 - 0.22} = 1303 \text{ kips}
\]
assume using 1 in-dia 7-wire 271-K low-relaxation strands \( (A_s = 0.013 \text{ in}^2) \)

Required number of strands:
\[
\frac{1303}{0.013 \times 202.5} = 44.6 \text{ strands}
\]

After two trials and adjustments, 48 strands with the configuration shown in Figure 15.11 are tried. Less than 48 strands result in tensile stresses at the bottom fibers at service which exceed the maximum allowable \( f_s = 484 \text{ psi} \). Twelve strands are harped at 0.4 L. Accordingly, 36 strands remain straight at the beam.

From data, \( c_s = 36.60 \text{ in.} \) and \( c_s = 72 - 2(36.60) = 35.40 \text{ in.} \)

\[
e = e_s - \left[ 2 \times 70 + 2 \times 68 + 2 \times 66 + 2 \times 64 + 2 \times 62 + 2 \times 60 + 4 \times 8 + 8 \times 6 + 12 \times 4 + 12 \times 2 \right]/48
\]
\[
= 36.60 - 19.42 = 17.18 \text{ in.}
\]

\[
e_s = e_s - \left[ 2 \times 12 + 12 \times 4 + 8 \times 6 + 8 \times 4 + 2 \times 10 + 2 \times 12 + 2 \times 14 + 2 \times 16 + 2 \times 18 + 2 \times 20 \right]/48
\]
\[
= 36.6 - 6.92 = 29.68 \text{ in.}
\]

Given \( f_s = 0.75 \times f_{ps} = 0.75 \times 202.5 = 151.875 \text{ psi} \),

\[
P_e = (48)(0.013)(202.5) = 1488 \text{ kips}
\]
After running a detailed step-by-step analysis of prestress losses, the total prestress loss was determined to be 36.4%.

\[ f_{pc} = 202.5(1 - 0.264) = 149.9 \text{ksi} \]

hence, \( P_r = 1381(1 - 0.264) = 1059.0 \text{ kips} \)

(2) Check of Concrete Unfrozen Stresses

(a) Stresses at Transfer

Initial \( f'_c = 0.7 f_c = 0.7 \times 270 = 202.5 \text{ ksi} \). Common practice assumes that initial relaxation losses at processing amount to 9 to 10%. Use 10% reduction in \( f'_c \).

\[ P_t = 0.90 \times 1059 = 1335 \text{ kips} \]

Hence, \( P_r = (0.9)(202.5)(0.153)(48) = 1338 \text{ kips} \)

(b) Support Section

From Chapter 14, Equation 14.9,

\[ f' = \frac{P}{A} \left( 1 - \frac{6 \sigma_c}{f_c} \right) \frac{M_{t}}{S_c} \]

\[ = \frac{1338}{767} \left( 1 - \frac{17.28 \times 35.4}{712} \right) = 0 = -0.25 \text{ ksi} \] (C), no tension, O.K.

\[ f_s = \frac{P}{A} \left( 1 + \frac{6 \sigma_c}{f_c} \right) \frac{M_{u}}{S_c} \]

\[ = \frac{1338}{767} \left( 1 + \frac{17.28 \times 36.8}{712} \right) = 0 \]

\[ = 3.29 \text{ ksi} < \text{allowable} f_s = 5.48 \text{ ksi} \] O.K.

(c) Midspan Section

\[ f' = \frac{1338}{767} \left( 1 - \frac{29.68 \times 36.6}{712} \right) = 1438 \times 12 \]

\[ = 0.917 - 1.119 = -0.202 \text{ ksi} \] (C), no tension allowed, hence, O.K.

\[ f_s = \frac{1338}{767} \left( 1 + \frac{29.68 \times 36.6}{712} \right) = 1438 \times 12 \]

\[ = -6.512 + 1.157 = -3.356 \text{ ksi} \] (C) < than allowable \( f_s = 5.50 \text{ ksi} \) O.K.

(b) Stresses at Service:

(6) Midspan Section:

From Chapter 14, Equations 14.10(a) and 14.10(b):

\[ f' = \frac{P}{A} \left( 1 - \frac{6 \sigma_c}{f_c} \right) \frac{M_s}{S_c} \leq f_s \]

\[ f_s = \frac{P}{A} \left( 1 + \frac{6 \sigma_c}{f_c} \right) \frac{M_s}{S_c} \geq f_s \]

Since the loads are placed at different stages of construction, for Service I prestress sections.

\[ f' = \frac{1059}{767} \left( 1 - \frac{29.68 \times 35.40}{712} \right) \frac{(1438 + 9690)/12}{15.42} \]

\[ = 0.679 - 2.411 - 0.183 = -1.835 \text{ ksi} \] (C)
Chapter 15  LRFD AASHTO Design of Concrete Bridge Deck Structures

Service allowable $f_s = 2925$ psi  O.K.

$$f_s = \frac{1095}{767} \left(1 + \frac{29.68 \times 36.60}{712}\right) \left(1 + \frac{1438 + 166012}{14.915}\right) \left(\frac{360 + 18012}{20.060}\right) = -3.605 + 2.493 + 0.323 = -3.609 \text{ ksi (O.K.)}

(3) Including stresses due to the transient lane and truck loads

$$f^* = -1.835 - 0.8 \left(\frac{1830 + 843112}{62,950}\right) = -1.835 - 0.8 \times 2234 \text{ ksi (C)} < \text{Service III} f_s = 3900 \text{ psi O.K.}

f_s = -1.835 + 20.060 = 0.8 \left(\frac{1830 + 843112}{62,950}\right)

= -0.579 + 1.270 = 0.691 \text{ ksi (T)} = \text{Allowable} f_s = 0.444, \text{ O.K.}

(4) Concrete stresses at top deck fibers

(i) Under permanent Service I loads

$$f_s = \frac{M_{x'} + M_{y}}{S_{d}} = \frac{360 + 18012}{55,264} = -0.111 \text{ ksi (C)} < \text{Allowable} f_s = 2.4 \text{ ksi O.K.}

(ii) Under permanent and transient lane and truck loads, Service I:

$$f_s = \frac{M_{x'} + M_{y}}{S_{d}} = \frac{M_{x'} + M_{y}}{S_{d}}

= -0.111 - \frac{1830 + 843112}{62,950}

= -0.697 \text{ ksi (C)} < \text{Allowable} f_s = 2.4 \text{ ksi O.K.}

(5) Concrete Stresses at beam bottom fibers, Service III (Use Step 3)

$$f_s = \frac{P}{A_e} \left(1 + \frac{c_0}{r'}\right) \frac{M_{x'} + M_{y}}{S_{d}} + (M_{x'} + M_{y}) + 0.8(M_{x'} + M_{y})

= 1095 \left(1 + \frac{29.68 \times 36.60}{712}\right) \left(\frac{1438 + 166012}{14.915}\right) \left(\frac{360 + 18012}{20.060}\right)

= -3.605 + 2.492 + 1.603 = 0.490 \text{ ksi (T)} \text{ in allowable} f_s = 0.494 \text{ ksi O.K.}

(6) Fatigue stresses

LRFD specifies that in regions of compressive stress due to permanent loads and prestress, fatigue is only considered if the compressive stress is less than twice the maximum tensile live load stress resulting from the fatigue.

Thus, for permanent loads only, the term $(M_{x'} + M_{y})S_{d}^e$ is taken out to give:

$$f_s = -\frac{P}{A_e} \left(1 + \frac{c_0}{r'}\right) \frac{M_{x'} + M_{y}}{S_{d}} + (M_{x'} + M_{y}) + 0.8(M_{x'} + M_{y})

= -3.605 + 2.492 + \frac{360 + 18012}{20.060} = -0.790 \text{ ksi (C) O.K.}

From Table 15.11, fatigue moment $M_{f} = 777 \text{ kips.}$
Tensile fatigue stress at the bottom flanges,

\[
f_t = \frac{0.75M_f}{S_{nc}} = \frac{0.75 \times 777 \times 12}{20,060} = 0.348 \text{ ksi (T)}
\]

Since twice 0.348 = 0.696 < 0.790 ksi (which is a conservative stress), a fatigue check is unnecessary.

From the foregoing computations, the flexural design is O.K. at the initial and service load conditions. To be complete and also determine the reserve strength available for overload conditions, the first state failure design is necessary as in the following section. The total design has to include shear, tension, if any, and serviceability as in Example 15.2.

8. Ultimate strength (Limit state of failure)

(a) Nominal flexural resistance moment

From Tables 15.1 and 15.2 total factored moment for Strength 1 Load:

\[M_u = 1.25DC + 1.5DW + 1.75(LL + IM)\]

From Table 15.10

\[M_u = 1.25(416 + 1600) + 1.5(360 + 180) + 1.75(1830 + 843) = 9316 \text{ ft.kip}
\]

Required \(M_u = \frac{M_p}{\phi} = \frac{9316}{1.0} = 9316 \text{ ft.kip}\)

Average stress in the posttensioning reinforcement when \(f_{pu} \geq 0.5 f_{ps} \) from equation 15.7:

\[f_{ps} = f_{ps}(1 - k \frac{\beta}{\bar{b}}) \text{ where } k = 2 \left(1.04 - \frac{f_{ps}}{f_{ps}}\right)
\]

For the depth of the compressive block, use slab \(f_{ps} = 4.0 \text{ ksi}\)

\[k = 0.28 \text{ for low-relaxation steel}
\]

\[d_p = (b - \text{cover to c.g.s.}) = (72 + 8) = 73.08 \text{ in.}
\]

\[b = \text{effective compression flange width} = 9' - 0'' = 108 \text{ in.}
\]

\[A_{ps} = 48 \times 0.153 = 7.344 \text{ in.}^2
\]

\[f_{ps} = 270 \text{ ksi}
\]

From equation 15.9,

\[
c = \frac{A_{ps} f_{ps} + A_{ds} f_{ds} - A_{ef} f_{ps}}{0.85 b \beta + k A_{ps} f_{ps}}
\]

\[
= \frac{7.344 \times 0 - 0}{0.85 \times 4.0 \times 0.85 \times 108 + 0.28 \times 7.344(270)} = 0.20 \text{ in. } < c = 7.3 \text{ in.}
\]

\[
\sigma = \beta c = 0.85 \times 8.20 = 5.27 \text{ in.}
\]

hence, neutral axis is within the flange and the section is considered rectangular.

Average design reinforcement strength \(f_{ps}^d\)

\[f_{ps}^d = 270 \left(1 - \frac{0.28 \times 5.27}{73.08}\right) = 263.6 \text{ ksi}
\]

nominal flexural resistance \(M_u = A_{ps} f_{ps} \left(\frac{d - c}{2}\right)\)

\[M_u = M = 7.344 \times 263.6 \times 73.08 \times \frac{5.27}{2} \times \frac{1}{12}
\]
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\[ \epsilon_c = 0.42 \text{ for ductile behavior discussed in section 15.2.} \]

Actual \( \epsilon_c = \frac{6.20}{73.08} = 0.085 < 0.42 \text{ O.K.} \)

(b) Minimum Reinforcement

As discussed in section 15.2.4, the minimum reinforcement has to be the lesser of 1.2 \( M_{fc} \) or 1.33 \( M_{re} \) required by the applicable load combinations.

\[ f_c = 7.5 \sqrt{f_y} = 7.5 \sqrt{6500} = 605 \text{ psi} = 0.6 \text{ ksi} \]

\( f_c \) = compressive stress due to effective prestress only at the bottom fibers as defined in section 15.2.4.

\[ \frac{f_{pc}}{f_y} \left( 1 + \frac{f_{pc}}{f_y} \right) = -3000 \text{ kips from before.} \]

Non-composite \( M_{pc} = M_{re} + M_{fc} = 1438 + 1600 = 3038 \text{ ft-kip} \)

\( S_{re} = 20.060 \text{ in.}^4 \)

\( S = 14.915 \text{ in.}^4 \)

From equation 15.12.

\[ M_{pc} = \frac{f_{pc} A_T}{f_y} - M_{re} \left( \frac{\epsilon_c}{\epsilon_Y} - 1 \right) \]

\[ = (0.6 + 3.6) \frac{14312}{12} - 3038 \left( \frac{2000}{14615} - 1 \right) \]

\[ = 520 - 2069 = 4151 \text{ ft-kip} \]

1.2 \( M_{pc} = 1.2 \times 4151 = 4981 \text{ ft-kip} \)

1.33 \( M_{pc} = 1.33 \times 4316 = 12,390 \text{ ft-kip} > 4981 \text{ ft-kip} \)

hence, the lesser of the two moments controls, namely, 1.2 \( M_{pc} = 4981 \text{ ft-kip} \).

\( M_{fc} \) or \( M_{re} = 11,364 > 4981 \text{ O.K.} \)

9. Premixed Anchor Zone

The zone reinforcement is designed using the force in the strands just prior to release tension. The LRFD specifications require that the bursting resistance, \( P_b \), should not be less than 60% of the force in the strands, \( F_{pc} \), before release. namely:

\[ P_b = 0.6 f_y A_s = 0.6 f_y \frac{f_{pc}}{f_y} \]

\[ P_b = 45 \times 0.153 = 202.5 = 1488 \text{ kips} \]

\[ P_b = 0.04 \times 1488 = 59.5 \text{ kips} \]

Use a stress, \( f_{pc} \) in the anchorage reinforcement not exceeding 20 ksi.

Required area = 59.500 = 2.98 in.²

Try No. 5 closed ties; \( A_s = 2 \times 0.31 = 0.62 \text{ in.}^2 \)

Number of ties = 2.98/0.62 = 4.8

Distance within which anchorage reinforcement has to be provided from beam end = 4.85 = 72.9 = 14.4 in.

Use No. 5 closed ties at 3 in. center-to-center, with the first tie starting at 2 in. from the beam end.

Conclusion:

Accept the design of the bulb-tee bridge for flexure. For the design to be complete, design for shear, interface shear transfer and deflection/camber checks have to be performed as in Example 15.2.
15.8 LRFD SHEAR AND DEFLECTION DESIGN: EXAMPLE 15.2

Design the web shear reinforcement for the built-up beam in Example 15.1 at the critical section near the supports and the interface shear transfer reinforcement at the interface plane between the precast section and the deck situ-cast concrete. Also, verify if the span deflection is within the allowable limits.

Solution:

1. Web Shear Design

(a) Strain at centroid level of reinforcement

$$\epsilon = 0.9$$

$$\epsilon_c = 17.28 \text{ in.}$$

Provide web steel when $$V_c > 0.5 \frac{P}{W_c} + V_a$$

Critical section is the greater of 0.3 $$d_c \cos \theta$$ or $$d_e$$:

$$d_e = d_c = 8 - 0.22 - 8.0 = 6.18 \text{ in.}$$

$$d_e = \left( \frac{d_c - \frac{v}{2}}{2} \right) = 6.72 - \frac{2}{2} = 6.08 \text{ in.}$$

$$\geq 0.9 \epsilon_c = 0.9 \times 6.72 = 60.45 \text{ in.}$$

$$\geq 0.72 \epsilon_c = 0.72 \times 8.0 = 57.6 \text{ in.}$$

$$d_e = 60.08 \text{ in., controls at the largest of the three values}$$

Assume $$\theta = 22\degree$$ for the first trial

$$0.5 \epsilon_c \cos \theta = 0.5 \times 60.08 \cos 22\degree$$

$$= 74.53 > 60.08 \text{ in., use } d_e = 74.53 \text{ in. in the } E$$ equation.

As the support bearing width is not yet determined, assume it conservatively = 0. Consequently, the critical section for shear is 74.35 in. = 63 ft from the support, being larger than the dimension $$d_e = 60.08 \text{ in.}$$ as stipulated by the LRFD AASHTO requirement, hence distance 74.35 in. controls for the critical shear section.

$$\frac{X}{L} = \frac{6.2}{120} = 0.052$$ from the support face.

From equation 15.20,

$$V_c = \frac{2}{\sqrt{1 - \nu^2}} A_{se} f_{se}$$

In order to determine the value of $$\theta$$ several computations have to be performed.

Reinforcement strain $$\epsilon_r$$ from equation 15.24 is

$$\epsilon_r = \frac{M_a}{d_e \left( E_r A_r + E_s A_s + E_p A_p \right)} = 0.002$$

At plane 0.05L, from Table 15.10

$$M_a = 1.25(275 + 315 + 34) + 1.5(62) + 1.75(388 + 160) = 1802 \text{ ft-kip}$$

Corresponding shear:

$$V_e = 1.25(43 + 50 + 5) + 1.5(11) + 1.75(74 + 31) = 323 \text{ kips}$$

$$N_e = \text{applied normal force at 0.05L, plane } = 0$$
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$f_{c0} =$ jacking stress  = 0.70 $f_{ck}$

$= f_{c0} + \frac{f_{p,e}E_t}{E_t}$  it can however, be conservatively taken

as the effective prestress $f_{p,e}$.

$f_c =$ concrete compressive stress at the centroid of the composite section due to both prestressing and the bending moments resisted by the precast section acting alone.

Distance from the e.g. of the composite section to the e.g. of the precast section.

c_1 = c_0 = 54.5 - 36.5 = 18.0 in.

At the critical section $\sigma_2 = 18.9$ in.

Section modulus $S_2 = \frac{L_2}{c_1} = \frac{L_2}{18.9} = 30.327$ in.\(^3\)

$c_2 = 17.28$ in.

\(r^2 = 712\) in.\(^2\)

\(f_{p,e} = \frac{P_{critical}}{A_{section}} \left(1 - \frac{c_2}{r^2}\right) \frac{(M_{net} + M_{cut})}{S_2}\)

\(= \left(\frac{1095}{712} \left(1 - \frac{18.9}{712}\right) \right) \frac{(275 + 135(12))}{30.327}\)

\(= 0.746 \times - 0.253 = -0.197\) ksi (c)

\(f_{p,c} = 48 \times 0.153 = 149.0\) ksi

\(E_p = 4990\) ksi

\(V_c = 323\) kips

\(M_c = 1802\) kip-ft

\(A_{c0} = 46 \times 0.153 = 7.144\)

\(f_{c0} = 0.70 \times 270 = 189\) ksi

($f_{c0}$ could be conservatively taken at $f_{c0} = 149.0$ ksi)

\(= \frac{1802 \times 12}{74.35} + 0.5 \times 232 \times cu. 22'' = 7.344 \times 189\)

\(e_s = -1.935 \times 10^{-3}\) in/in.  \(<0.002\) O.K.

Since the value of strain $e_s$ at the level of the reinforcement centroid is negative, its value has to be adjusted by the factor $F_e$

\(F_e = \frac{E_iA_i + E_pA_p}{E_iA_i + E_cA_c + E_{ct}A_{ct}}\)

From Figures 15.9 and 15.11, $h = 80$ in.

\(A_c = 26 \times 6 + 2 \times 0.5 \times 13 + 4.5 = 0.5 \times (6 + 4.5) = 378\) in.\(^2\)

\(F_r = \frac{4890 \times 378 + 0 + 20,000 \times 7.344}{4890 \times 378 + 0 + 20,000 \times 7.344} = 0.101\)
Adjusted $e_s = (-1.935 \times 10^{-3})(0.101) = -0.194 \times 10^{-1} = 0.002$ O.K.

(b) Web shear strength, $V_w$ from $\theta - \beta$ analysis.

$V_w = 323$ kips

From equation 15.14(b)

$V_w = (V_r - 0) \phi b d_s$

where $\phi = 0.90$ for shear

$f_s$, from before = 140 ksf

Figure 15.12 shows the inclination angle $\theta$ of the 12 harped strands.

$\sin \theta = \frac{65 - 155}{48.5 \times 12} = 0.006$

Harped tendon force: $12 \times 1.353 \times 0.149 = 273.6$ kips

$V_s = 273.6 \sin \theta = 273.6 \times 0.006 = 23.5$ kips

Required $V = \frac{323 - 0.9 \times 23.5}{0.9 \times 6 \times 74.35} = 0.75$ ksf

$\frac{V_s}{f_s} = \frac{0.75}{1.353} = 0.115$

$\epsilon_s$ from before $= -0.194 \times 10^{-7}$ in/in.

Entering the values of $\epsilon_s$ and $f_s$, in Table 15.9,

$\theta = 22^\circ$ (assumed $\theta = 22^\circ$ accept)

$\beta = 5.10$ (factor indicating the ability of the compression strut, namely, the diagonal crack to transmit tension)

Hence, $V_w = 0.0316 \times \sqrt{2.1 | b | d_s}$

$= 0.0316 \times 3.10 \times 5.5 \times 6 \times 74.35 = 111$ kips

(c) Selection of web reinforcement

From equation 15.23, check whether web reinforcement is needed, namely, if $V > 0.24 (V_r + V_s)$

$0.5 \phi (V_r + V_s) = 0.5 \times 0.9 (111 + 23.5) = 60.5$ kips, $V_s = 323$ kips

Use web steel.

$\frac{V}{\phi} - V_s = \frac{323}{0.9} - 60.5 - 23.5 = 250.3$ kips

Plane of crack

12 strands

38 strands

$\theta = 22^\circ$

$\epsilon_s = -1.935 \times 10^{-3}$

Beam Length = 121.0 ft

Figure 15.12 Beam tendon geometry
From Equation 15.2,

\[
V_r = \frac{A_{f, s} \cdot f_{c}}{s}
\]

Trying No. 4 stirrups, \(A_s = 2 \times 0.20 = 0.40\) in.\(^2\).

\[
\cot \theta = \cot 22^\circ = 2.475
\]

\[
\frac{250.3 \times 0.4 \times 66.0 \times 74.35 \times 2.475}{s} = 17.3 \text{ in.}
\]

(i) Maximum allowable web stirrup spacing:

\[
0.10f_{c, s}/d_s = 0.1 \times 6.5 \times 6 \times 74.35 = 313 \text{ kips} < V_r = 323 \text{ kips}
\]

hence, maximum allowable spacing \(s = 12\) in.

If \(0.10f_{c, s}/d_s > V_r\), maximum \(s = 24\) in.

(ii) Minimum area of transverse reinforcement:

\[
A_{s, w} = 0.016 \sqrt{\frac{f_{c}}{f_{y}}}
\]

\[
= 0.016 \sqrt{\frac{6.5}{0.12}} = 3.10 \text{ in.}^2/\text{ft}
\]

Use No. 4 stirrups at 12 in. center-to-center with the spacing to be increased along the span.

(iii) Minimum shear resistance:

To ensure that the concrete in the web does not crush prior to yielding of the

\[
(V_r - V_s) = 0.23/\beta_d
\]

\[
(V_r - V_s) = V_r - V_s = 62.8 + 252.6 = 315.4 \text{ kips}
\]

0.23/\beta_d = 0.25 \times 6.5 \times 6 \times 74.5 = 728.7 \text{ kips} > 315.4 \text{ kips, O.K.}

2. Interface shear transfer

(a) Dowel reinforcement design

Assume that the critical section for shear transfer is the same as the vertical shear

at plane 0.05\(L\) from the support face.

From flexural combination Strength:

\[
V_r = 1.25(3.5) + 1.5(10.8) + 1.75(73.8 + 30.4)
\]

\[
= 255.5 \text{ kips}
\]

\[
d = 74.35 \text{ in.}
\]

\[
V_w = 255.5 \times 74.35 = 2.76 \text{ kip/in.}
\]

Required \(V_s = \frac{V_w}{\phi} = \frac{2.76}{0.9} = 3.07 \text{ kip/in.}\)

From Equation 15.31:

\[
V_s = cA_{w, s} + \mu [A_{f, s} \cdot f_{c}]
\]

If no concrete placed clean, bond reinforced concrete with interface contact not intentionally toughened.

\[
c = 0.025 \text{ ksi} \quad \mu = 0.6
\]

\[
h_s = \text{ contact width between slab and prestressed flange top} = 42 \text{ in.}
\]

\[
\frac{A_{w, s}}{A_{s, s}} = 42 \times 1.0 = 42 \text{ in.}^2
\]
hence, $3.80 = 0.075 \times 42.0 + 0.6(A_f + 60 - 0)$ to give $A_f = 0.081$ in. $A_f = 0.04$ in. at 12 in. of vertical stirrups.

On this basis, no special additional dowel reinforcement is needed. LRFD, however, also requires that if the width exceeds 36 in., a minimum of four bars are required in dowel reinforcement. Thus, use also two No. 3 dowels at 12 in. c/c in addition to the No. 4 vertical stirrups at 12 in. c/c to give total $A_f = 0.02$ in. c/c.

(b) Maximum and minimum dowel reinforcement

$f' = 4.0$ ksi for the deck concrete

Actual provided $V_c = 0.075 \times 42 = 0.6(0.62 \times 60) = 5.01$ kips/in.

From Equations 15.35 and 15.36, the maximum allowable:

$0.5% A_f = 0.2 \times 4.0 \times 42.0 = 33.6$ kips/in.

In both cases, more than provided $V_c$. O.K.

3. LRFD Minimum Longitudinal Reinforcement

The longitudinal reinforcement at each beam section along the span has to satisfy equation 15.28:

$$A_f + A_r = \frac{M}{d_f} + 0.5 \frac{N}{\phi} \left(\frac{V_c}{6} - 0.5V_c - V_f\right) cot \theta$$

From Tables 15.10, 15.11 at $s = 0$ from support, $V_c = 1.25(42.0 + 533 + 6.0) = 1.62(92.0)$ kips

$V_c$ based on only the No. 4 stirrups = 261.4 kips

$M_s = 0$

$N_s = 0$

cut $\theta = cut \ 22^\circ = 2.475$

hence:

$$\frac{M_s}{d_f} + 0.5 \frac{N_s}{\phi} \left(\frac{V_c}{6} - 0.5V_c - V_f\right) cot \theta$$

$$= 0 + 0 \left(\frac{330}{0.9} - 0.5 \times 257.5 - 23.3\right)(2.475) = 586.2$ kips

Number of straight strands at the support = 36

Number of draped strands at the support = 12

From the assumed crack plane intersection with the strands in Figure 15.13, the distance of the intersection from the support = 6.422 cut 22° = 16.4 in. where the strand stops at 6 in. from face of the support.

Transfer length = 60 x strand diameter = 30 in.

The available prestress of the 36 straight strands at the support face is a portion of the effective prestress, $f_p$

Hence, use $f_p = 149.0 \times \frac{15.4}{30.0} = 81.5$ ksi

For the top-draped strands, the crack in Figure 15.12 intersects the strands at a distance $= 140$ in. from the support face (compute from geometry of the dimensions in Figure 15.12). Consequently, the effective prestress can be approximated at $f_p = 149.0$ ksf.

$$A_f + A_r = 0 \times 0.5(0.55, 5) + 12 \times 0.125(149.0)$$

$$= 443.4 + 273.6 = 717.0$ kips

$> 586.2$ kips, hence, no additional longitudinal reinforcement is needed.
4. Deflection and camber
   
   (i) Immediate deflection due to permanent loads

   \[
   \delta_0 = \frac{P_0 L^4}{8EI} \left[ \varepsilon_1 + \frac{(v - e_0)}{3} \right]
   \]

   here, \( n = \frac{L}{2} \)

   \[
   \delta = \frac{P_0 L^4}{8EI} \left[ \frac{v - e_0}{24} \right]
   \]

   From Example 13.1:

   \[ P_0 = 1488 \text{ kips} \]

   \[ e_0 = 0.0922 \text{ kips/ft} \]

   \[ v = 34.68 \text{ in.} \]

   \[ \varepsilon_1 = 767 \text{ in.} \]

   \[ E_0 = 4620 \text{ kips} \]

   \[ S = 14.05 \text{ in.} \]

   \[ S' = 15.42 \text{ in.} \]

   \[ w = 0.779 \text{ kips} \]

   \[ f = 545.87 \text{ in.} \]

   \[
   \delta = \frac{1488(120 \times 120)}{4620 \times 545.87} \left[ \frac{34.68}{24} + \frac{(767 - 29.68)}{3} \right]
   \]

   \[
   = 1.22 (3.71 - 0.39) \quad \text{in.} \quad \text{(curvature)}
   \]

   Photo 15.4 West Kowloon Expreway Viaduct, Hong Kong, 1997. 4.2-Km dual
   three-lane causeway connecting Western Harbor Crossing to new airport (Courtesy
   Institution of Civil Engineers, London).
15.8 LRFD Shear and Deflection Design: Example 15.2

\[ w_{pc} \text{ per inch} = 0.799/12 = 0.065 \text{ kip/in.} \]

\[ \delta_p = \frac{500.065 \times 120 \times 12'}{384 \times 4625 \times 543.894} = 0.14 \text{ in.} \]

\[ w_{psl} = 0.077 \text{ kip/ft} = 0.077 \text{ kip/in.} \]

\[ \delta_p = \frac{500.077 \times 120 \times 12'}{384 \times 4808 \times 543.894} = 0.14 \text{ in.} \]

\[ w_{bcr} = 0.100 \text{ kip/ft} = 0.025 \text{ kip/in.} \]

\[ \delta_p = \frac{500.025 \times 120 \times 12'}{384 \times 4888 \times 1,095.290} = 0.025 \text{ in.} \]

(ii) Immediate deflection due to transient loads

Live load deflection limit = 0.005.

LRFD specifications require that all the bridge deck beams be assumed to deflect equally under applied live load and impact. They also stipulate that the long-term deflection may be taken as four times the immediate deflection. This stipulation is too general and the designer is well-advised to use other more refined methods. The larger is the span the more is the needed accuracy. It should be emphasized that computed deflection values can differ from actual deflections by as much as 30 to 40% depending on the concrete modulus and stress-strain relationship assumed and the degree of accuracy of the method used in the computation.

From Figure 15.9 in Example 15.1, the number of bridge beams = 4 and the number of lanes = 4

DFD = distribution factor for deflection

\[ \text{DFD} = \frac{\text{number of lanes divided by number of beams}}{4} = \frac{4}{4} = 0.667 \text{ lanes/beam} \]

It is more conservative to use moment distribution factor DFM = 0.732.

Design live load, W = 0.84 DFM

<table>
<thead>
<tr>
<th>Transfer</th>
<th>Non-composite PCI Multipliers</th>
<th>Composite PCI Multipliers</th>
<th>Moment (in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{psl} )</td>
<td>1.80</td>
<td>2.20</td>
<td>8.677</td>
</tr>
<tr>
<td>( w_{bcr} )</td>
<td>1.85</td>
<td>2.40</td>
<td>8.417</td>
</tr>
<tr>
<td>Net 2.48</td>
<td>1.85</td>
<td>2.40</td>
<td>8.24</td>
</tr>
<tr>
<td>( w_{bcr} )</td>
<td>-1.61</td>
<td>1.85</td>
<td>2.30</td>
</tr>
<tr>
<td>( w_{bcr} )</td>
<td>-0.94</td>
<td>-1.78</td>
<td>-0.41</td>
</tr>
<tr>
<td>Total 8</td>
<td>2.48</td>
<td>-0.60</td>
<td>-0.41</td>
</tr>
</tbody>
</table>
Chapter 15 LRFD AASHTO Design of Concrete Bridge Deck Structures

\[
\begin{align*}
R & = 0.64 \text{ kip/ft (0.732)} = 0.468 \text{ kip/ft/beam} \\
R_i & = 0.039 \text{ kip/ft} \\
\beta_{12} = \frac{a_i}{384E_l} = \frac{5(0.009)(0.7 \times 12)^3}{384 \times 4880 \times 1.995.399} = 0.01 \text{ in.} \\
\end{align*}
\]

The transient truck load and impact deflection is determined from influence lines of wheel position for maximum moment. For a 120 ft span, the 72 kip resultant of the total loads falls at 2.33 ft from the midpoint. The deflection at midspan is 0.3 in.

\[
\beta_{11} = 0.3(1.3)(0.732) = 0.28 \text{ in.}
\]

Using the PCI multipliers from Ref. 15.22, a summary of the long-term camber and deflections are given in Table 15.12.

Allowable deflection \( R = \frac{12 \times 12}{800} = 1.80 \text{ in.} \) (down)

\[
\beta_{11} > 0.09 = 0.49 \text{ in.} \quad \text{O.K.}
\]

Adopt the bridge deck design of the interior beam in Example 15.1 and 15.2.

SELECTED REFERENCES


15.4. ACI, "Building Code Requirements for Structural Concrete (ACI 318-05) and Commentary (ACI \#318-05), American Concrete Institute, Farmington Hills, MI.


15.11. PCI, "Bridge Design Manual, Prestressed/Prextressed Concrete Institute, Chicago, 1999.


PROBLEMS FOR SOLUTION

15.1. Design for flexure a 100 ft (30.5m) simply supported AASHTO-PCI bulb-tee composite bridge deck with no skewing using the LRFD AASHTO specifications. The superstructure is composed of six precast girders at 9'-0" (2.74 m) on centers. The bridge has a 5.5 in. (203 mm) self-sustained concrete deck with the top one half inch to be considered as wearing surface. The design live load is the HL-90 AASHTO-LRFD fatigue loading. Assume the bridge is to be located in a low seismicity zone. Given the following maximum allowable stresses:

**Deck**

$f_d = 4000$ psi, normal weight

$f_{cd} = 0.60 f_d = 2400$ psi

**Bulb-tee**

$f_{ct} = 6500$ psi

$f_{ct} = 5500$ psi

$f_{ct} = 0.60 f_{ct} = 3300$ psi, Service III

$f_{ct} = 0.45 f_{ct} = 2925$ psi, Service I

$f_{ct} = 0.60 f_{ct} = 3480$ psi

$f_{ct} = 6 \sqrt{f_{ct}} = 484$ psi

$f_{cr} = 270,000$ psi

$f_{cr} = 0.90 f_{cr} = 243,000$ psi

$f_{cr} = 0.75 f_{cr} = 202,500$ psi

$f_{cr} = 60,000$ psi

$E_s = 26.3 \times 10^6$ psi

$E_c = 24.0 \times 10^6$ psi.

15.2. Design the web shear reinforcement for the bulb-tee beam in Problem 15.1 at the critical section near the supports and the web shear transfer reinforcement at the interface plane between the prestressed section and the deck skin. A concrete. Also, verify if the span deflection is within the allowable limits.
16.1 INTRODUCTION. MECHANISM OF EARTHQUAKES

Earth’s crust is composed of several layers of hard tectonic plates, called lithospheres, that float on the softer, underlying fluid medium called mantle. These plates or rock masses, when fractured, form fault lines. The adjoining plates or rock masses are prevented by the interacting frictional forces from moving past one another most of the time. However, when this frictional ultimate resistance is reached because of the continuous motion of the underlying fluid, any two plates can impact on one another, generating seismic waves that can cause large horizontal and vertical ground motions. These ground motions translate into inertia forces in structures.

The length and width of a fault are interrelated to the magnitude of the earthquake. The fault is the cause rather than the result of the earthquake. A fault can cause an earthquake due to the following reasons (Ref. 16.5):

1. Cumulative strain in the fault over a long period of time reaches the rupture strain.
2. Slip of the tectonic plates at the fault zones causes a rebound, as in Figure 16.1a.

Photo 16.1 Northridge, California, 1994 earthquake structural failure. (Courtesy Dr. Mural Sareen.)
16.1 Introduction: Mechanism of Earthquakes

Figure 16.1 Mechanism of earthquakes: (a) slip of tectonic plates; (b) reverse moment couples.

3. Suddenly push and pull forces at the fault lead to reverse moment couples, as in Figure 16.1b. The moment caused by these couples is a measure of earthquake size and can be termed the seismogenic moment. Their magnitude is equal to rock rigidity $\times$ fault area $\times$ amount of slip. The range of slip velocity in such faults as the San Andreas fault in California is 20 to 100 mm per year. On this basis, a slippage or horizontal motion of 3 m at such faults in one single earthquake is expected to occur at intervals of 38 to 100 years.

Earthquakes may be characterized by three categories: low, moderate, and high intensity. The intensity is governed by ground motion accelerations, represented by re-
response spectra and coefficients derived from such spectra. A structure is expected to respond essentially elastically to low-intensity earthquakes, where the stresses are expected to remain within the elastic range, with a slight possibility of developing limited inelasticity that is not expected to cause appreciable structural or nonstructural damage.

Structural response is expected to be inelastic under high-intensity earthquakes having an intensity of 5 or higher on the Richter scale and in regions close to the epicenter. For the design of structures in seismic zones, two methods are presented in the IBC 2000 and 2003 codes: the spectral response method and the equivalent lateral force method. The latter has certain limitations that will be discussed later.

A detailed discussion of the subject of earthquakes is beyond the scope of this book since the primary aim of this chapter is the proportioning of seismic resistant components of concrete structures. However, some of the basic underlying characteristics are important to cover. They are intended to help define the magnitude of the lateral seismic base shear forces that determine the geometry and form of the earthquake-resisting components of a structure, namely, the lateral force resisting system (LFRS).

Such a system has two components: horizontal and vertical. The horizontal elements are the components that resist the seismic forces. They can be diaphragms, coupling beams and shear walls. The vertical component comprises the walls and vertical frames of the structure.

15.1.1 Earthquake Ground Motion Characteristics

Ground motion, caused by seismic tremors, involves acceleration, velocity, and displacement. These are in the majority amplified, thereby producing forces and displacements, which can exceed those which the structure is able to sustain (Ref. 16.13). The maximum value of the ground motion magnitude, namely, the peak ground velocity, peak ground acceleration, and peak ground displacement become the principal parameters in the seismic design of structures.
Additional factors also affect the response of a structure. They include frequency, amplitude of motion, shaking duration, and site soil characteristics. These can all be represented by a response spectrum which idealizes a structure into a damped, single degree of freedom system (SDF) oscillating at various periods and frequencies. The maximum vibration magnitude reached anytime during the base ground motion is its spectral value.

16.1.2 Fundamental Period of Vibration

The basic natural period $T$ of a simple one-degree-of-freedom system is the time required to complete one whole cycle during dynamic loading. In other words, it is the time required for a phase angle $\omega t$ to travel from $0$ to $2\pi$, where $\omega$ is the angular frequency of the system. Hence $\omega t = 2\pi$, leading to the expression

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$  (16.1)

where $m = \text{mass of system}$
$k = \text{spring constant}$ and damping is not considered.

Most reinforced concrete structures are multi degrees-of-freedom systems, as in Figure 16.2. In this case, the structural mass can be assumed to be concentrated in the vertical spring element at the floor level, resulting in multiple modes with frequencies (periods) for each mode. The compound natural period $T$ is then evaluated with due con-
Consideration given to the distribution of mass and stiffness. Codes require that these be established using the structural properties and deformation characteristics of the resisting elements in a properly substantiated analysis using expressions such as those given by the International Building Code—IBC 2000 (Ref. 16.2), or The Uniform Building Code (Ref. 16.3) integrated into the IBC provisions.

Since a structure is composed of a series of single degrees of freedom components subjected to the same base motion, a series of maximum values related to the SDF system's fundamental periods, T, would ensue. These in turn form a spectral curve for the base ground motion. By knowing the base motion, the SDF fundamental period, and the percent critical damping, one can obtain from the applicable curve the maximum acceleration, velocity, and displacement relative to the base (Ref. 16.4). Evidently, computer use is needed to obtain a complete spectral response of the multi-degree of freedom system.

It should be recognized that a structure is designed to resist earthquake motion such that it is able to sustain and survive the earthquake through large inelastic deformations and energy dissipation through cracking and limited local material failure, but without loss of stability. It would be highly uneconomical to design the lateral force-resisting system to the earthquake force such that the structure deforms only elastically as a result of these forces. The codes have this as a basic philosophy, particularly for major earthquakes in which some structural damage can result.

16.1.3 Design Philosophy

The International Building Code (IBC 2000 and 2003) on seismic design consolidates the three major existing regional codes:

1. Building Officials Code Administration International (BOCA)
3. Southern Building Code Congress International (SBCCI)

Underlying its seismic design provisions are:

1. Recommended design levels related to effective peak accelerations that can resist minor earthquakes without damage, moderate earthquakes without structural damage, and major earthquakes in which some structural damage can result.
16.2 Spectral Response Method

2. Minimum design criteria for all types of buildings, low and high rise, with and without shear walls.
3. Spectral response values for various ground motion intensities, mainly within the elastic range.
4. The provision of design criteria for lateral ground motion, unidirectional and bidirectional, addressing them one at a time.
5. Limit the story drift and displacement magnitudes of the building structures within acceptable ranges, through control of stiffness of components and shear walls, diaphragms and coupling beams.

16.2 SPECTRAL RESPONSE METHOD

16.2.1 Spectral Response Acceleration Maps

As discussed in Ref. 16.15, prior to the Northridge and Kobe earthquakes, the Uniform Building Code (UBC) provisions performed satisfactorily in the United States in past earthquakes. The failures in these two cases were determined to be due to "related configurations, structural systems, inadequate connection detailing, incompatibility of deformations and design or construction deficiencies. They were not due to deficiency in strength" (Structural Engineers Association of California, 1995).

The UBC provisions incorporated in the International Building Code (IBC) are based on consideration of the site conditions of the structure and the application of max-
The maximum considered earthquake ground motion maps for site class B, prepared by the United States Geological Survey (USGS). The equivalent maximum considered earthquake ground motion values for the ceiling were determined to be 1.50 g for the short period and 0.60 g for the long period (Ref. 16.15).

In high seismicity regions, where the maximum considered earthquake ground motion values are greater than 0.75 g for the 1.0 sec peak acceleration, additional requirements are imposed on irregular structures exceeding 5 stories in height and a period T in excess of 0.5 sec, such as increasing the ground motion spectral acceleration values by 56 percent. The USGS large-scale maps for the 1.0 sec and the 0.2 sec levels of spectral response acceleration, site B class, and 5% critical damping are condensed and abridged in Figs. 16.3 (a) and (b) for general guidance. They show the relative values of the peak spectral response accelerations at the two ground motion levels of 0.2 and 1.0 sec. Values have to be extrapolated linearly from the USGS large-scale maps for use in the seismic design of structures.

16.2.2 Design Parameters
Both the spectral response method and the equivalent lateral force method are based on the same code principles and formulations presented in this chapter. Sites are classified into six categories A, B, C, D, E, and F as shown in Table 16.1 on site properties.

Ground motion accelerations and the maximum considered earthquake spectral response acceleration are considered at 1.0 sec period ($S_1$) and at short periods ($S_2$) such as 0.2 sec obtained from seismic contour maps discussed in Sec. 16.2.1.
### Table 16.1 Site Classifications

<table>
<thead>
<tr>
<th>Site Class</th>
<th>Soil Profile Name</th>
<th>Soil Shear Wave Velocity, ( V_s ) (ft/s)</th>
<th>Standard Penetration Resistance, ( N )</th>
<th>Soil Unconfined Shear Strength, ( S_u ) (PSF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Hard rock</td>
<td>( V_s &gt; 5,000 )</td>
<td>not applicable</td>
<td>not applicable</td>
</tr>
<tr>
<td>B</td>
<td>Rock</td>
<td>( 2,000 \leq V_s \leq 5,000 )</td>
<td>not applicable</td>
<td>not applicable</td>
</tr>
<tr>
<td>C</td>
<td>Very dense soil and soft rock</td>
<td>( 1,200 \leq V_s \leq 1,500 )</td>
<td>( N &gt; 50 )</td>
<td>( S_u &gt; 2,000 )</td>
</tr>
<tr>
<td>D</td>
<td>Stiff soil profile</td>
<td>( 600 \leq V_s \leq 1,000 )</td>
<td>( 15 \leq N \leq 50 )</td>
<td>( 1,000 \leq S_u \leq 2,000 )</td>
</tr>
<tr>
<td>E</td>
<td>Soft soil profile</td>
<td>( V_s &lt; 600 )</td>
<td>( N &lt; 15 )</td>
<td>( S_u &lt; 1,000 )</td>
</tr>
<tr>
<td>F</td>
<td>Any profile with more than 10 ft of soil having the following characteristics:</td>
<td>—plasticity index PI &gt; 20; —moisture content w &gt; 40%; and —unconfined shear strength ( S_u &lt; 500 ) psf</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

F1 (SI units): 1 ft = 0.305 m.
1 psf = 0.049 kPa.
1 l/s = 365 mm.

---

The design spectral response accelerations at short periods (\( S_1 \)) and at 1.0 sec (\( S_t \)) are to be adjusted for site-class effect (\( S_sp \)) at short periods and (\( S_sp \)) for 1.0 sec, based on Table 16.1 in conjunction with Tables 16.2a and 16.2b for site coefficients.

The maximum considered earthquake spectral response for short and one second periods are respectively defined by the following expressions:

### Table 16.2a Values of Site Coefficient \( F_s \) as a function of site class and mapped spectral response acceleration at short periods (\( S_1 \))

<table>
<thead>
<tr>
<th>Site Class</th>
<th>( S_1 \leq 0.25 )</th>
<th>( S_1 = 0.50 )</th>
<th>( S_1 = 0.75 )</th>
<th>( S_1 = 1.02 )</th>
<th>( S_1 = 1.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>B</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
<td>1.2</td>
</tr>
<tr>
<td>D</td>
<td>1.6</td>
<td>1.4</td>
<td>1.1</td>
<td>1.1</td>
<td>1.0</td>
</tr>
<tr>
<td>E</td>
<td>2.5</td>
<td>1.7</td>
<td>1.7</td>
<td>0.9</td>
<td>Note a</td>
</tr>
<tr>
<td>F</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
</tr>
</tbody>
</table>

Note: For site class F, Table 16.2b should be consulted.

---

For site class F, Table 16.2b should be consulted.
Table 16.2b Values of Site Coefficient $F_s$ as a function of site class and mapped spectral response acceleration at 1.0 sec periods ($S_r$)

<table>
<thead>
<tr>
<th>Site Class</th>
<th>$S_r = 0.1$</th>
<th>$S_r = 0.2$</th>
<th>$S_r = 0.3$</th>
<th>$S_r = 0.4$</th>
<th>$S_r = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>B</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>C</td>
<td>1.7</td>
<td>1.6</td>
<td>1.3</td>
<td>1.3</td>
<td>1.3</td>
</tr>
<tr>
<td>D</td>
<td>2.4</td>
<td>2.0</td>
<td>1.8</td>
<td>1.6</td>
<td>1.5</td>
</tr>
<tr>
<td>E</td>
<td>3.5</td>
<td>3.2</td>
<td>2.8</td>
<td>2.4</td>
<td>Note a</td>
</tr>
<tr>
<td>F</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
<td>Note b</td>
</tr>
</tbody>
</table>

Notes:  
a—Linear interpolation for intermediate values are to be made.  
b—Site geotechnical investigation and dynamic site response analysis are to be performed.

\[ S_{tot} = F_s S_r \]  
\[ S_{hi} = F_s S_r \]  

where:

$F_s$ = Site coefficient from Table 16.2a

$F_s$ = Site coefficient from Table 16.2b

$S_r$ = Mapped Spectral acceleration for short periods

(See Ref. 16.2 for map contour values)

$S_r$ = Mapped Spectral acceleration for 1.0 sec. periods

(See Ref. 16.2 for map contour values)

For 5% damped design, the spectral response acceleration becomes:

\[ S_{sp} = \frac{3}{2} S_{tot} \]  
\[ S_{sp} = \frac{3}{2} S_{hi} \]

16.2.3 Earthquake Design Load Classifications

The International Building Code (IBC 2000 and IBC 2003) classifies the seismic design categories into three “seismic use groups” in lieu of the former zones 0 through 4 of the UBC Code. The three groups for short-period and 1.0-sec period response acceleration:

Table 16.3a Seismic Design Category based on short-period response accelerations

<table>
<thead>
<tr>
<th>Value of $S_{sp}$</th>
<th>I</th>
<th>II</th>
<th>III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{sp} &lt; 0.107$ g</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>0.107 g $S_{sp} &lt; 0.33$ g</td>
<td>B</td>
<td>B</td>
<td>C</td>
</tr>
<tr>
<td>0.33 g $S_{sp} &lt; 0.50$ g</td>
<td>C</td>
<td>C</td>
<td>D</td>
</tr>
<tr>
<td>0.50 g $S_{sp}$</td>
<td>D+</td>
<td>D+</td>
<td>D+</td>
</tr>
</tbody>
</table>
16.2 Spectral Response Method

Spectral Response Method

<table>
<thead>
<tr>
<th>Table 16.3b</th>
<th>Seismic Design Category based on 1-s sec period response accelerations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of $S_m$</td>
<td>Seismic Use Group</td>
</tr>
<tr>
<td>0.067 g ≤ $S_m &lt; 0.133$ g</td>
<td>A</td>
</tr>
<tr>
<td>0.133 g ≤ $S_m &lt; 0.20$ g</td>
<td>C</td>
</tr>
<tr>
<td>0.20$g ≤ S_m</td>
<td></td>
</tr>
</tbody>
</table>

Note: 1. Seismic Use Groups I and II structures located on sites with seismid maximum considered earthquake spectral response acceleration of 1.0 sec period, $S_m$ equal or greater than 0.75 g shall be assigned to Seismic Design Category F and Seismic Use Group III structures located on such sites shall be assigned to Seismic Design Category G.

are given in Tables 16.3a and 16.3b, respectively. These seismic use groups can be defined as follows (Ref. 15.3):

(i) Seismic Use Group I: These are structures that are not assigned to either seismic use group II or III

(ii) Seismic Use Group II: The structures in this group are those the failure of which would result in substantial public hazard due to occupancy or use described in Table 16.5.

(iii) Seismic Use Group III: The structures in these groups are those the failure of which would result in having essential facilities for post-earthquake recovery and those containing substantial quantities of hazardous substances in jeopardy.

In Tables 16.3a and b, categories B and C range from low to moderate-risk regions, whereas categories D and E are designated as high-risk seismic regions.

These tables enable the designer to choose from the spectral maps the $S$ and $S_m$ values pertinent to the structure site location. They are based on classifying the map regions into three as follows (Ref. 16.15, Part 2):

Region 1: Regions of Negligible Seismicity with Very Low Probability of Collapse of the Structure (No Spectral Values)

Region definition: Regions for which $S > 0.25$ g and $S_m < 0.10$ g from Step 2b.

Design values: No spectral ground motion values required. Use a minimum lateral force level of $1\%$ of the dead load for seismic design category A.

Region 2: Regions of Low and Moderate to High Seismicity (Probabilistic Map Values)

Region definition: Regions for which $0.25$ g < $S < 1.5$ g and $0.25$ g < $S_m < 0.60$ g.

Maximum considered earthquake map values: Use $S$ and $S_m$ map values.

Transition Between Regions 2 and 3: Use values of $S_m = 1.5$ g and $S_m = 0.60$ g.

Region 3: Regions of High Seismicity Near Known Faults (Deterministic Values)

Regional definition: Regions for which $1.5$ g < $S$ and $0.60$ g < $S_m$.

The structural analysis based on the worst-load combinations should be the basis for determining the seismic forces $F$ for combined gravity and seismic load effects when they
are additive and the maximum seismic load effect $E_s$. The value of $E$ and $E_s$ are determined from the following expressions detailed in Ref. 16.2 for additive seismic force and dead load:

$$E = \mu Q_h + 0.2 S_{sp} D$$  \hspace{1cm} (16.4a) \\
$$E = \mu Q_v + 0.2 S_{sh} D$$  \hspace{1cm} (16.4b) \\

For countering seismic forces and dead load:

$$E = \mu Q_h - 0.2 S_{sp} D$$  \hspace{1cm} (16.5a) \\
$$E = \mu Q_v - 0.2 S_{sh} D$$  \hspace{1cm} (16.5b) \\

where $E$ is the combined effect of horizontal and vertical earthquake-induced forces, $\mu$ is a reliability factor based on system redundancy = 1.0 for categories A, B, and C, $Q_h$ is the effect of horizontal seismic forces, $S_{sh}$ is the spectral response acceleration at short periods obtained from IBC Sec. 1615.1.3 or 1615.2.2.5, $Q_v$ is System over-strength factor given in Table 16.4.

16.2.4 Redundancy

A redundancy coefficient $\rho$ has to be assigned to all structures based on the extent of structural redundancy inherent in the lateral force-resisting system. For structures in seismic design categories A, B, and C, the value of the redundancy coefficient $\rho$ is to be taken as 1.0. For structures in seismic design categories D, E, and F, the redundancy coefficient $\rho$ has to be taken as the largest of the values $\rho$, calculated at each story level “i” of the structure in accordance with the expression

$$\rho_i = 2 \left( \frac{20}{r_{max} \sqrt{A_i}} \right)$$  \hspace{1cm} (16.6a) \\

In SI units, the expression becomes

$$\rho_i = 2 \left( \frac{6.1}{r_{max} \sqrt{A_i}} \right)$$  \hspace{1cm} (16.6b) \\

where $r_{max}$ is ratio of the design story shear resisted by the most heavily loaded single element in the story to the total story shear for a given loading condition, $A_i$ is floor area in square feet (m$^2$) of the diaphragm level immediately above the story. The value of $\rho$ cannot be less than 1.0 and need not exceed 1.5.

16.2.5 General Procedure Response Spectrum

The design response can be idealized by the fundamental period-response acceleration relationship shown in Figure 16.4 for three fundamental periods levels:

1. For periods in seconds less than or equal to $T_p$, the design spectral response acceleration $S_p$ is determined from the following equation:

$$S_p = 0.6 \frac{S_{sp}}{T_p} T + 0.4 S_{sh}$$  \hspace{1cm} (16.7a)
<table>
<thead>
<tr>
<th>Basic Seismic-Force-Resisting System</th>
<th>Response Modification Coefficient ( R )</th>
<th>System Overstrength Factor ( \Omega_s )</th>
<th>Deflection Amplification Factor, ( c_d )</th>
<th>( A ) &amp; ( B )</th>
<th>( c )</th>
<th>( D^* )</th>
<th>( E^* )</th>
<th>( F^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bearing Wall System</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special reinforced concrete shear wall</td>
<td>5.5</td>
<td>2.5</td>
<td>3</td>
<td>NL</td>
<td>NL</td>
<td>160</td>
<td>160</td>
<td>100</td>
</tr>
<tr>
<td>Ordinary reinforced concrete shear wall</td>
<td>4.5</td>
<td>2.5</td>
<td>4</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Detailed plain concrete shear walls</td>
<td>2.5</td>
<td>2.5</td>
<td>2</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Ordinary plain concrete shear walls</td>
<td>2.5</td>
<td>2.5</td>
<td>1.5</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Building Frame System</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ordinary reinforced concrete shear wall</td>
<td>5</td>
<td>2.5</td>
<td>4.5</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Detailed plain concrete shear walls</td>
<td>3</td>
<td>2.5</td>
<td>2.5</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
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<tr>
<td>Ordinary plain concrete shear walls</td>
<td>2</td>
<td>2.5</td>
<td>2</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Moment Resistant Frames</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special reinforced concrete moment frames</td>
<td>8</td>
<td>3</td>
<td>5.5</td>
<td>NL</td>
<td>NL</td>
<td>NL</td>
<td>NL</td>
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<tr>
<td>Intermediate reinforced concrete moment frames</td>
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<td>4.5</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
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<td>NP</td>
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<tr>
<td>Ordinary reinforced concrete moment frames</td>
<td>3</td>
<td>3</td>
<td>2.5</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Dual System with Special Moment Frames</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Special reinforced concrete shear walls</td>
<td>8</td>
<td>2.5</td>
<td>6.5</td>
<td>NL</td>
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</tr>
<tr>
<td>Ordinary reinforced concrete shear walls</td>
<td>7</td>
<td>2.5</td>
<td>4</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Dual System with Intermediate Moment Frames</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Special reinforced concrete shear walls</td>
<td>6</td>
<td>2.5</td>
<td>5</td>
<td>NL</td>
<td>NL</td>
<td>160</td>
<td>160</td>
<td>100</td>
</tr>
<tr>
<td>Ordinary reinforced concrete shear walls</td>
<td>5.5</td>
<td>2.5</td>
<td>4.5</td>
<td>NL</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
<tr>
<td>Shear wall-frame hysteretic system with ordinary reinforced concrete moment frames and ordinary reinforced concrete shear walls</td>
<td>5.5</td>
<td>2.5</td>
<td>5</td>
<td>NL</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
<td>NP</td>
</tr>
</tbody>
</table>

---

1Unit SI: ft = 305 mm
2Response modification coefficient, \( R \), for use throughout
3Deflection amplification factor, \( c_d \)
4NL = not limiting and NP = not permitted
5Limit to buildings with a height of 240 feet or less.
6Limit to buildings with a height of 300 feet or less.
7Ordinary moment frame is permitted to be used in lieu of intermediate moment frame in seismic design category B and C.
8The reduced value of the overstrength factor, \( \Omega_s \), may be reduced by subtracting 1/2 for structures with flexible diaphragms but shall not be taken as less than 2.0 for any structure.
9Ordinary moment frames of reinforced concrete are not permitted at part of the seismic-force-resisting system in seismic design category B structures founded on Site Class F soils.
2. For periods greater than or equal to $T_o$, and less than or equal to $T_s$, the design spectral response acceleration $S_a$ is taken equal to $S_{mp}$.

3. For periods greater than $T_s$, the design spectral response acceleration, $S_a$, is determined from the expression:

$$S_a = \frac{S_{mp}}{T}$$  \hspace{1cm} (16.7b)

where:

- $S_{mp}$ = the design spectral response acceleration at short periods
- $S_{mp}$ = the design spectral response acceleration at 1 second periods
- $T$ = fundamental period (in seconds) of the structure
- $T_o = 0.2 S_{mp}/S_{mp}$
- $T_s = S_{mp}/S_{mp}$

The site have to be classified for determining the shear wave velocity and the maximum considered earthquake ground motion. Details are given in the IRC (Ref. 16.2) section 16.15.

16.3 EQUIVALENT LATERAL FORCE METHOD

16.3.1 Horizontal Base Shear

In this method, a building is considered to be fixed at the base. The seismic base shear, $V$, in a given direction is determined from the expression (Ref. 16.2):

$$V = C_s W$$  \hspace{1cm} (16.8)

where:

- $C_s$ = seismic response coefficient
- $W$ = The effective seismic weight of the structure, including the total dead loads and other loads listed below;

1. In areas used for storage, a minimum of 25% of the reduced floor live load (floor live load is 2000 psf) or parking structures, whichever is greater.
2. Where an allowance for partition load is included in the floor load design, the actual partition weight or a minimum weight of 10 psf (500 Pa/m²) of floor area, whichever is greater.
3. Total operating weight of permanent equipment.
4. 20% of flat roof snow load where the flat roof snow load exceeds 30 psf.

\[ C_s = \frac{S_{0.5}}{R/I} \]  

(16.9)

But \( C_s \) cannot exceed the value:

\[ C_s = \frac{S_{0.5}}{R/I} \]  

(16.10)

nor can it be taken less than:

\[ C_s = 0.044 S_{50} \]  

(16.11)

where,

\( S_{0.5} \) = Design spectral response acceleration at short period as determined in Sec. 16.2.2
\( R \) = Response modification factor from Table 16.4
\( I \) = Occupancy importance factor from Table 16.5
\( T \) = Fundamental period of building (seconds)

For buildings and structures in seismic design categories E or F and in buildings and structures for which the 1-sec spectral response, \( S_r \), is equal to or greater than 0.6 g, the value of the seismic coefficient \( C_s \) should not be taken less than:

\[ C_s = 0.55 \frac{R/I}{T^2} \]  

(16.12)

The fundamental period \( T \) in the direction under consideration has to be determined by analysis basis of the structural and deformational characteristics of the resisting element. In lieu of an analysis, an approximate fundamental period \( T_f \), in seconds, can be used from the following expression:

\[ T_f = C_f h^{0.4} \]  

(16.13)

where,

\( C_f \) = Building Period Coefficient
- 0.035 for moment-resisting frame systems of steel in which the frames resist 100% of the required seismic force and are not enclosed or braced by more rigid components that will prevent the frames from deflecting when subjected to seismic forces (the metric coefficient is 0.085)
- 0.020 for moment-resisting frame systems of reinforced concrete in which the frames resist 100% of the required seismic force and are not enclosed or braced by more rigid components that will prevent the frames from deflecting when subjected to seismic forces (the metric coefficient is 0.073)
- 0.010 for eccentrically braced steel frames (the metric coefficient is 0.075)
- 0.010 for all other building systems (the metric coefficient is 0.049)

\( h_b \) = the height (ft or m) above the base to the highest level of the building.
Table 16.5 Occupancy importance factor classification of buildings and other structures for importance factors

<table>
<thead>
<tr>
<th>Category</th>
<th>Nature of Occupancy</th>
<th>Seismic Factor $I_s$</th>
<th>Snow Factor $I_s$</th>
<th>Wind Factor $I_v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Building and other structures except those listed in Categories II, III, and IV</td>
<td>1.00</td>
<td>1.0</td>
<td>1.00</td>
</tr>
</tbody>
</table>
| II       | Buildings and other structures that represents a substantial hazard to human life in the event of failure, including, but not limited to:  
- Buildings and other structures where more than 300 people congregate in one area  
- Buildings and other structures: elementary school, secondary school or day-care facilities with capacity greater than 250  
- Buildings and other structures with a capacity greater than 500 for colleges or adult education facilities  
- Health care facilities with a capacity of 50 or more resident patients but not having surgery or emergency treatment facilities  
- Jail or detention facilities  
- Any other occupancy with an occupant load greater than 5,000  
- Power generating stations, water treatment for portable water, waste water treatment facilities, and other public utility facilities not included in category IV  
- Buildings and other structures not included in category IV containing sufficient quantities of toxic or explosive substances to be dangerous to the public if released. | 1.25 | 1.1 | 1.15 |
| III      | Buildings and other structures designated as essential facilities including, but not limited to:  
- Hospitals and other health care facilities having surgery or emergency treatment facilities  
- Fire, rescue, and police stations and emergency vehicle garages  
- Designated earthquake, hurricane, or other emergency shelters  
- Designated emergency preparedness, communication, and operation centers and other facilities required for emergency response  
- Power-generating stations and other public utility facilities required as emergency back-up facilities for category IV structures  
- Structures containing highly toxic material  
- Aviation control towers, air traffic control centers and emergency aircraft hangers  
- Buildings and other structures having critical national defense functions  
- Water treatment facilities required to maintain water pressure for fire suppression | 1.50 | 1.2 | 1.15 |
| IV       | Buildings and other structures of low hazard to human life such as agricultural facilities and minor storage facilities | 1.00 | 0.5 | 0.87 |

*Category is equivalent to Seismic Use Group for the purpose of Section 16.2.2

In cases where moment resisting frames do not exceed twelve stories in height and having a minimum story height of 10 ft (3 m), an approximate period $T_s$ in seconds in the following form can be used:

$$T_s = 0.1 \times N$$

(16.14)

where $N$ = number of stories

The calculated fundamental period, $T_s$, cannot exceed the product of the coefficient, $C_0$, in Table 16.6 for the upper limit or the calculated period times the approximate fundamental period. $T_s$. The base shear $V$ is to be based on a fundamental period, $T_s$, in seconds, of 1.2 times the coefficient for the upper limit; the calculated value, $C_0$, taken from Table 16.6 is times the approximate fundamental period.
Table 16.6 Coefficient for Upper Limit on Calculated Period

<table>
<thead>
<tr>
<th>Design Spectral Response Acceleration at 1-Second Period, $S_0$</th>
<th>Coefficient $C_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\geq 0.4$</td>
<td>1.2</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3</td>
</tr>
<tr>
<td>0.2</td>
<td>1.4</td>
</tr>
<tr>
<td>0.15</td>
<td>1.5</td>
</tr>
<tr>
<td>$\leq 0.1$</td>
<td>1.7</td>
</tr>
</tbody>
</table>

16.3.2 Vertical Distribution of Forces

The lateral force $F_i$ (kips or kN) induced at any level can be determined from the following expressions:

$$F_i = C_v V$$  \hspace{1cm} (16.15a)

$$C_v = \frac{W_i h_i^k}{\sum W_j h_j^k}$$  \hspace{1cm} (16.15b)

where

- $C_v$ = vertical distribution factor
- $V$ = total design lateral force or shear at the base of the building (kips or kN),
- $W_i$ and $W_j$ = the portion of the total gravity load of the building, $W$, located or assigned to Level $i$ or $j$,
- $h_i$ and $h_j$ = the height (ft or m) from the base to level $i$ or $j$, and
- $k$ = a distribution exponent related to the building period as follows:
  - For buildings having a period of 0.5 sec or less, $k = 1$
  - For buildings having a period of 2.5 sec or more, $k = 2$
  - For buildings having a period between 0.5 and 2.5 sec, $k$ shall be 2 or shall be determined by linear interpolation between 1 and 2

16.3.3 Horizontal Distribution of Story Shear $V_i$

The seismic design story horizontal shear in any story, $V_i$ (kips or kN) should be determined from the following expression:

$$V_i = \sum_{i=1}^{n} F_i$$  \hspace{1cm} (16.16)

Where

- $F_i$ = the portion of the seismic base shear, $V$ (kips or kN) introduced at level $i$.

16.3.4 Rigid and Flexible Diaphragms

(a) Rigid Diaphragms: The seismic design story shear, $V_i$, has to be distributed to the various vertical elements of the system in the story under consideration. This distribution is to be based on the relative stiffness of the vertical resisting elements and the diaphragms.

(b) Flexible Diaphragms: The seismic design story shear, $V_i$, in this case has to be distributed to the various vertical elements based on the tributary area of the di-
aphrags to each line of resistance. The vertical elements of the lateral force resis-
ting system can be considered to be in the same line of resistance, if the maximum out of plane offset between such elements is less than 5% of the building’s dimen-
sion perpendicular to the direction of the lateral load.

16.3.5 Torsion
If the diaphragms are not flexible, the design has to include the torsional moment \( M_t \) (ft-Kip of KN-m) resulting from the difference in location between the center of mass and the center of stiffness. Dynamic amplification of torsion for structures in seismic design cate-
gory C, D, E, or F has to be accounted for by multiplying the torsional moments by a tor-
sional amplification factor presented in Ref. 16.2, Sec. 16.17.4.

16.3.6 Story Drift and the P-Delta Effect
(a) Drift: The design story drift, \( \Delta \), is computed as the difference between the deflec-
tions of the center of mass at the top and bottom of the story being considered. If allowable stress design is used, \( \Delta \) is computed using earthquake forces without di-
viding by 1.4.

The deflection of level \( X \) is to be determined from the following expression,

\[
\delta_x = \frac{CA_x}{I}
\]

where,

- \( C_x \): Deflection amplification factor (Table 16.4)
- \( \delta_x \): Deflections (in. or mm) determined by an elastic analysis of the seismic forces resisting system
- \( I \): Occupancy importance factor (Table 16.5)

The design story drift, \( \Delta \), has to be increased by an increment factor relating to the P-delta effects. The redundancy coefficient, \( p \), in the case of drift should be taken as 1.0.

(b) P-Delta effects: The P-Delta effects can be disregarded if the stability coefficient, \( \theta \), from the following expression is equal or less than 0.10.

\[
\theta = \frac{P_s \Delta}{V \mu_s C_x}
\]

where,

- \( P_s \): The total unfactored vertical design load at and above level \( x \) (kip or kN);
- \( V \): When calculating the vertical design load for purposes of determining P-Delta, the individual load factors need not exceed 1.0
- \( \Delta_s \): The design story drift (in. or mm) occurring simultaneously with \( V \)
- \( V_s \): The seismic shear force (kip or kN) acting between level \( x \) and \( x-1 \)
- \( \mu_s \): The story height (ft or m) below level \( x \)
- \( C_x \): The deflection amplification factor in Table 16.4

The stability coefficient, \( \theta \), shall not exceed \( \theta_{\text{max}} \) determined as follows:
where:

\[ \beta = \frac{\beta_{\text{req}}}{C_\text{r}} \leq 0.25 \]

\( \beta \) = The ratio of shear demand to shear capacity for the story between level \( x \) and \( x-1 \). Where the ratio \( \beta \) is not calculated, a value of \( \beta = 1.0 \) shall be used.

When the stability coefficient, \( \theta \), is greater than 0.10 but less than or equal to \( \theta_{\text{req}} \), inter-story drifts and element forces shall be computed including \( P-\Delta \) effects. To obtain the story drifts for including the \( P-\Delta \) effect, the design story drift shall be multiplied by 1.0/(1-\( \theta \)).

Where \( \theta \) is greater than \( \theta_{\text{req}} \), the structure is potentially unstable and must be redesigned.

The allowable story drifts are given in Table 16.7.

### 16.3.7 Overturning

Ground motion can result in overturning of a structure. At any story, the increment of overturning moment in the story under consideration would have to be distributed to the various vertical force-resisting elements, in the same proportion as the distribution of the horizontal shear forces to these elements. The overturning moment at level \( x \), \( M_x \) (kip-ft or kN-m), is determined from the following expression:

\[ M_x = \tau \sum_{i=1}^{x} F_i (h_i - h_x) \]  

(16.19)

where,

- \( F_i \) = Portion of the seismic base shear, \( V_i \), induced at level \( i \)
- \( h_i \) and \( h_x \) = Height (ft or m) from the base to the level \( i \) or \( x \)
- \( \tau \) = overturning moment reduction factor

### Table 16.7 Allowable Story Drift, \( \Delta \) (in. or mm)

<table>
<thead>
<tr>
<th>Building</th>
<th>Seismic Use Group</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>Buildings, other than masonry shear wall or masonry wall frame buildings, four stories or less in height with interior walls, partitions, ceilings, and exterior wall systems that have been designed to accommodate the story drifts</td>
<td>0.025( h_x )</td>
</tr>
<tr>
<td>Masonry cantilever shear wall buildings( ^* )</td>
<td>0.015( h_x )</td>
</tr>
<tr>
<td>Other masonry shear wall buildings</td>
<td>0.007( h_x )</td>
</tr>
<tr>
<td>Masonry wall frame buildings</td>
<td>0.017( h_x )</td>
</tr>
<tr>
<td>All other buildings</td>
<td>0.020( h_x )</td>
</tr>
</tbody>
</table>

\( ^* \) There shall be no drift limit for single-story buildings with interior walls, partitions, ceilings, and exterior wall systems that have been designed to accommodate the story drifts.

\( h_x \) is the story height below level \( x \).

\( ^* \) Buildings in which the basic structural system consists of masonry shear walls as vertical elements cantilevered from their base or foundation support which are so constructed that moment transfer between shear walls (coupling) is negligible.
16.4 SIMPLIFIED ANALYSIS PROCEDURE FOR SEISMIC DESIGN OF BUILDINGS

This procedure can be used for structures in seismic use group I, subject to the following limitations, otherwise either the method in sec. 16.2 or this section has to be used.

2. Buildings of any construction other than light framed, not exceeding two stories in height, excluding basement.

The seismic base shear, $V$, can be computed from the following expression.

$$V = \frac{1.2S_{0x}}{R} W$$  \hspace{1cm} (16.20)

where

$S_{0x} =$ Design elastic response acceleration at short periods as determined from Sec. 16.2

$R =$ Response modification factor from Table 16.4

$W =$ The effective seismic weight of the structure, including the total dead load and other loads listed below:

1. In areas used for storage, a minimum of 25% of the reduced floor live load (floor live load in public garages and open parking structures need not be included.)
2. Where an allowance for partition load is included in the floor load design, the actual partition weight of 10 psf of floor area, whichever is greater.
3. Total weight of permanent operating equipment
4. 20% of flat roof snow load where flat snow load exceeds 30 psf (1.44 kN/m$^2$).

The vertical distribution of forces at each level would be computed from the following expression:

$$F_s = \frac{1.2S_{0x}}{R} W_s$$  \hspace{1cm} (16.21)

where

$W_s =$ The portion of the effective seismic weight of the total structure, $W$, at story level $s$.

For structures satisfying this section, the design story drift, $\Delta$, is taken as 1% of the story height unless a more exact analysis is made.
16.5 OTHER ASPECTS IN SEISMIC DESIGN

The discussion presented in the previous sections is intended only to highlight the most important basic considerations for establishing the seismic basic shear force values and their distribution over the height of a structure, at all story levels. The scope of this book does not permit more coverage of other essential topics such as modeling, model forces, deflections and drifts, diaphragms, coupling beams, interconnecting shear walls, connections, irregularity of structures, out-of-plane loading and torsion and foundations.

Through a careful review of the details presented, the numerical examples and solving the assignments, the reader becomes well equipped to handle the design requirement aspects of the topics listed. The International Building Code—IBC 2003’s (Ref. 16.2) detailed provisions give all the additional provisions and guidance needed for safe complete designs of concrete structures that can successfully resist severe earthquakes. The ensuing sections will present ACI 318-05 code provisions for proportioning and detailing of reinforced concrete elements that can withstand seismic loading through conformity with the IBC 2003-2013 requirements.

16.6 FLEXURAL DESIGN OF BEAMS AND COLUMNS

Moment-resisting ductile frames of reinforced concrete structures are designed for strength and ductility. During a strong earthquake, it is anticipated that the critical regions of frame members will develop plastic hinges to dissipate seismically induced energy. In a well-designed frame structure, the energy dissipation occurs in the plastic hinges that form at the ends of the beams, while the columns remain elastic and provide
overall strength and stability to the stories above. This can be achieved if the sequence of
plastification in the structure can be controlled.

In an effort to control the seismic response of structures, the ACI 318-05 building
code calls for "strong columns and weak beams," although it may be difficult to prevent
hinging of the lower-story columns. All the potential hinging regions are required to be
detailed by special confining reinforcement for improved ductility and energy absorption
capacity.

16.6.1 Seismic Shear Forces in Beams and Columns

Shear failure in reinforced concrete members is regarded as brittle failure. Therefore, in
designing earthquake-resistant structures, it is important to provide excess shear capacity
over and above that corresponding to flexural failure. The ACI 318-05 requirements are
based on the strong column-weak beam concept subsequently discussed. Hence plastifi-
cation of the critical regions at the ends of the beams will have to be considered as a pos-
able loading condition.

The shear forces are then computed based on the moment resistances in the de-
veloped plastic hinges, labeled as probable moment resistance $M_{pl}$, developed when the lon-
gitudinal flexural steel enters into the hardening stage. Consequently, the computation of
the probable moment resistance $M_{pl}$ is used as the stress in the longitudinal reinforce-
ment. In order to absorb the energy that can cause plastic hinging, the earthquake-
resistant frame has to be ductile in part through confinement of the longitudinal rein-
fforcement of the columns and the beam-column joints and in part through the provision
of the excess shear capacity previously discussed.

Figure 16.5 shows the deformed geometry of and the moment and shear forces for a
beam subjected to gravity loading and reversible side-way. If the intensity of gravity load
is $w_g$, then ACI 318-05 stipulates

$$ w_c = 1.2D + 1.6E + 1.4F $$

(16.22a)

The IBC (Sec. 1605.2) supplants the following load combinations:

$$ 1.4D $$

$$ 1.2D + 1.6E + 0.5(L, or S or R) $$

$$ 1.2D + 0.6E(L, or S) + (g_fL or 0.8W) $$

$$ 1.2D + 0.3W + f_L + 0.5(L, or S or R) $$

$$ 1.2D + 1.0E + (f_fL or f_S) $$

$$ 0.9D = (1.0E or 1.3W) $$

(16.22b)

\[ \text{Figure 16.5 Seismic moments and shears at beam ends: (a) side-way to the left, (b) side-way to the right.} \]
where,

- $f_1 = 1.0$ for floors in places of public assembly, for live loads in excess of 100 lb per square foot (4.99 kN/m²) and for parking garage live load.
- $f_1 = 0.5$ for other live loads.
- $f_2 = 0.7$ for roof configurations (such as saw tooth) that do not shed snow off the structure.
- $f_2 = 0.2$ for other roof configurations.
- $L = \text{Live load except roof load}$
- $L_e = \text{Roof live load including any live load reduction}$
- $R = \text{Rain load}$
- $S = \text{Snow load}$
- $W = \text{Wind load}$

The seismic shear forces are

\[
V_L = \frac{M_{LE} + M_{LE}^*}{\ell} + \frac{1.2D + 1.6L}{2} \tag{16.23a}
\]

\[
V_R = \frac{N_{RE}^* + M_{RE}^*}{\ell} - \frac{1.2D + 1.6L}{2} \tag{16.23b}
\]

where $\ell = \text{span, } L \text{ and } R \text{ subscripts = left and right ends, and } M_{LE,RE} = \text{probable moment strength at the end of the beam based on steel reinforcement tensile strength of 1.25,}$ and $\sigma = 1.0$.

These instantaneous moments $M_{LE,RE}$ should be computed on the basis of equilibrium of moments at the joint where the beam moments are equal to the probable moments of resistance.

The shear forces in the columns are computed in a similar manner, so the horizontal shear forces (at top and bottom of the columns) are

\[
V_c = \frac{M_{LE} + M_{RE}}{h} \tag{16.24}
\]

except that end moments for columns ($M_{LE}$ and $M_{RE}$) need not be greater than the moments generated by the $M_{LE}$ of beams framing into the beam-column joint. $h = \text{column height, and the subscripts 1 and 2 indicate the top and bottom column end moments, respectively, as seen in Figure 16.6.}$

**16.6.2 Strong Column–Weak Beam Concept**

As previously stated, U.S. seismic codes require that earthquake-induced energy be dissipated by plastic hinging of the beams, rather than the columns. This hypothesis is due to the fact that compression members such as columns have lower ductility than flexuraldominant beams. If columns are not stronger than beams framing into a joint, inelastic action can develop in the column. Furthermore, the consequence of a column failure is far more severe than a local beam failure. Therefore, the ACI 318-05 Code as well as the IRC stipulate "strong columns and weak beams." This is ensured by the following inequality:

\[
\sum M_{LE} \geq \frac{6}{5} \sum M_{RE} \tag{16.25}
\]
where $\Sigma M_{lo} = \text{sum of nominal flexural strength of columns framing into joint, calculated}
\text{for factored axial forces consistent with the direction of forces considered, resulting in lowest flexural strength.}$

$\Sigma M_{ho} = \text{sum of moments at the faces of the joint corresponding to the nominal flexural strengths of the beams framing into that joint. In T-beam con}-\text{struction, where the slab is in tension under moments at the face of the joint, slab reinforcement within the effective slab width has to be assumed to contribute to flexural strength if the slab reinforcement is developed at the critical section for flexure.}$

For a joint subjected to reversible base shear forces, as shown in Fig. 16.6b, Eq. 16.25 becomes

$$\{\phi M_{o}^{*} + \phi M_{s}^{*}\}_{ho} = \frac{6}{3} \{\phi M_{o}^{*} + \phi M_{s}^{*}\}_{lo}$$

(16.26)

where $\phi = 0.90$ for beams, 0.65 for tied columns, and 0.70 for spiral columns. For beam-columns, $\phi = 0.90$ to 0.65.
16.7 SEISMIC DETAILING REQUIREMENTS FOR BEAMS AND COLUMNS

16.7.1 Longitudinal Reinforcement

1. In seismic design, when the factored axial load \( P \) is negligible or significantly less than \( A_f f_{y} / 10 \), the member is considered a flexural member (beam). If \( P > A_f f_{y} / 10 \), the member is considered a beam-column, because it is subjected to both axial and flexural loads as columns and shear walls are.

2. The shortest cross-sectional dimension \( \geq 12 \text{ in.} \ (300 \text{ mm}) \).

3. The limitation on the longitudinal reinforcement ratio in the beam-column element is \( 0.01 \leq \rho \leq A_f / A_x \leq 0.05 \). For practical considerations, an upper limit of 6% is too excessive, because it results in impractical congestion of longitudinal reinforcement. A practical maximum total percentage \( \rho \) of 3.5% to 4.0% should be a reasonable limit.

4. A minimum percentage of longitudinal reinforcement in flexural members (beams) is (a) For sections requiring tensile reinforcement

\[
\rho \geq \frac{3 \sqrt{f_y}}{f_y} \geq \frac{200}{f_y} \quad (16.27a)
\]

(b) For statically determinate T-sections with flanges in tension

\[
\rho \geq \frac{6 \sqrt{f_y}}{f_y} \geq \frac{200}{f_y} \quad (16.27b)
\]
Chapter 16 Seismic Design of Concrete Structures

But under no condition should \( p \) exceed 0.025. The stresses \( f'_c \) and \( f_c \) in these expressions are in psi units. All reinforcement has to be continued through the joint. At least two bars have to be continuously provided both at top and bottom.

5. Main reinforcement should be chosen on the basis of the strong column–weak beam concept of the ACI Code:

\[
\sum M_{un} = \frac{6}{5} \sum M_{uc}
\]

6. The nominal moment strength requirements are:

(a) \( M' \) at joint face \( 2l/3 \) \( M' \) at that face.

(b) Neither the negative nor the positive moment strength at any section along the spans can be less than one-quarter the maximum moment strength provided at the face of either joint. Hence at joint face

\[
M_{n} = \frac{1}{2} M' \tag{16.28a}
\]

At any section,

\[
M_{n} \leq \frac{1}{4} (M_{uc})_{max} \tag{16.28b}
\]

\[
M_{n} \leq \frac{1}{4} (M_{uc})_{max} \tag{16.28c}
\]

7. For coupling beams with aspect ratio \( l/h < 2 \), and with factored shear force \( V' \), exceeding \( 4\sqrt{f_c A_{cm}} \), has to be reinforced with two intersecting groups of diagonally placed bars, symmetrical about the midspan, where \( A_{cm} = \) area of concrete-resisting shear.

16.7.2 Transverse Confining Reinforcement

Transverse reinforcement in the form of closely spaced hoops (ties) or spirals has to be adequately provided. The aim is to produce adequate rotational capacity within the elastic hinges that may develop as a result of the seismic forces.

1. For column spirals, the minimum volumetric ratio of the spiral hoops needed for the concrete core confinement cannot be less than the larger of:

\[
\rho_s = \frac{0.125}{f_{tn}} \tag{16.29a}
\]

or

\[
\rho_s \geq 0.45 \left( \frac{A_s}{A_{cm}} - 1 \right) \frac{f_c}{f_{tn}} \tag{16.29b}
\]

whichever is greater, where

- \( \rho_s = \) ratio of volume of spiral reinforcement to the core volume measured out-to-out
- \( A_s = \) gross area of the column section
- \( A_{cm} = \) core area of section measured to the outside of the transverse reinforcement (in.²)
- \( f_{tn} = \) specified yield strength of transverse reinforcement, psi

2. For column rectangular hoops, the total cross-sectional area within spacing \( s \) cannot be less than the larger of:
\[ A_{x,\text{tot}} \geq 0.09 h b_i \frac{P_e}{f_{y,\text{req}}} \]  
\[ \text{or} \]  
\[ A_{x,\text{tot}} \geq 0.3 h b_i \left( \frac{A_{x,\text{req}}}{A_{x,\text{tot}}} - 1 \right) \frac{P_e}{f_{y,\text{req}}} \]  
whichever is greater, where

\( A_{x,\text{tot}} \) = total cross-sectional area of transverse reinforcement (including cross ties) within spacing \( s \) and perpendicular to dimension \( b_i \)

\( b_i \) = cross-sectional dimension of column core measured c-c of confining reinforcement, in.

\( h_i \) = maximum horizontal spacing of hoops or cross-ties on all faces of the column, in.

\( A_{x,\text{req}} \) = cross-sectional area of structural member, measured out-to-out transverse reinforcement

\( s \) = spacing of transverse reinforcement measured along the longitudinal axis of the member, in.

\( L_e \) = longitudinal spacing of transverse reinforcement within length \( L_e \), in.

\( t_{\text{min}} \) = one-quarter of the smallest cross-sectional dimension of the member, 6 times diameter of longitudinal reinforcement. Also, \( S \), in \( (24 - b_i)/3 \)

where \( s \), in longitudinal spacing of the transverse reinforcement within length \( L_e \), its value should not exceed 6 in. and need not be taken less than 4 in.

Additionally, if the thickness of the concrete outside the confining transverse reinforcement exceeds 4 in., additional transverse reinforcement has to be provided at a spacing not to exceed 12 in. The concrete cover on the additional reinforcement should not exceed 4 in.

3. The confining transverse reinforcement in columns should be placed on both sides of a potential hanger over a distance \( L_e \). The largest of the following three conditions governs the length \( L_e \):

(a) depth of member at joint face
(b) one-sixth of the clear span
(c) 18 in. Increase \( L_e \) by 50% or more in locations of high axial load and flexural demands such as the base of a building.

When transverse reinforcement is not provided throughout the column length, the remainder of the column length has to contain spiral or hoop reinforcement with spacing not exceeding the smaller of 6 times the diameter of the longitudinal bars or 6 in.

4. For beam confinement, the confining transverse reinforcement at beam ends should be placed over a length equal to twice the member depth \( h \) from the face of the joint on either side or of any other location where plastic hinges can develop. The maximum hoop spacing should be the smallest of the following four conditions:

(a) One-fourth effective depth \( d \)
(b) 8 \times diameter of longitudinal bars
(c) 24 \times diameter of the hoop
(d) 12 in. (300 mm)

IBC Sec. 1908.1.19 however, requires that confining reinforcement spacing not exceed 4 in.

Figure 16.7 (Ref. 16.8) summarizes typical detailing requirements for a confined column.
Figure 16.7 Typical detailing of seismically reinforced column: (a) spirally confined; (b) confined with rectangular hoops; (c) cross-sectional detailing of ties, x ≤ 14 in. Consecutive cross ties should have 90° hooks on opposite sides.
5. Reduction in confinement at joints: a 50% reduction in confinement and an increase in the minimum tie spacing to 6 in. are allowed by the ACI Code if a joint is confined on all four faces by adjoining beams with each beam wide enough to cover three-quarters of the adjoining face.

6. The yield strength of reinforcement in seismic zones should not exceed 60,000 psi.

16.8 HORIZONTAL SHEAR IN BEAM-COLUMN CONNECTIONS (JOINTS)

Test of joints and deep beams have shown that shear strength is not as sensitive to joint (shear) reinforcement as for that along the span. On this basis, the ACI Code has a reduced the joint strength as a function of only the compressive strength of the concrete and requires a minimum amount of transverse reinforcement in the joint. The effective area \( A_j \) in Figure 16.8 from the ACI 318 Commentary should in no case be greater than the column cross-sectional area.

The maximum shear strength of the joint should not be taken greater than the force \( V_o \) specified below for normal-weight concrete.

1. Confined on all faces by beams framing into the joint:
\[
V_c \approx 20 \sqrt{f'_c A_j}
\]  
(16.31)

2. Confined on three faces or on two opposite faces:
\[
V_c \approx 15 \sqrt{f'_c A_j}
\]  
(16.31)

![Figure 16.8 Seismic effective area of joint (Ref. 16.1)]
3. All other cases:

\[ V_c \leq 12 \sqrt{f_t} A_t \quad (16.31c) \]

A framing beam is considered to provide confinement to the joint only if at least three-quarters of the joint is covered by the beam.

The value of allowable \( V_c \) should be reduced by 25% if lightweight concrete is used. Also, test data indicate that the value of Eq. 16.31c is unconservative when applied to corner joints. \( A_t \) = effective cross-sectional area within a joint, as in Figure 16.6, in a plane parallel to the plane of reinforcement generating shear at the joint. The ACI Code assumes that the horizontal shear in the joint is determined on the basis that the stress in the flexural tensile steel = 1.25\( f_t \). Figure 16.9 shows the forces acting on a beam-column connection at the joint. These forces are the result of the equilibrium forces shown in the deformed wall of Figure 16.10.

For slab-column connections of two-way slabs without beams, slab shear reinforcement should be based on \( V_s \leq 6 \sqrt{f_t} b \Delta \), provided that \( V_s \leq 3 \sqrt{f_t} b \Delta \), and should exceed at least 4 times the slab thickness from the face of the support.

16.8.1 Development of Reinforcement at the Joint

For bars of sizes Nos. 3 through 11 terminating at an exterior joint with standard 90° hooks in normal concrete, the development length \( \ell_d \) beyond the column face, as required by the ACI 318 Code, should not be less than the largest of the following:

\[ \ell_d \geq f_y d_b / (65 \sqrt{f_t}) \quad (16.32a) \]
\[ \ell_d \geq 8 d_b \quad (16.32b) \]
\[ \ell_d \geq 6 \text{ in.} \quad (16.32c) \]

where \( d_b \) = bar diameter.

The development length provided beyond the column face must be no less than \( \ell_d = 2.5 \ell_0 \) when the depth of concrete cast in one lift beneath the bar is 12 in. or \( \ell_d = 3.5 \ell_0 \) when the depth of concrete cast in one lift beneath the bar exceeds 12 in.

All straight bars terminated at a joint are required to pass through the confined core of the column or shear wall boundary member. Any portion of the straight embedment length not within the confined core should be increased by a factor of 1.6.

Figure 16.9 Reversible forces at beam-column joint connection. (\( V_c = \) horizontal shear at joint).
16.9 Design of Shear Walls

Structural walls in a frame building should be so proportioned that they possess the necessary stiffness needed to reduce the relative interstory distortions caused by seismic-induced motions. Such walls are termed shear walls. Their additional function is to reduce the possibility of damage to the nonstructural elements that most buildings contain.

Buildings stiffened by shear walls are considerably more effective than rigid-frame buildings with regard to damage control, overall safety, and integrity of the structure. This performance is due to the fact that shear walls are considerably stiffer than regular frame elements and thus can respond to or absorb greater lateral forces induced by the earthquake motions, while controlling interstory drift.

16.9.1 Forces and Reinforcement in Shear Walls and Diaphragms

Shear walls, that is, structural walls, with height-to-depth ratio in excess of 2.0 essentially act as vertical cantilever beams. As a result, their strength is determined by flexure rather than by shear.

Reinforcement considerations:

(1) Displacement-based Approach. For walls or piers continuous in cross-section from the base of the structure to the top of the wall and designed to have a single critical section for flexure and axial loads, the compressive zones have to be reinforced with boundary elements with a geometry defined as follows:

\[ c = \frac{f_s}{0.0005b h_s} \]  \hspace{1cm} (16.33a)

but that \( h_s / b \) is taken not less than 0.007. The reinforcement has to extend vertically along the wall a distance not less than the larger of \( t \), or \( M_y / 4V_s \) from the critical section.
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c = distance from the extreme compression fibers to the neutral axis calculated from the factored axial force and nominal moment strength.

h = height of entire wall.

\( \delta \) = design placement.

(6) Stress-based Approach: This alternative design procedure requires that boundary elements have to be provided whenever the extreme fiber compressive stresses exceed 0.20 \( f'_{c} \). The boundary elements have to extend along the vertical boundaries of the entire wall and around the edges of openings. They can be discontinued where the calculated compressive stress is less than 0.15 \( f'_{c} \). The stresses are calculated for factored forces using a linearly elastic model and gross-section properties.

Note that when boundary elements are required, the wall is essentially detailed in a similar manner in both approaches.

Shear Considerations:

If the shear wall is subjected to factored in-plane seismic shear forces \( V_{s} > A_{c} \sqrt{f'_{c}} \), then it should be reinforced with a reinforcement percentage \( \rho_{s} \geq 0.005 \% \). Spacing of the reinforcement each way should not exceed 18 in. c to c. If \( V_{s} < A_{c} \sqrt{f'_{c}} \) the reinforcement percentage can be reduced to 0.0012 for No. 5 bars or less in diameter and 0.0015 for larger deformed bar sizes. Reinforcement provided for shear strength has to be continuous and distributed across the shear plane.

At least two curtains of reinforcement are needed in the wall if the in-plane factored shear forces exceed a value of 2\( A_{c} \sqrt{f'_{c}} \).

\( \rho_{s} = A_{s}/A_{c} \)

\( A_{c} = \) net area of concrete cross section = thickness \times length of section in direction of shear considered

\( A_{s} = \) projection on \( A_{c} \) of area of distributed shear reinforcement crossing the plane of \( A_{s} \).

The nominal shear strength \( V_{n} \) of structural walls and diaphragms of high-rise buildings with aspect ratio greater than 2 should not exceed the shear force calculated from

\[ V_{n} = A_{c} (2 \sqrt{f'_{c}} + \rho_{f} f_{c}) \]

where \( \rho_{f} = \) ratio of distributed shear reinforcement on a plane perpendicular to the plane of \( A_{c} \).

For low-rise walls with aspect ratio \( h'/h \), less than 2, the ACI Code requires that the coefficients 2 in Eq. 16.33b be increased linearly up to a value of 3 when the \( h'/h \), ratio reaches 1.5 in order to account for the higher shear capacity of low-rise walls. In other words,

\[ V_{n} = A_{c} (a_{0} \sqrt{f'_{c}} + \rho_{f} f_{c}) \]

where \( a_{0} = 2 \) when \( h'/h \leq 2 \) and \( a_{0} = 3 \) when \( h'/h = 1.5 \).

\( \phi = 0.6 \) for designing the joint. If nominal shear is less than the shear corresponding to the development of the nominal flexural strength of the member.

The nominal flexural strength is determined considering the most critical factored axial loads including earthquake effects. The maximum allowable nominal unit shear strength in structural walls is \( 8A_{c} \sqrt{f'_{c}} \), where \( A_{c} \) is the total cross-sectional area (in.) previously defined and \( f'_{c} \) is in psi. However, the nominal shear strength of any one of the individual wall piers can be permitted to have a minimum value of 10\( A_{c} \sqrt{f'_{c}} \), where \( A_{c} \) is the cross-sectional area of the individual pier.

Figure 16.10 shows the forces acting on a shear wall (structural wall). Figure 16.11 shows typical failure modes of low- and high-rise structural walls.
16.9.2 Coupling Beams

Coupling beams are structural elements connecting structural walls to provide additional stiffness and energy dissipation. In many cases, geometrical limits result in coupling beams whose depth to clear span ratio is high (Ref. 16.1, 16.2). Hence, they can be controlled by shear and subjected to strength and stiffness deterioration in earthquakes. To reduce the extent of the deterioration, the span to depth ratio $L/d$ is limited to a value of 4.0 except in cases of special moment frames in which the width to depth ratio cannot be less than 0.30. Coupling beams should only be used in locations where damage to them would not impair the vertical load carrying capacity of the structure or the integrity of the non-structural components and their connection to the structure (Ref. 16.1).

If the factored shear force $V_s$ exceeds $4V_p/A_{op}$, two intersecting groups of diagonally placed bars symmetrical about the midpoint have to be used. This requirement can be waived if it can be demonstrated that their stiffness and stiffness loss does not impair the vertical load-carrying capacity of the structure. The nominal shear strength $V_{sn}$ is determined from the following expression:

$$V_{sn} = 2A_{op}(f_{c})_{min} = 10\sqrt{f_p/A_{op}}$$

(16.34)
where $A_{xx}$ is the cross-sectional area of the beam, and $A_{xy}$ is the total area of reinforcement in each group of diagonally reinforced coupling beams. $\alpha$ is the angle between the diagonally placed bars and the longitudinal axis of the coupling beam.

A typical illustration of a diagonally reinforced coupling beam is shown in Figure 16.12. The diagonally placed bars have to be developed in tension within the wall and also considered to contribute to the nominal flexural strength of the coupling beam.

16.10 DESIGN PROCEDURE FOR EARTHQUAKE-RESISTANT STRUCTURES

Figure 16.13 gives a logic flowchart for the following operational steps.

1. Determine the earthquake seismicity region, namely whether it is in a low, moderate, or high seismicity region and the site classification (A, B, C, D, E, and F) from Table 16.1.
Figure 16.13 Flowchart for seismic design of ductile structures.
2. Determine from the maximum considered earthquake ground motion maps, the maximum spectral response $S_x$ for 0.2 sec. and $S_y$ for 1 sec., site class B, Figure 16.3a and b, respectively using the large scale FEMA maps of USGS (Ref. 16.15).

3. Compute for the particular seismic Use Group (Table 16.3), the design spectral response $S_{xw}$ and $S_{yw}$ from Equations 16.2 and 16.3:

$$S_{xw} = \frac{2}{3} S_{xv}$$

Where, $S_{xv} = F_x S_x$

$$S_{yw} = \frac{2}{3} S_{yv}$$

Where, $S_{yv} = F_y S_y$

There are three seismic use groups I, II, and III with groups II and III structures requiring full seismic design considerations.

4. Compute the seismic base shear $V = C_b W$

$$C_b = \frac{S_{xw}}{R/I}$$

But $C_b$ cannot exceed $C_b = S_{xw}/(R/I)$ or less than $C_b = 0.064 S_{xw}$

$R$ = Response modification factor from Table 16.4

$I$ = Occupancy importance factor from Table 16.5

$T$ = Fundamental period of vibration of a structure, Sec. 16.3.1, $T_s = C_T h^{0.85}$

where, $C_T$ = building period coefficient ranging between 0.035 – 0.020 as given in the text.

In cases where moment resisting frames do not exceed twelve stories in height, an approximate period $T_s = 0.1 N$ can be used where $N$ = number of stories.

For structures in seismic design categories E or F and for other structures having a spectral response $S_x > 0.6$, the value of $C_s = 0.55 f(R/I)$

5. Vertically distribute the base shear force $V$ to the floors above the base level:

$$F_i = C_{on} V$$

$$C_{on} = \frac{W_i h_i^2}{\sum_i W_i h_i}$$

6. Horizontally distribute the shear $V_e$

$$V_e = \sum_i F_i$$

Where $F_i$ = the portion of the seismic base shear, $V_e$, introduced at level $i$.

7. Tabulate these forces at all story levels.

8. Evaluate the torsional moments, story drift, the P-Delta effect and the overturning moment to ensure they are within permissible limits.

9. Execute a structural frame analysis to determine all shears and moments in the frame beams, columns, shear walls, diaphragms and/or coupling beams if these are used to connect shear walls.

10. Proprietary members for the ductile moment-resisting frame. In rare cases the designer has the uneconomical and inefficient alternative of choosing a brittle system using a low response modification factor $R$. 

11. Using the strong column–weak beam concept, plastic hinges are assumed to form in the beams.

**Seismic beam shear forces**

\[
V_L = \frac{M_{bc} + M_{bl}}{c} + \frac{1.2D + 1.6L}{2} \\
V_R = \frac{M_{bc} + M_{br}}{c} + \frac{1.2D + 1.6L}{2}
\]

\(c = \text{beam span}, M_p = \text{probable moment of resistance}, \text{and } L, R = \text{left and right}.

**Seismic column shear force**

\[
V_c = \frac{M_{ct} + M_{cr}}{h}
\]

where \(h = \text{column height} \)

\[\sum M_s \geq \frac{6}{5} M_{bc} \]

at joint to ensure hinges form in the beams; hence

\[
(\delta M_c^* + 4\delta M_c^*)_{\text{int}} = \frac{6}{5} (\delta M_c^* + \delta M_c^*)_{\text{int}}
\]

The nominal moment strengths \(M_p\) have to be evaluated and the member proportioned prior to evaluating the seismic beam shear forces.

Beam: flexural design, \(P_x\) insignificant

Column: combined bending and axial load \(P_x\)

Beam–column: \(P_y > A_f f_{10} \)

Shortest cross-sectional dimension \(\geq 12\) in.

12. **Longitudinal reinforcement**

Beam–column or columns

\[0.01 \leq \rho_y = \frac{A_y}{A_x} \leq 0.05\]

For practical considerations, \(\rho_y \leq 0.05\).

Beam (positive reinforcement):

\[\rho_{\text{min}} \geq \frac{200}{f_y} \geq \frac{3\sqrt{f_y}}{f_t} \]

T-Beam (flange in tension):

\[\rho_{\text{min}} \geq \frac{200}{f_t} \geq \frac{6\sqrt{f_y}}{f_t} \]

where \(f_y\) is in psi units. \(\rho\) should never exceed 0.025.

The factor value, 6, is in the numerator instead of 3 because a flange width twice the web width is used.

For proportioning reinforcement in beams, the nominal moment strength requirements are

(a) \(M_{p1}^* \) at face of joint \(\geq 1 \frac{2}{3} M_{bc} \) at that face.

(b) \(M_c^* \) or \(M_f^* \) at any section \(\geq \frac{2}{3} M_{bc} \) at the face.
13. Transverse confining reinforcement

(a) Spirals

\[ p_s \geq \frac{0.12 f_y}{f_y'} \quad \text{or} \quad p_s \geq 0.45 \left( \frac{A_\alpha}{A_{\alpha \text{a}}} - 1 \right) f_y' \]

whichever is greater.

- \( A_\alpha \) = gross area
- \( A_{\alpha \text{a}} \) = core area to outside of spirals
- \( f_y' \) = specified yield strength

(b) Rectangular hoops in columns: Total cross sectional within spacing \( e \):

\[ A_\alpha \geq 0.02 \pi e \frac{f_c}{f_y} \]

\[ \geq 0.3 h (\frac{A_\alpha}{A_{\alpha \text{a}}} - 1) \frac{f_c}{f_y} \]

whichever is greater.

Photo 16.A NC USA Tower, Charlotte, North Carolina, 9000 psi concrete. (Courtesy Portland Cement Association.)
$A_{tr}$ = total cross-sectional area of transverse reinforcement (including cross ties) within spacing $s$ and perpendicular to dimension $h$.

$h_c$ = cross-sectional dimension of column core, in.

$s$ = spacing of transverse hoops

$x_{min}$ = one-quarter of the smallest cross-sectional dimension, should not exceed 6 in. or be less than 4 in., whichever is smallest.

**Placement of confining reinforcement:** Place confining reinforcement on either side of potential hinge over a distance the largest of:

(i) Depth of member at joint face
(ii) One-sixth clear span
(iii) 18 in.

The spacing of the ties in the balance of column height follows normal column tie requirements.

**Confining reinforcement in beam webs:** Should be placed on a length $= 2h$ on both sides of the joint if it is internal; otherwise, maximum hoop spacing, smallest of:

(i) One-quarter effective depth $d$
(ii) 8 x diameter of longitudinal bar
(iii) 24 x diameter of hoop
(iv) 12 in.

IBC requires that the spacing not exceed 4 in. in the plasticity zone.

The ties in the balance of the beam span follow the standard shear web reinforcement requirements. If the joint is confined on all four sides, 50% reduction in confinement and increase in minimum tie spacing to 6 in. in the columns are allowed. No smooth bar reinforcement is allowed in seismic structures.

### 14. Beam-column connections (joints): Normal concrete nominal shear strength $V_{d}$ at a joint:

(a) Confined on all faces: $V_{d} \leq 20 \sqrt{f_{c}}A_{j}$

(b) Confined on three faces or two opposite faces: $V_{d} \leq 15 \sqrt{f_{c}}A_{j}$

(c) All other cases: $V_{d} \leq 12 \sqrt{f_{c}}A_{j}$

where $A_{j}$ is effective area at joint (Figure 16.8). The value of allowable $V_{d}$ should be reduced by 25% for lightweight concrete. Note from Figure 16.9 that the horizontal shear in the joint is determined by assuming a stress $= 1.25f_{c}$ in the tensile reinforcement.

### 15. Development length of reinforcing bars: For bar sizes No. 3 to 11 without hooks, the largest of

- $l_{d} = 2.5l_{w}$ when concrete below bars $\leq 2$ in.
- $l_{d} = 3.5l_{w}$ when concrete below bars $\geq 2$ in.

where for normal-weight concrete

$$l_{w} = \frac{f_{d,w}(0.5\sqrt{f_{c}})}{f_{c}} \geq 6d_{b} \geq 6 \text{ in.}$$

When standard 90° hooks are used, $l_{d} = l_{w}$. Any portion of straight embedment length not within the confined core should be increased by a factor of 1.6.
16. Shear walls: height/depth \( > 2.0 \)
   (i) Minimum \( p_x = 0.0025 \) if \( V_{ew} > A_{ew} \sqrt{f_c} \). At least two curtains of reinforcement needed if in-plane factored shear force \( V_{ew} > 2A_{ew} \sqrt{f_c} \), where \( A_{ew} \) = net area of concrete cross section = thickness \( \times \) length of section in direction of the considered shear.

   (ii) If extreme fiber compressive stresses exceed \( 0.2f_c \), shear walls have to be provided with boundary elements along their vertical boundaries and around the edges of openings.

   (iii) Available \( V_s = A_{es} (2\sqrt{f_c} + p_y) \) for \( h_{ew} \geq 2.0 \). For \( h_{ew} < 2 \), the factor of 2 inside the parentheses varies linearly from 3.0 for \( h_{ew} = 1.5 \) to 2.0 for \( h_{ew} = 2.0 \). \( V_s = 0.5V_s \) where \( \phi = 0.60 \).

   (iv) Maximum allowable nominal unit shear \( V_s = 8A_{es} \sqrt{f_c} \) for total wall; but can be increased to \( V_s = 10A_{es} \sqrt{f_c} \) for an individual pier, where \( A_{es} \) is the cross-sectional area of the individual pier.

16.10.1 SI Seismic Design Expressions

compressive strength \( f_c \geq 20 \) MPa

\[
E_c = \frac{w_1}{1000} + 0.043 \sqrt{f_c} \text{ MPa}
\]

\[
E_s = 200,000 \text{ MPa}
\]

Equation 16.22b: \( w_1 = 1.2D + 1.6L + 1.4E \)

Equation 16.23a: \( V_s = \frac{M_{ew}^c + M_{ew}^p}{t} + \left( \frac{1.2D + 1.6L}{2} \right) \)
Equation 16.23b: \[ V_0 = \frac{M_{pl.} + M_{pl.}}{\ell} - \left( \frac{1.2D + 1.6L}{2} \right) \]

Equation 16.24: \[ V_0 = \frac{M_{pl.} + M_{pl.}}{\ell} \]

Equation 16.26: \[ (6M_{pl.} + 4M_{pl.})_{xx} \geq \frac{6}{5} (6M_{pl.} + 4M_{pl.})_{xx} \]

\( \phi = 0.9 \) for beams and 0.65 or 0.70 for columns.

Equation 16.27:

For positive moment: \[ \rho \geq \frac{\sqrt{f_c}}{2f_c} \geq \frac{1.4}{f_c} \]

where \( f_c, f_t \) are in MPa

For T-beam with negative moment: \[ \rho \geq \frac{\sqrt{f_c}}{2f_c} \geq \frac{1.4}{f_c} \]

Equation 16.28a:

At joint face: \[ M^* \geq \frac{1}{2} M_{pl.} \]

At any section:

\[ M_c^* = \frac{1}{3} (M_{pl.})_{max} \]

\[ M_c^* = \frac{1}{5} (M_{pl.})_{max} \]

Equation 16.29a: \[ \rho_i \geq 0.17 \frac{f_{ct}}{f_{ct}} \]

or Eq. 16.29b: \[ \rho_i \geq 0.45 \left( \frac{A_s}{A_{cs}} - 1 \right) \frac{f_{ct}}{f_{ct}}, \text{ whichever is greater} \]

Equation 16.30a: \[ A_{cs} \geq 1.292h \frac{f_{ct}}{f_{ct}} \]

or Equation 16.30b: \[ A_{cs} \geq 3.3 \left( \frac{A_s}{A_{cs}} - 1 \right) \frac{f_{ct}}{f_{ct}}, \text{ whichever is greater} \]

(same stipulations for max \( s \) as in Section 15.5.2).

Equation 16.31a: \[ V_{cs} \leq 1.74f_{ct} A_{cs} \]

Equation 16.31b: \[ V_{cs} \leq 1.28f_{ct} A_{cs} \]

Equation 16.31c: \[ V_{cs} \leq 1.0f_{ct} A_{cs} \]

Stress in the flexural reinforcement at the joint has to be taken as \( 1.25f_{ct} \) (see Fig. 15.9).

Equation 16.32a:

\[ \epsilon_{th} = \frac{f_{ct} d_s}{5.4 \sqrt{f_{ct}}} \]

for bar sizes No. 10 M through No. 35 M.
16.11 EXAMPLE 16.1: SEISMIC BASE SHEAR AND LATERAL FORCES AND MOMENTS BY THE INTERNATIONAL BUILDING CODE (IBC) APPROACH

A moment resisting, five story building with shear walls idealized as in Figure 16.2. Each floor has a weight \( W_f \) and a height \( h = 9.6 \) ft (2.9 m). Calculate the seismic base shear, \( V \), and the overturning moment, \( M \), at each story level in terms of single floor weight \( W_f \), assuming the idealized mass of each floor is \( W_f \). Consider the structure a building category II in site class B and seismic use group II. Given: Response modification factor \( R = 3.0 \), Occupancy importance factor \( I = 1.25 \). Use the equivalent lateral force method in the solution.

Solution: (a) Spectral response period and base shear

Total building height = \( 5 \times 9.5 = 47.5 \) ft

From the FEMA ground motion maps (Figure 16.3), spectral response accelerations \( \tilde{a}_f = 0.42 \) sec and \( \tilde{a}_L = 0.85 \) sec, with a site B class and 5% damping. Adjusted spectral response accelerations for site class effects from Tables 16.2a and b:

- For \( S_1 = 0.42 \) sec, \( F_1 = 1.0 \), and for \( S_2 = 0.85 \) sec, \( F_2 = 1.0 \).

From Equations 16.2a and b,

\[
S_{B,1} = F_1 S_1 = 1.0 \times 0.42 = 0.42 \\
S_{B,2} = F_2 S_2 = 1.0 \times 0.85 = 0.85
\]

For 5% damping design spectral response acceleration, using Eqs. 16.3a and b, becomes:

\[
S_{B,1} = \frac{2}{3} S_{D,1} = \frac{2}{3} \times 0.42 = 0.28 \\
S_{B,2} = \frac{2}{3} S_{D,2} = \frac{2}{3} \times 0.85 = 0.567
\]

The seismic base shear \( V \) from Eq. 16.8 is \( V = C_s W = C_s (5 W_f) \) for the five stories where \( W_f \) is the idealized weight of each story.

From Table 16.4, the response modification coefficient for an ordinary reinforced concrete moment frame is given as \( R = 3 \).
The occupancy importance factor for building category II from Table 16.5 is \( I = 0.25 \).

From Eq. 16.9, \( C_0 = \frac{S_{100}}{(R/T)} = \frac{0.507}{3/1.25} = 0.336 \).

But \( C_0 \) cannot exceed the value: \( C_0 = \frac{S_{100}}{(R/T)} \) from Eq. 16.10.

For \( S_{100} = 0.278 \) and from Table 16.6, \( C_0 = 1.32 \).

For this moment-resistant frame, the building period coefficient \( C_T = 0.022 \).

From Eq. 18.13, the approximate fundamental period,

\[ T_p = C_T \frac{h^2}{s^2} = 0.022 \times 37.5 \times 37.5 = 0.390 \text{ sec} \]

Maximum allowable \( T_p = C_T \times T_p = 1.32 \times 0.390 = 0.523 \text{ sec} \).

For computing the base shear \( V = T_p = 1.2 \times 0.523 \times 0.6 = 0.328 \).

\[ C_T = \frac{S_{100}}{(R/T)} = \frac{0.278}{3/1.25} = 0.184 \text{ sec} \]

From Eq. 18.11, \( C_T \) cannot be less than

\[ C_T = \frac{0.044 \times S_{100}}{0.044 \times 0.587} = 0.025 \]

Hence, \( C_T = 0.184 \) sec. controls.

Base Shear \( V = C_s W = C_T (5W) = 0.184 \times 5 \times W = 0.92 W \).

(b) Vertical Distribution of Forces and Overturning Moments

From Eqs. 18.13 (a) and (b), the lateral force induced at any story level is:

\[ F_i = C_m V \]

where,

\[ C_m = \frac{W_i h_i^2}{\sum W_i h_i^2} \]

\[ k = \frac{0.63 - 0.50}{2.30 - 0.50} \times 1.0 = 1.07 \text{ (by linear interpolation)} \]

Since \( h \) is constant for all the stories, \( C_m \) becomes \( \frac{W_i}{\sum W_i} \) where \( i = 5 \) at the top floor.

\[ \sum_{i=1}^{5} W_i = 1W_1 + 2W_2 + 3W_3 + 4W_4 + 5W_5 = 15W_1 \]

Lateral force \( F_i = C_m V = 0.92 C_W W_1 \).

Overturning moment from Eq. 16.19 is \( M_r = \sum F_i (h_i - h_5) \)

for the top ten stories, overturning moment reduction factor \( r = 1.0 \).

Hence \( M_r = \sum F_i (h_i - h_5) \).

Computing and tabulating the story forces \( F_i \) and the overturning moment \( M_r \) for all stories in the following table:
### Example 16.1: Seismic Base Shear and Lateral Forces and Moments by the ISCC 2000

<table>
<thead>
<tr>
<th>Floor</th>
<th>$C_i$</th>
<th>Lateral force $F_i = 0.82, W_i C_i$</th>
<th>Story Shear</th>
<th>Story Moment</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td></td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{3}{15}$</td>
<td>$0.2064, W_i$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{2}{15}$</td>
<td>$0.2456, W_i$</td>
<td>0.3064, W_i</td>
<td>0.3064, W_i,0</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{3}{15}$</td>
<td>$0.1860, W_i$</td>
<td>0.3064, W_i</td>
<td>0.8364, W_i,0</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{2}{15}$</td>
<td>$0.1254, W_i$</td>
<td>0.7304, W_i</td>
<td>1.5944, W_i,0</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{3}{15}$</td>
<td>$0.0616, W_i$</td>
<td>0.8356, W_i</td>
<td>2.4528, W_i,0</td>
</tr>
<tr>
<td>Wall base</td>
<td>$C_b = 0$</td>
<td>0</td>
<td>0.9200, W_i</td>
<td>3.3728, W_i,0</td>
</tr>
</tbody>
</table>

Hence seismic base shear $V = 0.010\, W_i$. The moments at each story level are tabulated in column (5).

---

**Photo 16.10** Overpass collapse in 1971 Los Angeles earthquake. (Courtesy Portland Cement Association.)
16.12 EXAMPLE 16.2: DESIGN OF CONFINING REINFORCEMENT FOR BEAM-COLUMN CONNECTIONS

Design the transverse reinforcing of joint A in a ductile moment-resistant frame of a building as shown in Figure 16.14. The structure is located in seismic use group III. Given:

- Working: $W_1 = 25,000$ lb (118 kN)
- $W_2 = 60,000$ lb (267 kN)
- $f' = 4000$ psi (27.6 MPa)
- $f = 60,000$ psi (414 MPa)

All beams are 12 in. x 24 in. (305 mm x 610 mm) with four No. 8 bars (6.6-mm diameter) top and bottom and columns 15 in. x 24 in. (381 mm x 610 mm). Probable moment = 4,400,000 in.-lb.

Solution:

1. Web shear reinforcement along beam span outside the inelastic zone.

   Figure 16.15 shows a schematic of the lines of action of the beam-column joint forces.

\[
\begin{align*}
\delta &= 24.0 - 2.5 = 21.5 \text{ in. (546 mm)} \\
A &= 4 \times 0.79 = 3.16 \text{ in.}^2 (20.0 \text{ mm}^2)
\end{align*}
\]

- Frame elevation

\[
\begin{align*}
M_A &= 35,000 \text{ ft-lb} \\
M_B &= 40,000 \text{ ft-lb}
\end{align*}
\]

- Beam $A$ equilibrium forces:
  
\[
\begin{align*}
V_A &= 17,500 \text{ lb (6)} \\
V_B &= 20,000 \text{ lb (6)}
\end{align*}
\]

- Beam $B$ equilibrium forces:

\[
\begin{align*}
V_A &= 17,500 \text{ lb (6)} \\
V_B &= 20,000 \text{ lb (6)}
\end{align*}
\]

Figure 16.14 Ductile frame joint A design, Ex. 16.2: (a) frame elevation; (b) beam $A$ equilibrium forces.
Figure 16.15 Shear forces at beam-column joint of example 16.2.

\[ V_a = \frac{40}{12} = 3.33 \text{ kips} \]

\[ V_a < V_{pl} = 0.016 \text{ kips} \]

By ACI318 O.K.

\[ M_{pl} = 1.25 A_f f_c \left( \frac{d - 0.5}{2} \right) = M_c \]

\[ f_c = 0.85 f_{c'} \]

\[ f_{c'} = 4000 \times 12 = 48,000 \text{ psi} \]

\[ M_c = 3.16 \times 1.25 \times 60,000 \times \frac{21.5 - \frac{5}{2}}{2} = 4,080,000 \text{ in} \cdot \text{lb} (4900 \text{ kN} \cdot \text{m}) \]

\[ > M_{pl} = 4,000,000 \] O.K.

Hence, 4 No. 8 bars assumed at top and bottom are O.K.

2. Beam Transverse Confining Reinforcement in the Inelastic Zone of Plastic Hinging

From Eq 16.23

\[ V_t = \frac{M_p}{C_b} \left( \frac{1.2D + 2.5L}{2} \right) \times \frac{1}{4} \]

\[ = \frac{4,000,000}{24 \times 12} \left( \frac{1.2 \times 35,000 + 1.6 \times 60,000}{2} \right) \times \frac{1}{0.75} \]

\[ = 30,555 + 92,000 = 122,555 \text{ lb (540 kN)} \]

\[ V_t = 2\sqrt{f_{c'} b_d} = 2\sqrt{4000 \times 12} = 33,634 \text{ lb (149 kN)} \]

Note that if \( V_t \) can be assumed zero under certain ACI318-02 Code Sec. 21.3.2 conditions.

\[ V_{pl} = \frac{12}{12} = 104.257 \text{ lb (464 kN)} \]

\[ V_t = V_t - V_{pl} = 104.257 - 33,634 = 70,623 \text{ lb (319 kN)} \]

Try No. 4 hoop. \( A_t = 2 \times 0.20 = 0.40 \text{ in}^2 \)

\[ \frac{A_t f}{V_{pl}} = \frac{0.40 \times 60,000 \times 21.5}{71,623} = 7.20 \text{ in (183 mm)} \]
Chapter 16  Seismic Design of Concrete Structures

Maximum allowable spacing within a distance 2d = 2 × 24 = 48 in. at d = 21.5 in. 8 in. smallest longitudinal bar = 8 × 10 = 80 in. 24 in. dia of loop = 24 × 0.5 = 12 in.; max. 12 in. IBC requirement (Sec. 1908.1.11) for maximum loop spacing = 4

Use No. 4 confining hoops at 4 in. c-e over this distance. Place the confining hoops in

Use No. 4 closed hoops in 7 in. c-e beyond critical section. Increase to d/2 = 10.75

3. Confining reinforcement in the column at beam-column joints: From Eqs. 16.29(a) and (b)

(a) Joint Shear Strength

Column horizontal shear forces should not exceed those based on the probable

Hence.  \[ V_c = \frac{\text{Probable } M_{cr}}{b/d} = \frac{4,400,000}{60/12} = 30,555 \text{ lb (136 kN)} \]

\[ V_c \text{ at each joint} = A_f f_r = 3.16 \times 60,000 - 30,555 \]

\[ = 159,845 \text{ lb (707 kN)} \]

From Eq. 16.30b, allowable \( V_c \) within column joint ≤ 15√\( V_c \), \( A_d = 15 \times 24 = 360 \text{ in.}^2 (2,322,000 \text{ mm}^2) \)

allowable \( V_c = 15 \times 60,000 \times 360 = 351,526 \text{ lb (1,594 kN)} \)

> actual \( V_c = 159,845 \text{ lb, O.K.} \)

Hence, the confinéd column joint is adequate to resit the seismic shear.

(b) Column Confinement in the Inelastic Zone

Column d = 24 - 2.5 = 21.5 in.

At the \( A_d \) plane:

\[ V_c = 2 \sqrt{6000} \times 15 \times 21.5 = 40,700 \text{ lb} \]

\[ V_c = 159,845 - 40,700 = 119,055 \text{ lb (530 kN)} \]

\[ s = \frac{A_d f_d}{V_c} = \frac{119,055}{40,700} = 4.33 \text{ in. (110 mm)} \]

\[ A_d = 0.066, \frac{f_d}{f_r} \]

or

\[ A_d = 0.3 \ln \left( \frac{A_d}{A_{cr}} - 1 \right) \frac{f_r}{f_c} \]

whichever is greater.

\( b_c = \) column core dimension = 24 - 2(1.5 + 0.5) = 20 in.

Trying \( s = 3 \text{ in.} \):

\[ A_d = 0.09 \times 3.5 \times 20 \times \frac{4000}{60,000} = 0.42 \text{ in.}^2 \]

\[ A_d = 0.3 \times 3.5 \times 209 \times \frac{15 \times 24}{11^2 \times 20} = \frac{4000}{60,000} = 0.89 \text{ in.}^2 \text{ controls} \]
Example 16.2: Design of Confining Reinforcement for Beam-Column Connections

Figurz 16.16 Confining hoop at joint in Ex. 16.2.

max. allowable $\sigma = 0.7$ at joint column dimension or $4\text{ in.} = 0.25 \times 16 = 3.75\text{ in.}$
or: $\leq 6$ dia. longitudinal bar $= 6 \times 1.0 = 6$ in.

or: $\leq 4 + (d - h)/3$ where $b_c$ = maximum horizontal spacing of hoop or encased leg on all sides of the column.

Assuming that spacing $h = \frac{20 - 2}{2} = 8.5\text{ in.}$, $\delta \leq 4 + (14 - 8.5)/3 = 6.8$ in. Hence, controlling hoop spacing = $3.75$ in. a. to c.
16.13 EXAMPLE 16.3: TRANSVERSE REINFORCEMENT IN A BEAM POTENTIAL HINGE REGION

Design the transverse reinforcement for the potential hinge region of the earthquake resistant 20 ft span (6.1 m) beam in the monolithic reinforced concrete frame (Ref. 16.9) shown in Figure 16.17. The beam is subjected to a service live load intensity of 1000 lb per ft (17.3 kN/m) and a service dead load of 1750 lb per ft (25.7 kN/m). Given:

- $f_c' = 4000$ psi (27.6 MPa)
- $f_y = f_r = 60,000$ psi (414 MPa)
- $b = 18$ in. (457 mm)
- $h = 24$ in. (610 mm)
- clear concrete cover = 1.5 in. (38 mm)
- Support section $A_s = 5 \# 4$ bars, $A_s' = 3 \# 9$ bars (38.7 mm dia. bars)

Solution:

- $W = (1.2 \times 570 \times 1.6 \times 1100) = 3660$ lb/ft
- $A_s = 5.0$ in.²

Assume using #3 closed links, diameter = 0.375 in. (9.5 mm)

- $d = 24 - 1.5 - 0.375 - 1.128/2 = 21.6$ in.

Probable negative moment:

$$M = \frac{Wd^2}{8} = \frac{3660 \times 21.6^2}{8} = 118,800 \text{ in.-lb} = 118.8 \text{ kN-m}$$

**Figure 16.17** Monolithic frame beam geometry.
16.14 Example 16.4. Probable Shear Strength of Monolithic Beam-Column Joint

\[
\begin{align*}
J^* &= \frac{2.5A_{ld}f_y}{0.85f_{yP}} = \frac{2.5 \times 4.0 \times 60,000}{0.85 \times 60,000} = 6.13 \text{ in.} (156 \text{ mm}) \\
M_{pc} &= 12.54 f_y \left( \frac{d - \frac{h}{2}}{2} \right) \\
&= 12.5 \times 5.0 \times 60,000 \left( 115 - 6.13/2 \right) = 6,950,000 \text{ in.-lb} \\
&= 579,167 \text{ ft-lb} (865 \text{ kN-m})
\end{align*}
\]

Probable positive moment:

\[
\begin{align*}
\Delta &= 12.5 \times 5.0 \times 60,000 \\
&= 0.85 \times 4,000 \times 18 = 3.65 \text{ in.} (94 \text{ mm}) \\
M_{pc} &= 12.5 \times 5.0 \times 60,000 \left( \frac{26.6 - 6.65}{2} \right) = 4,446,000 \text{ in.-lb} \\
&= 371,839 \text{ ft-lb} (502 \text{ kN-m})
\end{align*}
\]

Probable shear \( V_p \):

\[
\begin{align*}
V_p &= \frac{w_e}{2d} \left[ \frac{M_{pc}^2 + M_{pc}^2}{l_c} \right] \\
&= \frac{3800 \times 20}{2 \times 0.75} \left[ \frac{371,839^2 + 371,839^2}{20} \right] = 61,467 \approx 47,083
\end{align*}
\]

Hence, \( (V_p)_{max} = 98,950 \text{ lb (449 kN)} \)

\[
(V_p)_{min} = 49,470 \text{ lb}
\]

\[
\left( \frac{M_{pc}}{l_c} \right) = \frac{47,483}{(V_p)_{max}} = 1 \text{ in.}
\]

Use, \( V_c = 0 \) in the design, as the probable shear is close to \( V_p \).

Transverse reinforcement in the plasticity length \( l_c \) should be proportioned to resist shear, assuming that \( V_c = 0 \), when the seismic induced shear represents one-half or more of the maximum required shear within that length or the axial compressive force including earthquake effects is less than \( A_{pc} f_y / 20 \).

Hence, \( V_c = 98,950 \text{ lb (449 kN)} \)

Confining hoop spacing \( s \):

\[
\begin{align*}
\frac{A_{ld} f_y}{V_p} &= \frac{3 \times 0.11 \times 60,000 \times 21.6}{90,500} = 4.3 \text{ in. (109 mm)}.
\end{align*}
\]

Hoop spacing the smallest of:

\[
\begin{align*}
\frac{d_1}{s} &= 2.63 \approx 5.4 \text{ in.} \\
\frac{d_2}{s} &= 8 \times 1128 + 9 \text{ in.} \\
\frac{d_3}{s} &= 24 \times 0.375 = 9 \text{ in.}
\end{align*}
\]

or 13 in.

Hence, \( s = 6 \text{ in.} (15 \text{ mm}) \) controls over a zone length \( 2h = 2 \times 24 = 48 \text{ in.} \) from each support face.

\[
\begin{align*}
\text{Hoop confining steel @ 4 in. c/c.}
\end{align*}
\]

16.14 EXAMPLE 16.4: PROBABLE SHEAR STRENGTH OF MONOLITHIC BEAM-COLUMN JOINT

The monolithic column of the structural frame system in Example 16.3 have a clear height of 12"-0" (3.2 m) and the probable moment strength, \( M_{pc} \), of each column is 530,000 ft-lb (707 kN-m). Compute (a) the column probable shear \( V_p \) at the column extremity and (b) verify
that the joint shear strength of the section exceeds the actual probable horizontal stress in flexural tension shear.

Given
Main reinforcement allowable flexural stress is 1.25 $f_y$, assuming:

\[ f_y = 4000 \text{ psi} (27.6 \text{ MPa}) \]
\[ f_y' = 60,000 \text{ psi} (414 \text{ MPa}) \]

Column size 24 in. x 24 in. (610 mm x 610 mm)

Beam $A_c = 5.6 \text{ in.}^2$ and $A_t = 3.0 \text{ in.}^2$ at the support section.

Solution: From example 16.3, the probable beam moments are

\[ M_{cy} = 370,500 \text{ ft-lb} \]
\[ M_{cy}' = 579,167 \text{ ft-lb} \]

Probable moment transferred to the upper and lower columns at the joint is

\[ M_{cy} = \frac{370,500 + 579,167}{2} = 474,834 \text{ ft-lb} \]

This moment corresponds to a probable shear $V_e = \frac{2(474,834)}{2 \times 8} = 79,140 \text{ lb}$. 

Column shear $V_c$ associated with the formation of plastic hinges at the column extremities when the probable moment strengths, $M_{cy}$, are developed at these extremities is

\[ V_c = \frac{2(M_{cy})}{h} = \frac{2 \times 520}{12} = 86,667 \text{ lb} > \text{beam probable shear } V_e = 79,140 \text{ lb}. \]

Hence, use $V_e = 79,140 \text{ lb} (352 \text{ kN})$. See Fig. 16.18 for schematic details of the sense of probable moments and shears (from Ref. 16.16).

Joint probable compressive reinforcement force $C_t$ shown in Figure 16.18 is

\[ C_t = 1.25 A_t f_y = 1.25 \times 5.0 \times 60,000 = 375,000 \text{ lb} (1668 \text{ kN}) \]

Joint probable compressive reinforcement force $C_b$ at bottom fiber in Figure 16.18 is

\[ C_b = 0 \]

Figure 16.18 Joint probable moments and forces at joint faces in example 16.4:
(a) column probable moments at the upper and lower joints, (b) base-column joint sideways to the left.
16.15 Example 16.5: Seismic Shear Wall Design and Detailing

![Photo 16.11 Failure of piers at Hanshin Highway Bridge, Kobe Earthquake, Japan, 1995. (Courtesy Professor Masami Yamaguchi, Kyoto University, Japan.]

\[ C_1 = 1.25 \times 10 \times 60,000 = 225,000 \text{ lb (200 kN)} \]

Joint net horizontal shear \( V_n = T_2 + C_1 = V_n = 375,000 + 225,000 = 594,000 \text{ lb} \)

From Equation 16.21a, the available shear design strength of the joint, confined on all four sides is \( V_d = 20 \sqrt{f_c} A_t \)

\[ A_t = 24 \times 24 \text{ in}^2 \]

Hence, \( V_d = 20 \sqrt{f_c} A_t = 58 \times 32,590 \text{ lb} > \text{actual 594,000 lb, O.K.} \)

Accept the design of the column for seismic capacity.

Note that the framing beams are considered in this frame as having symmetrical reinforcement arrangement. Consequently, the computed joint shear associated with sideways to the right would be the same as that for the sideways to the left.

Table 16.4 in the Appendix gives coefficients for rapid computation of the probable seismic moment strength in terms of reinforcement percentage. Tables A-41 and A-42 respectively give volumetric ratios of spiral bars and areas of reinforcing hoops (Ref. 16.16) for confinement of the longitudinal reinforcement of structural elements in seismic regions.

16.15 EXAMPLE 16.5: SEISMIC SHEAR WALL DESIGN AND DETAILING

Design by the AIC 318 Code the reinforcement for a shear wall in a multistory, ductile frame, 12-story structure (adapted from Ref. 16.9) having a total height \( h = 148 \text{ ft (45 m)} \) and having equal span of 22 ft (6.7 m). Except for the ground story, which is 6 ft (1.8 m) high, all other stories have 12 ft (3.67 m) height. The total gravity factorized on the shear wall is \( W_t = 4,900,000 \text{ lb (21.4 MN)} \). The factored moment at the base of the wall due to seismic loads from the lateral load analysis of the transverse frame is \( M_b = 554 \times 10^6 \text{ in-lb (62.5 MN-m)} \). The maximum axial force on the boundary element is \( P_b = 4,900,000 \text{ lb (20 MN)} \). The horizontal shear force at the base is 585,000 lb (3,400 kN).
Given:
- wall length (horizontally) = 26' - 2" = 26.17 ft = 314 in. (7980 mm)
- thickness = 20 in. = 1.67 ft (508 mm)
- boundary element width = 12 in. (313 mm)
- depth = 20 in. (508 mm)
- $A_v = 39$ No. 11 bars (39 bars of 25-mm diameter) in each boundary element
- $f'_e = 4000$ psi (27.6 MPa), normal weight
- $f_v = f_c = 60,000$ psi (414 MPa)

Use $\phi = 0.60$ as the strength reduction factor for shear in this example.

Solution:
1. **Wall geometry and forces:**
   - $L_v = 22$ ft (6.7 m), $L'_v$ (horizontal dimension) = 26.17 ft
   - $b_{ext} = 20$ in. = 1.67 ft and $b_{int} = 12$ in. = 2.00 ft
   - factored $W_v = 4,000,000$ lb (21.4 MN)
   - $M_v = 554 \times 10^6$ lb-in (82.6 MN-m)
   - $P_v = 4,500,000$ lb (20 MN)
2. Boundary element check: \( f_c = 26.17 \text{ ft}, h = 1.67 \text{ ft}, f_p = 4,500,000 \text{ lbs}, \) and \( M_p = 550 \times 10^6 \text{ in}-\text{ft}. \) Assume that the wall will not be provided with confinement over its entire section.

- \( \text{gross } f_p = \frac{M_p}{h^2} = \frac{550 \times 10^6}{2617} = 210 \text{ ft} \)
- \( A_p = 1.67 \times 26.17 = 43.7 \text{ ft}^2 \)
- \( f_p = \frac{f_p}{A_p} = \frac{210}{43.7} = 4.8 \text{ ft} \)

Concrete compressive stress in the wall is

\[
\sigma_c = \frac{f_p}{f_{cp}} = \frac{210}{2494} = 8.4 \text{ ksi} (59 \text{ MPa})
\]

Maximum allowable \( f_p = 0.2f_{cp} = 0.2 \times 4000 = 800 \text{ psi} (5.52 \text{ MPa}) \) in compression if a boundary element is not required. Hence boundary elements are needed subject to the confinement and loading requirements of Section 16.9.

3. Longitudinal and transverse reinforcement: Check if 2 pairs of reinforcement are needed, that is, if in-plane factored shear \( > 2A_{wr} \sqrt{f_p} \) (Section 16.9.1).

- \( V_r = 885,000 \text{ lb} \)
- \( A_{wr} = \text{area bound by web thickness and length of section in direction of shear force} \)
- \( = 20 \times 314.4 = 6280 \text{ in}^2 \)

\[
2A_{wr} \sqrt{f_p} = 2 \times 6280 \sqrt{210} = 779,400 \text{ lb (533 kN)} < V_r = 885,000 \text{ lb}
\]

Hence two pairs of reinforcement are required.

- \( \min \sigma = \frac{f_p}{f_{cp}} = \frac{210}{2494} \text{ and } \max \sigma = 4.8 \text{ ksi} \)

- \( A_{wr} = \text{ required } \)

- \( A_{wr} = 0.025 \times 240 = 60 \text{ in}^2 \)

- \( A_{wr} = 20 \times 314.4 = 6280 \text{ in}^2 \)

- \( 0.62 \text{ in} \) required \( A_{wr}/12 \)

- \( 0.62/12 = 0.05 \text{ in} \)

- \( = 12.4 \text{ in} (315 \text{ mm}) < 18 \text{ in. limit} \)

Use \( s = 12 \text{ in.} \)

Check for shear reinforcement capacities

A check is needed in order to determine that the No. 5 bars in two columns at 12 in. e-e are adequate for the wall section to sustain the applied shear force at the base. The shear wall aspect ratio is

\[
\frac{A_{wr}}{T} = \frac{148}{26.37} = 5.66 > 2
\]

Hence from Eq. 16.33b

\[
4V_r = \phi A_{wr}(2\sqrt{f_p} + \rho f_p)
\]

where \( \phi = 0.60 \) in this example; otherwise, refer to the ACI 318-05 Code for other conditions.

- \( A_{wr} = 20(26.17 \times 12) = 6280 \text{ in}^2 \)
- \( \rho = \frac{2 \times 314.4}{20 \times 12} = 0.0026 \)
available $\phi V_c = 0.60 \times 6280 \left( 2 \sqrt[3]{4000 \times 0.0020 \times 60.000} \right) \\
= 1,645.000 \text{ lb} > V_c = 885.000 \text{ lb} \\
(47 \text{ MN} > \text{required } 3.9 \text{ MN})$

Hence the wall section is adequate. Therefore, use two curtains of No. 5 bars spaced at 12 in.
each in both horizontal and vertical directions.

4. Boundary element check if acting as a short column under factored vertical forces
due to gravity and factored loads $f_{\text{c}}$.

Compute $w = 4500 \times 1000 \times 1000 = 4,500,000 \text{ lb}$. From before, $h = 30$
in., $h = 50 \text{ in.}$, $A_s = 24 \text{ No. } 5$ bars, $A_s = 24 \times 0.56 = 30.72 \text{ in.}^2$, $133.09 \text{ mm}^2$ in each boundary element.

- $\rho_s = \frac{A_s}{A_w} = \frac{60.84}{32 \times 50} = 0.038$
- $\rho_{\text{con}} = 0.01 < \rho_s < \rho_{\text{con}} = 0.06$, O.K.

The axial load capacity of the boundary element acting as a short column is

$$ f_{\text{P,con}} = 0.80 \left( 0.85 f'_{c} \left( A_s - A_w \right) + A_s f_{c} \right) $$

$$ = 0.80 \times 0.85 \times 4000 \times (1600 - 46.8) + 60.84 \times 60.000 $$

$$ = 4,619.44 \text{ lb} > f_{\text{c}} = 4,500,000 \text{ lb}$$. O.K.

5. Boundary element transverse confining reinforcement $f_s = 20 \text{ in.}$, $b_s = 32 \text{ in.}$, $h_{\text{c}} = 314 \text{ in.}$, and $A_s = 1600 \text{ in.}^2$. From Eqs. 16.306 and 16.

$$ A_s = 0.60 f'_{c} $$

and

$$ A_{m} = 0.50 \left( \frac{A_s}{A_w} - 1 \right) f_{c} $$

Assume No. 5 hoops and hoops spaced at 4 in. c.c.

(a) Short direction

- $b_{h} = 50 - 2 \left( 1.3 + \frac{5}{16} \right) = 46.37 \text{ in.}$
- $h_{w} = 32 - 2 \left( 1.3 + \frac{5}{16} \right) = 28.37 \text{ in.}$
- $A_{h} = 46.33 \times 28.37 = 1314 \text{ in.}^2$ (core area)
- $A_{w} = 0.59 f_{w} f_{c} \left( 0.09 \times 4000 \times 4 \times 4.37 \right. \left. \text{ in.}^2 \right) = 1.11 \text{ in.}^2$
- $A_{m} = 6.0 \times 4 \times 4.37 \left( \frac{1600}{1314} - 1 \right) \text{ in.}^2 = 0.80 \text{ in.}^2$

$b_{w} = 11.1 \text{ in.} (716 \text{ mm})$ governs.

Use three No. 3 crossties, for a total of five legs being provided including the hook every 4 in. along the boundary length (wall length $h'_{w}$), $A_{s3}$ provided = $2 \times 0.31 = 1.55 \text{ in.}^2$, O.K.

(b) Longitudinal direction

- $b_{h} = 28.37 \text{ in.}, A_{h} = 1314 \text{ in.}^2$
- $A_{w} = 0.59 f_{w} f_{c} \left( 0.09 \times 4000 \times 4 \times 4.37 \right) = 0.69 \text{ in.}^2$
- $A_{m} = 0.3 \times 4 \times 4.37 \left( \frac{1600}{1314} - 1 \right) \text{ in.}^2 = 0.49 \text{ in.}^2$
A_s = 0.69 in^2 (445 mm^2) controls. With one No. 5 corner, a total of three legs is provided every 4 in. e-e. 

6. Check for maximum hoop spacing:

\[ s = \frac{1}{2} \times 32 = 8 \text{ in.} \]

\[ s = \text{dia. of longitudinal bar} = \frac{11}{8} \times 8.25 \text{ in.} \]

\[ h_s = \frac{1 - \left( \frac{1}{3} - \frac{1}{4} \right)}{4} + \frac{11}{8} \times \frac{11}{11} = 5.4 \text{ in.} \]

\[ z_s \approx 4 + \left( \frac{14 - h_s}{2} \right) + 4 + \left( \frac{14 - 5.4}{2} \right) = 6.9 \text{ in.} \neq 6.0 \text{ in. within length } \xi_0. \]

7. Development of reinforcement: Development length of No. 5 horizontal bars assuming no hooks are used within the boundary element. From Eqs. 16.32a, b, and c.

\[ t_{\rho} = 60,000 \times 0.625 = 9 \text{ in.} \]

\[ t_{\rho} = \frac{60,000 \times 0.625}{65 \sqrt{f'_c}} = 5 \text{ in.} \]

\[ t_{\rho} = 8 \times 0.625 = 5 \text{ in.} \]

\[ t_{\rho} = 5 \text{ in.} \]

\[ f_{y} = 3.5 \xi_{0} \times 3.5 \times 9 = 32 \text{ in. (815 mm)} \]
If bars are straight as in this example, ensure that development length is provided. If hooks are used, \( f_x \times f_y = \text{9 in.} \). Note that no lap splice should be allowed in the No. 5 horizontal bars.

8. Verify adequacy of shear wall sections at its base under combined axial load and bending in its plane. From before:

\[
\begin{align*}
M_e &= 480,000 \text{ lb FT} \\
M_c &= 554 \times 10^3 \text{ in.-lb} \\
\frac{M_e}{P_e} &= 113 \text{ in.} \\
b_{max} &= 20 \text{ in.} \\
\text{b}_{min} &= 32 \text{ in.} \\
\frac{M_e}{P_e} &= 554 \times 10^3 \times 0.65 = 352 \times 10^3 \text{ in.-lb} \\
e &= 2.17 \times 3.14 \times \text{wall height} = 148 \text{ ft} = 1776 \text{ in.} (45 \text{ m}) \\
P_e &= 0.65 \\
&= 7,364,615 \\
\text{no. of longitudinal bars in wall plane} &= 116 \text{ composed of two (30 No. 11 bars)} \\
&\text{for both boundary elements and two courses of No. 5 bars at 12 in. c-c over} \\
e &= 314 \text{ in.} \\
\text{total } A_v \text{ in the lateral cross section} &= 2 \times 60.84 \times (2 \times 0.5) = 132.8 \text{ in.}^2 \\
A_v &= 2(32 \times 50) + 2(314 - 2 \times 50) = 7480 \text{ in.}^2 (48,300 \text{ mm}^2) \\
p &= \frac{7480}{480,000} = 0.0154 > 0.01 \text{ and } < 0.06 \text{ O.K.} \\
P_e &= 7,364,615 \\
P_e &= \frac{7480 \times 7480}{400 \times 400} = 0.247 \\
\text{Enter the plot in Figure 9.24a with } \frac{P_e}{f_y A_y} = 0.247 \text{ and } p = 0.0154 \\
\text{This gives a value of } \frac{M_e}{P_e} = 0.19 \\
hence, \text{ the available } M_e = 0.19 \times 400 \times 7480 \times 314 = 1783 \times 10^3 \text{ in.-lb} \gg \text{required } M_e = 852 \times 10^3 \text{ in.-lb} \text{ O.K.} \\
\text{Figure 18.19 gives detailing of the shear wall longitudinal and boundary element confining reinforcement.}
\]

SELECTED REFERENCES

16.1. ACI Committee 318, Building Code Requirements for Structural Concrete (ACI 318-05) and Commentary (ACI 318R-05), American Concrete Institute, Farmington Hills, MI, 2005, 444 pp.


PROBLEMS FOR SOLUTION

16.1. A 3 × 18 panel, ductile, moment-resistant category II, the class B frame building has a ground story 15 ft high (4.6 m) and 10 upper stories of equal height of 11 ft 6 in. (3.5 m). Calculate the base shear V and the overturning moment at each story level in terms of the weight Wf of each floor. Use the Equivalent Lateral Force Method in the solution. Given:

\[ S = 0.34 \text{ sec, } S_f = 0.90 \text{ sec, } R = 5 \]

\[ W_{	ext{per floor}} = 2,000,000 \text{ lb (8,960 kN)} \]

16.2. If the building in Example 16.2 is constructed with components having the dimensions and data listed below, design the confining transverse reinforcement for the slabs of exterior beams and columns (three faces confined) joint for the bottom floor. Given:

floors have slabs of thickness \( h_s = 7 \text{ in. (178 mm)} \)

all beams: \( 18 \text{ in. } \times \text{ 24 in. (457 mm } \times 610 \text{ mm)} \)

exterior columns: \( 20 \text{ in. } \times \text{ 20 in. (508 mm } \times 508 \text{ mm)} \)

clear beam spans in both longitudinal and transverse directions = 20' 6" (6.2 m)

shear wall base length \( L_s = 25 \text{ ft (7.6 m)} \)

shear wall height \( h_s = 120 \text{ ft (36.6 m)} \)

Beams: \( A_s = A_{sl} = 4 \text{ No. 9 bars} \)

Columns: \( A_s = 8 \text{ No. 9 bars} \)

\( f_s = 5000 \text{ psi, normal weight (34.5 MPa)} \)

\( f_s = 60,000 \text{ psi (414 MPa)} \)

Sketch the joint reinforcement.

16.3. Design the confining reinforcement for the joint at an interior column in Problem 16.2 and sketch the joint reinforcement.

16.4. Solve Problem 16.3 if the beam spans are 28 ft, factored \( W_f = 3000 \text{ psf (143 kN/m)} \) and factored \( W_{df} = 1200 \text{ psf (57 kN/m)} \).

16.5. Design the shear wall in Problem 16.2 and find the boundary elements for the shear wall, assuming that the magnitude of loads, forces, and moments are 10% of the values used in Ex. 16.5.
TABLES AND NOMOGRAMS

A-1  Selected Conversion Factors to SI Units.
A-2a Geometrical Properties of Reinforcing Bars.
A-2b ASTM Standard Metric Reinforcing Bars.
A-3 Cross-sectional Area of Bars for Various Bar Combinations.
A-4 Area of Bars in a Jit Wide Slab Strip.
A-5 Gross Moment of Inertia of T-sections.
A-6 Cracked Section Moment of Inertia \( I^t \) for Rectangular Sections with Tension Reinforcement Only.
A-7 Effective Moment of Inertia \( I \).
A-8 Stirrup Design Requirements for Nonprestressed Beams with Vertical Stirrups and Normal-Weight Concrete Subjected to Flexure and Shear.
A-9 Nominal Strength Coefficients for Design of Rectangular Beams with Tension Reinforcement Only, \( f_c = 6000 \) psi.
A-10 Nominal Strength Coefficients for Rectangular Beams with Compressive Reinforcement in which \( f_t = f_c \) and for Flanged Sections with \( h_f = 6000 \) psi.
A-11 Nominal Moment Strength \( M_p \) for Compression Reinforcement in which \( f_p = f_c \).

Photo A.1  City Spires, one of the tallest concrete buildings in New York City. (Courtesy The Concrete Industry Board Inc., New York.)
Appendix A Tables and Nomograms

A-12 Nominal Strength Coefficients for Rectangular Beams with Compression Reinforcement in which $f'_c = f_c$ and for Flanged Sections with $h_c < d_c$: $f'_c = 5000$ and 6000 psi.

A-13 Coefficient $K_k$ for use in Comparing $A_s$ for a Flanged Section with $h_c < d_c$.

A-14 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 4000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-15 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 4000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-16 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 4000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-17 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 4000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-18 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 5000$ psi, $f_c = 60,000$ psi, $\gamma = 0.8$

A-19 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 6000$ psi, $f_c = 60,000$ psi, $\gamma = 0.9$

A-20 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 6000$ psi, $f_c = 60,000$ psi, $\gamma = 0.9$

A-21 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 9000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-22 Rectangular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 9000$ psi, $f_c = 75,000$ psi, $\gamma = 0.8$

A-23 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 4000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-24 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 5000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-25 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 5000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-26 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 6000$ psi, $f_c = 60,000$ psi, $\gamma = 0.7$

A-27 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 9000$ psi, $f_c = 75,000$ psi, $\gamma = 0.8$

A-28 Circular Column Nominal Load-Moment Strength Interaction Diagram: $f'_c = 9000$ psi, $f_c = 75,000$ psi, $\gamma = 0.8$


A-30 Moment Distribution Factors for Slabs with Drop Panels, $h_1 = 1.25h_s$.

A-31 Moment Distribution Factors for Slabs with Drop Panels, $h_1 = 1.50h_s$.

A-32 Stiffness and Carryover Factors for Columns.

A-33 Value of Torison Constant C for use in Equation for Torision Stiffness $K_t$.

A-34 Coefficient k for Determination of Torisonal Stiffness $K_t$.

A-35 Flexural Stiffness of Equivalent Column $K_c$, for Use in Coquating $\alpha_{et}$.

A-36 Nominal Strength $M'_c$ Chart for Slab Sections 12 in. wide; $f_c = 4000$ psi.

A-37 Nominal Strength $M'_c$ Table for Slab Sections 12 in. wide; $f_c = 4000$ psi.

A-38 Nominal Strength $M'_c$ Chart for Slab Sections 12 in. wide; $f_c = 6000$ psi.

A-39 Nominal Strength $M'_c$ Table for Slab Sections 12 in. wide; $f_c = 6000$ psi.

A-40 Probable Seismic Moment Strengths for Beams.

A-41 Volumetric Ratio of Spiral Reinforcement $e_1$ for Concrete Confinement.

A-42 Area Percentage of Reclination Hoop Reinforcement $e_1$ for Concrete Confinement.

A-43 Recommended Minimum Floor Loads.

A-44 Dead Weights of Floors, Ceilings, Roofs and Walls.

(Figures A-6 through A-42 are extracted from ACT SP-17, 1997, with permission)
## To convert from

<table>
<thead>
<tr>
<th>Unit</th>
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<td>millimeter (mm)</td>
</tr>
<tr>
<td><strong>Foot</strong></td>
<td>foot</td>
<td>meter (m)</td>
</tr>
<tr>
<td><strong>Yard</strong></td>
<td>yard</td>
<td>meter (m)</td>
</tr>
<tr>
<td><strong>Mile (statute)</strong></td>
<td>mile</td>
<td>kilometer (km)</td>
</tr>
<tr>
<td><strong>Square inch</strong></td>
<td>square inch</td>
<td>square centimeter (cm²)</td>
</tr>
<tr>
<td><strong>Square foot</strong></td>
<td>square foot</td>
<td>square meter (m²)</td>
</tr>
<tr>
<td><strong>Square yard</strong></td>
<td>square yard</td>
<td>square meter (m²)</td>
</tr>
<tr>
<td><strong>Ounce</strong></td>
<td>ounce</td>
<td>square centimeter (cm²)</td>
</tr>
<tr>
<td><strong>Gallon</strong></td>
<td>gallon</td>
<td>cubic meter (m³)</td>
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<tr>
<td><strong>Cubic inch</strong></td>
<td>cubic inch</td>
<td>cubic centimeter (cm³)</td>
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<td><strong>Cubic foot</strong></td>
<td>cubic foot</td>
<td>cubic centimeter (cm³)</td>
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<td>cubic yard</td>
<td>cubic meter (m³)</td>
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<td>kilogram-force</td>
<td>newton (N)</td>
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<tr>
<td><strong>Kip-force</strong></td>
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<td>newton (N)</td>
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### Pressure or Stress (Force per Area)

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</thead>
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<td>kilogram-force/square meter</td>
<td>pascal (Pa)</td>
</tr>
<tr>
<td><strong>Kilogram-force/square inch</strong></td>
<td>kilogram-force/square inch</td>
<td>pascal (Pa)</td>
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<tr>
<td><strong>Newton/meter squared</strong></td>
<td>newton/meter squared</td>
<td>pascal (Pa)</td>
</tr>
<tr>
<td><strong>Pound-force/square inch</strong></td>
<td>pound-force/square inch</td>
<td>pascal (Pa)</td>
</tr>
<tr>
<td><strong>Pound-force/square inch (psi)</strong></td>
<td>pound-force/square inch (psi)</td>
<td>pascal (Pa)</td>
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</table>

### Bending Moment or Torque

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<td><strong>Inch-pound</strong></td>
<td>inch-pound</td>
<td>newton-meter (Nm)</td>
</tr>
<tr>
<td><strong>Foot-pound</strong></td>
<td>foot-pound</td>
<td>newton-meter (Nm)</td>
</tr>
<tr>
<td><strong>Meter-kilogram-force</strong></td>
<td>meter-kilogram-force</td>
<td>newton-meter (Nm)</td>
</tr>
</tbody>
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### Mass

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</tr>
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<td><strong>Gram</strong></td>
<td>gram</td>
<td>kilogram (kg)</td>
</tr>
<tr>
<td><strong>Pound-mass (avoirdupois)</strong></td>
<td>pound-mass (avoirdupois)</td>
<td>kilogram (kg)</td>
</tr>
<tr>
<td><strong>Ton (metric)</strong></td>
<td>ton</td>
<td>megagram (Mg)</td>
</tr>
<tr>
<td><strong>Ton (short 2000 lb)</strong></td>
<td>ton (short 2000 lb)</td>
<td>megagram (Mg)</td>
</tr>
<tr>
<td><strong>Mass per Volume</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Kilogram/meter³</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Kilogram/cubic foot</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Kilogram/cubic yard</strong></td>
<td></td>
<td></td>
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### Temperature

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<td><strong>deg Fahrenheit</strong></td>
<td>deg Fahrenheit</td>
<td>deg Celsius (°C)</td>
</tr>
<tr>
<td><strong>deg Celsius</strong></td>
<td>deg Celsius</td>
<td>deg Fahrenheit (°F)</td>
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</tbody>
</table>

*E = English Unit

---

**Figure A.1** Selected conversion factors to SI units.
### Table A.2a: Geometrical properties of reinforcing bars.

<table>
<thead>
<tr>
<th>Bar Size</th>
<th>Nominal cross section area (sq. in)</th>
<th>Weight (lbf per ft)</th>
<th>Nominal diameter (in)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#3</td>
<td>0.11</td>
<td>0.378</td>
<td>0.375</td>
</tr>
<tr>
<td>#4</td>
<td>0.20</td>
<td>0.666</td>
<td>0.500</td>
</tr>
<tr>
<td>#5</td>
<td>0.31</td>
<td>1.043</td>
<td>0.625</td>
</tr>
<tr>
<td>#6</td>
<td>0.44</td>
<td>1.502</td>
<td>0.750</td>
</tr>
<tr>
<td>#7</td>
<td>0.60</td>
<td>2.044</td>
<td>0.875</td>
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<tr>
<td>#8</td>
<td>0.79</td>
<td>2.670</td>
<td>1.000</td>
</tr>
<tr>
<td>#9</td>
<td>1.00</td>
<td>3.400</td>
<td>1.128</td>
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<tr>
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<td>#14</td>
<td>2.25</td>
<td>7.850</td>
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<tr>
<td>#18</td>
<td>4.00</td>
<td>13.600</td>
<td>2.267</td>
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</table>

### Table A.2b: Nominal Dimensions of reinforcing bars.

<table>
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<tr>
<th>Bar Size Designation</th>
<th>Nominal Dimensions</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Mass (Kg/m)</td>
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<tr>
<td>10 M</td>
<td>0.725</td>
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<tr>
<td>15 M</td>
<td>1.576</td>
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<tr>
<td>20 M</td>
<td>2.355</td>
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<tr>
<td>25 M</td>
<td>3.925</td>
</tr>
<tr>
<td>30 M</td>
<td>5.495</td>
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<tr>
<td>35 M</td>
<td>7.850</td>
</tr>
<tr>
<td>45 M</td>
<td>11.775</td>
</tr>
<tr>
<td>55 M</td>
<td>19.525</td>
</tr>
</tbody>
</table>

Note: ASTM A615M Grade 300 is limited to size No. 10 M through 20 M; otherwise grades 400 or 500 MPa for all the sizes. Check availability with local suppliers for Nos. 45 M and 55 M.
<table>
<thead>
<tr>
<th>#4</th>
<th>#5</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Areas, $A_3$ (or $A_4$), in sq. in.

Columns headed with contain data for bars of one size in groups of one to ten.

Columns headed with contain data for bars of two sizes with from one to five of each size.

Figure A.3 Cross-sectional area of bars for various bar combinations.
### Figure A.4 Area of basc in a 1-ft-wide slab span.

<table>
<thead>
<tr>
<th>Spacing, in</th>
<th>#1</th>
<th>#3</th>
<th>#4</th>
<th>#5</th>
<th>#6</th>
<th>#7</th>
<th>#8</th>
<th>#9</th>
</tr>
</thead>
<tbody>
<tr>
<td>4’-0”</td>
<td>0.75</td>
<td>0.49</td>
<td>0.47</td>
<td>0.46</td>
<td>0.46</td>
<td>0.46</td>
<td>0.46</td>
<td>0.46</td>
</tr>
<tr>
<td>5’-0”</td>
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Note: The table above shows the area of basc in a 1-ft-wide slab span for different spacings. The values are given in inches.
Example: For the T-beam shown, find the moment of inertia $I_e$:

$$
I_e = \frac{1}{12} K_a \left( \frac{b_w}{2} \right)^3
$$

$$
K_a = 1 + (\alpha_a - 1)b_w^2 = \frac{3(1 - r_1^2)(b)(b_a - 1)}{1 - b_a/(\alpha_a - 1)}
$$

$a_u = b/b_w = 143/15 = 9.53$

$b_u = b/b_a = 8/36 = 0.22$

Interpolating between the curves for $\alpha_a = 0.2$ and 0.3, read $K_a = 2.28$

$$
I_e = \frac{K_a b_w^3}{12} = 2.28 \frac{1596^2}{12} = 133,000 \text{ in}^4
$$

Figure A.5  Gross moment of inertia of T-sections.
Figure A.6  Cracked section moment of inertia $I_c$ for rectangular sections with tension reinforcement only.

\[ K_c = \frac{665}{3} + \sqrt{(1 - Q/60) + (Q/24)} \]
Figure A.7 Effective moment of inertia $I_j$

Note: For $M_j$
For $I_j$

$I_j = \frac{3}{16}b^2 h^2$ or $K_d(h^3b^3)^{1/12}$
Figure A.8 Stirrup design requirements for nonprestressed beams with vertical stirrups and normal-weight concrete subjected to flexure shear (Ref. 93).
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*Values of $f$ above high rate are less than $f_{m,a} = \sqrt{f_{m,a} \cdot 1000};$

Figure A.9 Nominal strength coefficients for design of rectangular beams with tension reinforcement only. $f_m = 6000$ psi (Ref. 9.8)
\[ M_e = \frac{N_e}{f_c} \times A_e + \frac{10000M_{pl}}{f_yd^2d^2}, \text{ in.k}^3 \]

Note: To take into account the effect of the displaced concrete, multiply the value of \( A_e \) obtained from the graph by \( f_yA_f - 0.85f_y \).

Figure A.11  Nominal moment strength \( M_e \) for compression reinforcement in which \( f_y = \frac{f_y}{f_y} \) (Ref. 9.8).
For a rectangular beam with
App. A. Tables and Nomograms

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

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where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

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where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

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where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.

For a rectangular beam with

\[ A_x = \frac{A_0}{L_y} \]

\[ A_0 = \frac{I}{y} \]

\[ y = \frac{I}{A_0} \]

where \( I \) is the moment of inertia of the cross-section.
where $K_r = \frac{0.858}{D_e}$ (Ref. 9.6)

and $a = \frac{f}{12,330} \left( 1 - \frac{h}{2d} \right)$

($b$ and $a$ from FLEXURE Eqs. 3.1 and 3.3)

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Figure A.13: Coefficient $K_r$ for use in computing $A_p$ for a flanged section with $h < a$ (Ref. 9.8).
Figure A.14 Rectangular column nominal load-moment strength interaction diagram: $f_y' = 4000$ psi, $f_{y} = 60,000$ psi, $\gamma = 0.6$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.15 Rectangular column nominal load-moment strength interaction diagram: $f_c' = 4000$ psi, $f_y = 60,000$ psi, $\gamma = 0.7$ (ACI-SP17 and Peto, 9.8.3.10, 9.11).
Figure A.10 Rectangular column nominal load-moment strength interaction diagram: 
\( f' = 4000 \text{ psi}, f'' = 60,000 \text{ psi}, \gamma = 3.9 \) (ACI-SP17 and Pits, 9.8, 9.10, 9.11).
Figure A.17  Rectangular column nominal load-moment strength interaction diagram: $f_p = 6000$ psi, $f_y = 60$ ksi, $\gamma = 0.8$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.18 Rectangular column nominal load-moment strength interaction diagram:
$\gamma' = 5000$ psi, $f_c' = 60,000$ psi, $\gamma = 0.9$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.19 Rectangular column nominal load-moment strength interaction diagram:
$\phi = 6000 \text{ psi}, \phi_y = 60,000 \text{ psi}, \gamma = 0.7$ (ACI-SP17 and Refs. 9.6, 9.10, 9.11).
Figure A.20  Rectangular column nominal load-moment strength interaction diagram:

\[ f' = 6000 \text{ psi}, \quad f = 60,000 \text{ psi}, \quad \gamma = 0.9 \text{ (ACI SP17 and Refs. 3.8, 8.12, 5.11).} \]
Figure A.21 Rectangular column nominal load-moment strength interaction diagram: $f'_y = 9000$ psi, $f'_c = 60,000$ psi, $\gamma = 0.7$ (ACI-SP17 and Refs. 98, 9.10, 9.11).
Figure A.22 Rectangular column nominal load-moment strength interaction diagram:
$f_c = 9000$ psi, $f_y = 75,000$ psi, $\gamma = 0.8$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.23  Circular column nominal load-moment strength interaction diagram:
$f' = 4000$ psi, $f = 60,000$ psi, $\gamma = 0.7$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.24  Circular column nominal load-moment strength interaction diagram: $f'_c = 4000$ psi, $f_y = 60,000$ psi, $u = 0.7$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.35  Circular column nominal load-moment strength interaction diagram:
\( f_t = 6000 \text{ psi}, f_{c1} = 60,000 \text{ psi}, \gamma = 0.8 \) (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.26  Circular column nominal load-moment strength interaction diagram: $f_y = 6000$ psi, $f'_y = 60,000$ psi, $\gamma = 0.5$ (ACI-SP17 and Refs. 9.8, 9.10, 9.11).
Figure A.27  Circular column drift-vital load-moment strength interaction diagram: 
\[ f_c = 5000 \text{ psi}; f_y = 70,000 \text{ psi} ; \gamma = 0.8 \text{ (ACI-SP-17 and Refs. 9.8, 5.10, 5.11).} \]
Figure A-28. Circular column nominal load-moment strength interaction diagram:
$C_f = 5000$ psi, $f_y = 75,000$ psi, $\gamma = 0.6$ (ACI-SP17) and Refs. 3.8, 8.10, 9.11.
\[
\text{FEM (uniform load w) = } M \text{ at } E = F \text{ at } E^1 \\
\text{Carryover factor = } COF
\]

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$x = (1 - c_f/f_c)$

1.000, 0.906, 0.729, 0.503, 0.421, 0.343

$c_f$ and $c_e$ are the widths of the columns measured parallel to $f_e$ and $f_c$.

Figure A.29: Moment distribution factors for slabs without drop panels. (From S. H. Simmonds and M. Janis, "Design Factors for Equivalent Frame Method," Journal of the American Concrete Institute, Vol. 68, No. 11, November 1971. Figures A.30-A.32 are from this same source.)
FEM (uniform tool w) = M whore x \( K \) (uniform) = KEGA725x,
Carryover factor = COF

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FIGURE A.30  Moment distribution factor for slides with drop panels, \( h = 1.25h \).
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</table>

* b: Slab thickness; *b1*: load thickness in drop panel.

**Figure A.31** Moment distribution factors for slabs with drop panels; *h* = 1.58.
\[
K = \sum an^3 \text{ in}.
\]

where
\[
a = \frac{b}{(b + c)(b + d)}
\]

and \(a\) and \(b\) are in inches and are measured transverse to the direction of the span for which moments are being determined.

Figure A.34 Coefficient \(a\) for the determination of torsional stiffness \(K\) (Ref. 9.8).
Figure A.35  Flexural stiffness of equivalent column $K_{eq}$ (for structures in which $E_{eq} = E_{oo} = E_{el}$) for use in computing $e_{eq} = \frac{K_{eq}}{K_{el}/E}$ (Ref. 9.8).
Figure A.36 Nominal strength $M_n$ chart for slab sections 12 in. wide, $f'_c = 4000$ psi. (Ref. 9.8).
### Table A.37 Nominal Strength M\textsubscript{c} Table for Slab Sections 12 in. Wide, f\textsubscript{c} = 4000 psi (Ref. 9.8)

<table>
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<th>M\textsubscript{c} (Nominal Moment, k-in)</th>
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<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
<th>i</th>
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**Notes:**
- M\textsubscript{c} = 3.88A\textsubscript{c}f\textsubscript{c} \text{ k-in}
- A\textsubscript{cn} = 0.72 f\textsubscript{c} (1.85) \text{ in.}^2
- A\textsubscript{ct} = 0.75 (1.85) \text{ (in.}^2\text{)} (4)
- A\textsubscript{cr} = A\textsubscript{ct} + A\textsubscript{cn}
- Strain gage & Temperature instrumentation = 0.001 g/in
- M\textsubscript{p} = \frac{M}{M\textsubscript{c}}
Figure A.38  Nominal strength $M_y$ chart for slab sections 12 in. wide; $f = 3000$ psi (Ref. # R).
Appendix A  Tables and Nomograms

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</table>

Figure A.39  Nominal strength M, table for slab sections 12 in. wide; f_y = 5000 psi (Ref. 9.8).
<table>
<thead>
<tr>
<th>Column 1</th>
<th>Column 2</th>
<th>Column 3</th>
<th>Column 4</th>
<th>Column 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data 1</td>
<td>Data 2</td>
<td>Data 3</td>
<td>Data 4</td>
<td>Data 5</td>
</tr>
<tr>
<td>Data 6</td>
<td>Data 7</td>
<td>Data 8</td>
<td>Data 9</td>
<td>Data 10</td>
</tr>
<tr>
<td>Data 11</td>
<td>Data 12</td>
<td>Data 13</td>
<td>Data 14</td>
<td>Data 15</td>
</tr>
</tbody>
</table>

Note: The table contains sample data for illustrative purposes.
### Appendix A: Tables and Nomographs

**Figure A.41** Volumetric ratio of spiral reinforcement $\rho_v$ for concrete confinement (Ref. 16.16).

\[
p_v = \frac{A_s}{V_0} = 0.5 \left( \frac{A_s}{A_{c0}} - 1 \right)
\]

and $\approx 0.09 \frac{f_y}{f'_c}$

<table>
<thead>
<tr>
<th>$A_s/A_{c0}$</th>
<th>$f_c = 40,000$ psi</th>
<th>$f'_c = 60,000$ psi</th>
<th>$f_c = 8000$ psi</th>
<th>$f'_c = 40,000$ psi</th>
<th>$f_c = 60,000$ psi</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_s$</td>
<td>$f_c = 4000$ psi</td>
<td>$f'_c = 4000$ psi</td>
<td>$f_c = 4000$ psi</td>
<td>$f'_c = 6000$ psi</td>
<td>$f_c = 8000$ psi</td>
</tr>
<tr>
<td>1.1</td>
<td>0.012</td>
<td>0.018</td>
<td>0.034</td>
<td>0.006</td>
<td>0.012</td>
</tr>
<tr>
<td>1.3</td>
<td>0.014</td>
<td>0.023</td>
<td>0.027</td>
<td>0.009</td>
<td>0.014</td>
</tr>
<tr>
<td>1.4</td>
<td>0.018</td>
<td>0.032</td>
<td>0.036</td>
<td>0.012</td>
<td>0.018</td>
</tr>
<tr>
<td>1.5</td>
<td>0.022</td>
<td>0.044</td>
<td>0.045</td>
<td>0.015</td>
<td>0.023</td>
</tr>
<tr>
<td>1.6</td>
<td>0.027</td>
<td>0.061</td>
<td>0.054</td>
<td>0.018</td>
<td>0.037</td>
</tr>
<tr>
<td>1.7</td>
<td>0.032</td>
<td>0.074</td>
<td>0.063</td>
<td>0.021</td>
<td>0.040</td>
</tr>
<tr>
<td>1.8</td>
<td>0.036</td>
<td>0.084</td>
<td>0.075</td>
<td>0.024</td>
<td>0.048</td>
</tr>
<tr>
<td>1.9</td>
<td>0.041</td>
<td>0.091</td>
<td>0.088</td>
<td>0.027</td>
<td>0.054</td>
</tr>
<tr>
<td>2.0</td>
<td>0.045</td>
<td>0.099</td>
<td>0.090</td>
<td>0.030</td>
<td>0.060</td>
</tr>
<tr>
<td>2.1</td>
<td>0.050</td>
<td>0.104</td>
<td>0.099</td>
<td>0.033</td>
<td>0.066</td>
</tr>
<tr>
<td>2.2</td>
<td>0.054</td>
<td>0.109</td>
<td>0.108</td>
<td>0.036</td>
<td>0.069</td>
</tr>
<tr>
<td>2.3</td>
<td>0.058</td>
<td>0.112</td>
<td>0.117</td>
<td>0.039</td>
<td>0.072</td>
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<tr>
<td>2.4</td>
<td>0.063</td>
<td>0.116</td>
<td>0.126</td>
<td>0.042</td>
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<td>2.5</td>
<td>0.067</td>
<td>0.121</td>
<td>0.135</td>
<td>0.045</td>
<td>0.084</td>
</tr>
</tbody>
</table>

\[
p_v = \frac{A_s}{V_0} = \frac{A_s}{A_{c0}} - 1
\]

and $\approx 0.09 \frac{f_y}{f'_c}$

<table>
<thead>
<tr>
<th>$A_s/A_{c0}$</th>
<th>$f_c = 40,000$ psi</th>
<th>$f'_c = 60,000$ psi</th>
<th>$f_c = 8000$ psi</th>
<th>$f'_c = 40,000$ psi</th>
<th>$f_c = 60,000$ psi</th>
</tr>
</thead>
<tbody>
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<td>$f_c = 4000$ psi</td>
<td>$f'_c = 4000$ psi</td>
<td>$f_c = 4000$ psi</td>
<td>$f'_c = 6000$ psi</td>
<td>$f_c = 8000$ psi</td>
</tr>
<tr>
<td>1.1</td>
<td>0.009</td>
<td>0.014</td>
<td>0.015</td>
<td>0.006</td>
<td>0.009</td>
</tr>
<tr>
<td>1.2</td>
<td>0.009</td>
<td>0.014</td>
<td>0.018</td>
<td>0.006</td>
<td>0.009</td>
</tr>
<tr>
<td>1.3</td>
<td>0.010</td>
<td>0.014</td>
<td>0.018</td>
<td>0.006</td>
<td>0.009</td>
</tr>
<tr>
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<td>0.024</td>
<td>0.008</td>
<td>0.012</td>
</tr>
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<td>0.023</td>
<td>0.026</td>
<td>0.015</td>
<td>0.020</td>
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<tr>
<td>1.6</td>
<td>0.018</td>
<td>0.027</td>
<td>0.032</td>
<td>0.018</td>
<td>0.026</td>
</tr>
<tr>
<td>1.7</td>
<td>0.021</td>
<td>0.032</td>
<td>0.042</td>
<td>0.021</td>
<td>0.027</td>
</tr>
<tr>
<td>1.8</td>
<td>0.024</td>
<td>0.036</td>
<td>0.054</td>
<td>0.016</td>
<td>0.024</td>
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<tr>
<td>1.9</td>
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<td>0.042</td>
<td>0.060</td>
<td>0.019</td>
<td>0.027</td>
</tr>
<tr>
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<td>0.045</td>
<td>0.066</td>
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<td>0.030</td>
</tr>
<tr>
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<td>0.050</td>
<td>0.072</td>
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<td>0.035</td>
</tr>
<tr>
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<td>0.054</td>
<td>0.077</td>
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<td>0.039</td>
</tr>
<tr>
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<td>0.058</td>
<td>0.082</td>
<td>0.026</td>
<td>0.043</td>
</tr>
<tr>
<td>2.4</td>
<td>0.042</td>
<td>0.063</td>
<td>0.084</td>
<td>0.028</td>
<td>0.047</td>
</tr>
<tr>
<td>2.5</td>
<td>0.045</td>
<td>0.067</td>
<td>0.089</td>
<td>0.030</td>
<td>0.050</td>
</tr>
</tbody>
</table>

**Figure A.42** Area percentage of rectangular hoop reinforcement $\rho_v$ for concrete confinement (Ref. 16.16).
<table>
<thead>
<tr>
<th>Uniformly Distributed Loads</th>
<th>Uniformly Distributed Loads</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Occupancy or Use</strong></td>
<td><strong>Live Load (psf)</strong></td>
</tr>
<tr>
<td>Apartments (see Residential)</td>
<td>150</td>
</tr>
<tr>
<td>Assembly halls and other</td>
<td>60</td>
</tr>
<tr>
<td>places of assembly:</td>
<td>100</td>
</tr>
<tr>
<td>Fixed seats</td>
<td>100</td>
</tr>
<tr>
<td>Movable seats</td>
<td>125</td>
</tr>
<tr>
<td>Platform (assembly)</td>
<td>150</td>
</tr>
<tr>
<td>Balcony (exterior)</td>
<td>65</td>
</tr>
<tr>
<td>On one- and two-family</td>
<td>75</td>
</tr>
<tr>
<td>residences only and not</td>
<td></td>
</tr>
<tr>
<td>exceeding 100 sq ft</td>
<td></td>
</tr>
<tr>
<td>Bowling alleys, pinrooms,</td>
<td></td>
</tr>
<tr>
<td>and similar recreational</td>
<td></td>
</tr>
<tr>
<td>areas</td>
<td></td>
</tr>
<tr>
<td>Corridors</td>
<td></td>
</tr>
<tr>
<td>First floor</td>
<td>100</td>
</tr>
<tr>
<td>Other floors, except as</td>
<td></td>
</tr>
<tr>
<td>occupancy specified</td>
<td></td>
</tr>
<tr>
<td>Dance halls and ballrooms</td>
<td>100</td>
</tr>
<tr>
<td>Dining rooms and restaurants</td>
<td>100</td>
</tr>
<tr>
<td>Dwellings (see Residential)</td>
<td>100</td>
</tr>
<tr>
<td>Fire escapes</td>
<td>40</td>
</tr>
<tr>
<td>On multi- or single-family</td>
<td>40</td>
</tr>
<tr>
<td>residential buildings only</td>
<td>40</td>
</tr>
<tr>
<td>Garages (passenger cars only)</td>
<td>100</td>
</tr>
<tr>
<td>For trucks and buses use</td>
<td></td>
</tr>
<tr>
<td>A ASHTO limit load (1)</td>
<td></td>
</tr>
<tr>
<td>Grandstands (see Stadia and</td>
<td></td>
</tr>
<tr>
<td>areas bleachers)</td>
<td></td>
</tr>
<tr>
<td>Gymnasiums, main floors and</td>
<td></td>
</tr>
<tr>
<td>balconies</td>
<td></td>
</tr>
<tr>
<td>Hospitals</td>
<td></td>
</tr>
<tr>
<td>Operating rooms, laboratories</td>
<td>60</td>
</tr>
<tr>
<td>Private rooms</td>
<td>40</td>
</tr>
<tr>
<td>Ward rooms</td>
<td>40</td>
</tr>
<tr>
<td>Corridors, above first floor</td>
<td>80</td>
</tr>
</tbody>
</table>
**Figure A-43 Continued**

### Uniformly Distributed Loads

<table>
<thead>
<tr>
<th>Occupancy or Use</th>
<th>Live Load (psf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schools:</td>
<td></td>
</tr>
<tr>
<td>Classrooms</td>
<td>40</td>
</tr>
<tr>
<td>Corridors above first floor</td>
<td>80</td>
</tr>
<tr>
<td>Sidewalks, vehicular driveways, and yards, subject to</td>
<td></td>
</tr>
<tr>
<td>trucking (3)</td>
<td>250</td>
</tr>
<tr>
<td>Stadiums and arena bleachers (3)</td>
<td>100</td>
</tr>
<tr>
<td>Stairs and railways</td>
<td>106</td>
</tr>
<tr>
<td>Storage warehouses:</td>
<td></td>
</tr>
<tr>
<td>Light</td>
<td>125</td>
</tr>
<tr>
<td>Heavy</td>
<td>250</td>
</tr>
<tr>
<td>Stores:</td>
<td></td>
</tr>
<tr>
<td>Retail:</td>
<td></td>
</tr>
<tr>
<td>First floor</td>
<td>100</td>
</tr>
<tr>
<td>Upper floors</td>
<td>75</td>
</tr>
<tr>
<td>Wholesale, all floors</td>
<td>125</td>
</tr>
<tr>
<td>Walkways and elevated platforms (other than exitways)</td>
<td>60</td>
</tr>
</tbody>
</table>


### Concentrated Loads

<table>
<thead>
<tr>
<th>Location</th>
<th>Load (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Elevator machine room grazing (on area of 4 sq ft)</td>
<td>300</td>
</tr>
<tr>
<td>Finish light floor plate construction (on area of 1 sq ft)</td>
<td>200</td>
</tr>
<tr>
<td>Garages (4)</td>
<td></td>
</tr>
<tr>
<td>Office floors</td>
<td>2000</td>
</tr>
<tr>
<td>Skylights, skylight ribs, and accessible ceilings</td>
<td>200</td>
</tr>
<tr>
<td>Sidewalks</td>
<td>8000</td>
</tr>
<tr>
<td>Stair treads (on area of 4 sq ft at center of tread)</td>
<td>200</td>
</tr>
</tbody>
</table>

1. American Association of State Highway and Transportation Officials.
2. AASHTO base loads should also be considered where appropriate.
3. For detailed recommendations, see Assembly Seating, Stands and Aisle Supported Structures, AASHTO H-102-1978 [2.20.3].
4. Floors in garages or portions of buildings used for storage of motor vehicles shall be designed for the uniformly distributed live loads shown in the following concentration loads: (1) for passenger cars accommodating not more than nine passengers, 2000 pounds acting on an area of 20 in.*
5. (2) Mechanical parking structures without side or deck passenger cars only, 1500 pounds per wheel. (3) for trucks or buses, minimum live load on an area of 20 in.*
### Figure A-44  Dead Weights of Floors, Ceilings, Roofs, and Walls

#### Floors

<table>
<thead>
<tr>
<th>Flooring Description</th>
<th>Weight (psf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal weight concrete topping, per inch of thickness</td>
<td>12</td>
</tr>
<tr>
<td>Sand-lightweight (130 psf) concrete topping, per inch</td>
<td>10</td>
</tr>
<tr>
<td>Lightweight (90–100 psf) concrete topping, per inch</td>
<td>8</td>
</tr>
<tr>
<td>3 in. hardwood floor on sleepers clipped to concrete without fill</td>
<td>5</td>
</tr>
<tr>
<td>3 in. terrazzo floor finish directly on slab</td>
<td>19</td>
</tr>
<tr>
<td>3 in. terrazzo floor finish on 1 in. mortar bed</td>
<td>30</td>
</tr>
<tr>
<td>1 in. terrazzo finish on 2 in. concrete bed</td>
<td>38</td>
</tr>
<tr>
<td>1 in. ceramic or quarry tile on 1 in. mortar bed</td>
<td>16</td>
</tr>
<tr>
<td>1 in. ceramic or quarry tile on 1 in. mortar bed</td>
<td>22</td>
</tr>
<tr>
<td>1 in. linoleum or asphalt tile directly on concrete</td>
<td>1</td>
</tr>
<tr>
<td>1 in. linoleum or asphalt tile on 1 in. mortar bed</td>
<td>12</td>
</tr>
<tr>
<td>1 in. mastic floor</td>
<td>9</td>
</tr>
<tr>
<td>Hardwood flooring, 1 in. thick</td>
<td>4</td>
</tr>
<tr>
<td>Subflooring (soft wood), 1 in. thick</td>
<td>2/3</td>
</tr>
<tr>
<td>Asphaltic concrete, 1/8 in. thick</td>
<td>18</td>
</tr>
</tbody>
</table>

#### Ceilings

<table>
<thead>
<tr>
<th>Ceiling Description</th>
<th>Weight (psf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 in. gypsum board</td>
<td>2</td>
</tr>
<tr>
<td>1 in. gypsum board</td>
<td>24</td>
</tr>
<tr>
<td>1 in. plaster directly on concrete</td>
<td>5</td>
</tr>
<tr>
<td>1 in. plaster on metal lath furring</td>
<td>8</td>
</tr>
<tr>
<td>Suspended ceilings</td>
<td>2</td>
</tr>
<tr>
<td>Acoustical tile on wood furring strips</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Roofs

<table>
<thead>
<tr>
<th>Roof Description</th>
<th>Weight (psf)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ballasted inverted membrane</td>
<td>16</td>
</tr>
<tr>
<td>Five-ply felt and gravel (or slag)</td>
<td>64</td>
</tr>
<tr>
<td>Three-ply felt and gravel (or slag)</td>
<td>54</td>
</tr>
<tr>
<td>Five-ply felt composition roof, no gravel</td>
<td>4</td>
</tr>
<tr>
<td>Three-ply felt composition roof, no gravel</td>
<td>3</td>
</tr>
<tr>
<td>Asphalt strip shingles</td>
<td>3</td>
</tr>
<tr>
<td>Roof insulation, per inch</td>
<td>1</td>
</tr>
<tr>
<td>Gypsum, per inch of thickness</td>
<td>4</td>
</tr>
<tr>
<td>Insulating concrete, per inch</td>
<td>3</td>
</tr>
</tbody>
</table>

#### Walls

<table>
<thead>
<tr>
<th>Wall Description</th>
<th>Unplastered</th>
<th>One side plastered</th>
<th>Both sides plastered</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 in. brick wall</td>
<td>40</td>
<td>45</td>
<td>50</td>
</tr>
<tr>
<td>6 in. brick wall</td>
<td>80</td>
<td>85</td>
<td>90</td>
</tr>
<tr>
<td>12 in. brick wall</td>
<td>120</td>
<td>125</td>
<td>130</td>
</tr>
<tr>
<td>4 in. hollow normal weight concrete block</td>
<td>28</td>
<td>33</td>
<td>38</td>
</tr>
<tr>
<td>6 in. hollow normal weight concrete block</td>
<td>36</td>
<td>41</td>
<td>46</td>
</tr>
<tr>
<td>8 in. hollow normal weight concrete block</td>
<td>51</td>
<td>56</td>
<td>61</td>
</tr>
<tr>
<td>12 in. hollow normal weight concrete block</td>
<td>59</td>
<td>64</td>
<td>69</td>
</tr>
</tbody>
</table>
### Figure A-44 Continued

<table>
<thead>
<tr>
<th>Walls</th>
<th>Unplastered</th>
<th>One side plastered</th>
<th>Both sides plastered</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 in. hollow lightweight block or tile</td>
<td>19</td>
<td>24</td>
<td>29</td>
</tr>
<tr>
<td>6 in. hollow lightweight block or tile</td>
<td>22</td>
<td>27</td>
<td>32</td>
</tr>
<tr>
<td>8 in. hollow lightweight block or tile</td>
<td>33</td>
<td>34</td>
<td>34</td>
</tr>
<tr>
<td>12 in. hollow lightweight block or tile</td>
<td>44</td>
<td>49</td>
<td>54</td>
</tr>
<tr>
<td>4 in. brick 4 in. hollow normal weight block backing</td>
<td>60</td>
<td>73</td>
<td>78</td>
</tr>
<tr>
<td>4 in. brick 8 in. hollow normal weight block backing</td>
<td>91</td>
<td>96</td>
<td>101</td>
</tr>
<tr>
<td>4 in. brick 12 in. hollow normal weight block backing</td>
<td>119</td>
<td>124</td>
<td>128</td>
</tr>
<tr>
<td>4 in. brick 4 in. hollow lightweight block or tile backing</td>
<td>59</td>
<td>64</td>
<td>68</td>
</tr>
<tr>
<td>4 in. brick 8 in. hollow lightweight block or tile backing</td>
<td>75</td>
<td>78</td>
<td>82</td>
</tr>
<tr>
<td>4 in. brick 12 in. hollow lightweight block or tile backing</td>
<td>84</td>
<td>89</td>
<td>94</td>
</tr>
<tr>
<td>4 in. brick, steel or wood studs, 1 in. gypsum board</td>
<td>43</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Windows, glass, frame and sash</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4 in. stone</td>
<td>55</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel or wood studs, 2 in. plaster</td>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel or wood studs, 1 in. gypsum board each side</td>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Steel or wood studs, 2 layers 1 in. gypsum board each side</td>
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sets of working drawings, end-of-chapter problems, pictures of actual structural tests to failure, and flowcharts
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